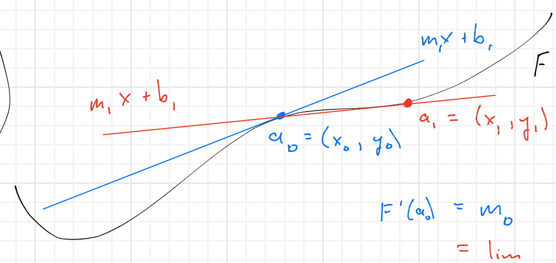
F is diff $\Rightarrow$ F' exist "everywhere"

$\Rightarrow F'(x) = \lim_{y \to x} \frac{F(x)-F(y)}{x-y} \left(= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \right)$

If $\mathrm{Dom}(F)$ is $\underset{\text{closed \& bounded}}{\text{compact}}$ then diff $\Rightarrow$ Lipschitz.

this has to do w/ uniform continuity — $\boxed{\text{look it up}}$



$F'(a_0) = m_0$

$= \lim_{a_1 \to a_0} \frac{F(x_0)-F(x_1)}{x_0 - x_1}$

$m_0 \approx m_1 = \frac{y_1 - y_0}{x_1 - x_0}$

$y_0 = F(x_0)$
$y_1 = F(x_1)$

Lip $\Rightarrow$ cts?

cts means that $\lim_{x\to a} F(x) = F(a)$

$\Leftrightarrow \lim_{x \to a} |F(x) - F(a)| = 0$

$$\mathrm{Lip}\, F = L = \sup_{x \neq y} \frac{|F(x)-F(y)|}{|x-y|}$$

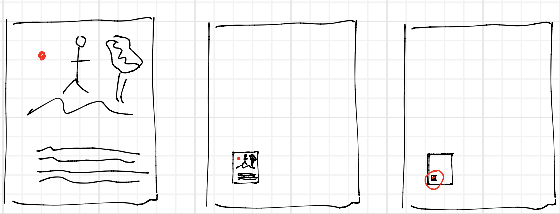
$$\geq \frac{F(x)-F(y)}{|x-y|} \quad \text{for any fixed } x, y$$

$\Leftrightarrow L|x-y| \geq |F(x)-F(y)| \geq 0$

$\Leftrightarrow \underbrace{\lim_{x\to y} L|x-y|}_{=0} \geq \lim_{x\to y} |F(x)-F(y)| \geq 0$

$\xrightarrow{\text{by Squeeze Thm}} \lim_{x\to y} |F(x)-F(y)| = 0 \quad \forall x, y$

$\Rightarrow F$ is cts. ☺



Candidate Fixed Point:

Choose some x_0
Define $x_{n+1} = F(x_n)$
Let $x_* = \lim_{n\to\infty} x_n$
Claim: $x_* = F(x_*)$

existence: key observation is that (x_n) is Cauchy

$d(F(x), F(y)) \leq c|x-y|$

$\Rightarrow d(x_m, x_n) \leq d(F^m(x_0) - F^n(x_0))$
$\leq c^{m-n} d(x_0, F^n(x_0))$

Key Property of an Orthonormal Transformation O is that

$$d(Ox, Oy) = d(x,y)$$

$d(\tau_b x, \tau_b y)$
$= d(x-b, y-b)$
$= |(x-b)-(y-b)|$

$d(x-b, x-y)$

$= \sqrt{[(x_1 - \cancel{b_1}) - (y_1 - \cancel{b_1})]^2 + [(x_2 - \cancel{b_2}) - (y_2 - \cancel{b_2})]^2 + \cdots + [(x_n - \cancel{b_n}) - (y_n - \cancel{b_n})]^2}$

$= \sqrt{(x_1-y_1)^2 + \cdots + (x_n - y_n)^2}$

$= d(x,y)$ in $\mathbb{R}^n$

$\boxed{\text{Read §2.4 — come up with at least 3 examples}}$

$X = \{0\}, \; Y = \{-1, 1\} \subseteq \mathbb{R}$

Argument: Suppose that $d(F(x), F(y)) \le r\, d(x,y)$ for all x, y in some complete metric space, $r<1$. Fix some x_0 in that space and for each $n \in \mathbb{N}$, define $x_n = F(x_{n-1})$.

Claim: (x_n) is Cauchy.

Proof: Fix $\varepsilon > 0$ and choose $M > \boxed{???}$. For $n > m > M$

$$
\begin{aligned}
d(x_m, x_n) &= d(F^m(x_0), F^n(x_0)) \\
&= r\, d(F^{m-1}(x_0), F^{n-1}(x_0)) \\
&\;\vdots \\
&= r^m d(x_0, F^{n-m}(x_0)) \\
&\le r^m \sum_{k=0}^{n-m-1} \underbrace{d(F^k(x_0), F^{k+1}(x_0))}_{= r^k d(x_0, F(x_0))} \qquad \text{(triangle inequality)} \\
&= r^m \sum_{k=0}^{n-m-1} r^k d(x_0, F(x_0)) \\
&= r^m d(x_0, F(x_0)) \underbrace{\sum_{k=0}^{n-m-1} r^k}_{\text{geometric}} \\
&= r^m d(x_0, F(x_0)) \frac{1-r^{n-m}}{1-r} \\
&\le r^m \frac{d(x_0, F(x_0))}{1-r} \\
&\le r^M \frac{d(x_0, F(x_0))}{1-r} \\
&< \varepsilon .
\end{aligned}
$$

$$
\begin{cases}
S := 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^{n-m-1}} \\
rS = \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cancel{r^4} + \cdots + r^{n-m} \\
\Rightarrow (1-r)S = 1 + r^{n-m} \\
\Rightarrow S = \dfrac{1-r^{n-m}}{1-r}
\end{cases}
$$

So we want to choose M so large that

$$
r^M \frac{d(x_0, F(x_0))}{1-r} < \varepsilon
$$

$$
\Rightarrow r^M < \frac{\varepsilon(1-r)}{d(x_0, F(x_0))}
$$

$$
\Rightarrow M \log(r) > \log\left(\frac{\varepsilon(1-r)}{d(x_0, F(x_0))}\right)
$$

$$
\Rightarrow M > \frac{\log\left(\frac{\varepsilon(1-r)}{d(x_0, F(x_0))}\right)}{\log(r)} = \log_r\left(\frac{\varepsilon(1-r)}{d(x_0, F(x_0))}\right)
$$

Pick M this way at $\boxed{???}$

Therefore (x_n) is Cauchy.

Since (x_n) is Cauchy in a complete metric space, the sequence (x_n) converges to some point x_*.

Claim: $F(x_*) = x_*$

Proof: Observe that

$$
x_* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}) = F(\lim_{n\to\infty} x_{n-1}) = F(x_*).
$$

Therefore $d(x_*, F(x_*)) = 0$. That is, $x_* = F(x_*)$. ∎