Def: Let $\{a_n\}$ be a sequence. We say that $\{a_n\}$ is Cauchy if for all $\varepsilon > 0$ there exists some N so large that $m, n > N$, then

$$d(a_m, a_n) < \varepsilon.$$

Ex: $\{a_n = \frac{1}{n}\}$

Fix $\varepsilon > 0$. WLOG $m < n$

$$|a_m - a_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} < \varepsilon$$

$$\Rightarrow m, n > \frac{1}{\varepsilon}.$$

Ex: $\{a_n = \sum_{i=1}^{n} \frac{1}{n}\}$ is not Cauchy

HW: Prove this.

In $\mathbb{R}$, Cauchy $\Rightarrow$ convergent. What is $\lim_{n\to\infty} a_n$ when $a_n = \frac{1}{n}$?

In $\mathbb{R} \setminus \{0\}$, Cauchy does not imply convergent (necessarily). For example, $\{a_n = \frac{1}{n}\}$ does not converge in $\mathbb{R} \setminus \{0\}$, but it is (still) Cauchy.

UR-EXAMPLE: $\mathbb{Q}$ is not complete

HW: Prove this

$\boxed{\text{HW}}$ Find examples of sequences which are not Cauchy.

Let $f(x) = |x|$ on $\mathbb{R}$. This is cts but not diff. Integrate:

$$F(x) = \int_0^x f(t)\,dt$$

Then F is C^1 but not C^∞

In general, take any funct w/ a cusp or corner, and integrate.

$$C^1 \supseteq C^2 \supseteq C^3 \supseteq \cdots \supseteq C^\infty$$

<u>Examples of C^∞:</u>

polynomials

sin, cos

$x \mapsto e^x$

Ex of an application of p-tuples to representing rational numbers w/ finite length decimal expansions.

e.g $\frac{1}{4} = 0.25 = 2 \cdot 10^{-1} + 5 \cdot 10^{-2}$

$\sim \langle 2, 5 \rangle$

Ex] $\alpha < \beta$

$$\left.\begin{array}{l} \alpha = \langle 2, 5 \rangle \\ \beta = \langle 2, 5, 1 \rangle \end{array}\right\} \quad \alpha \lneq \beta$$

here, $p = 2$, $q = 1$

$\gamma = \langle 1, 2, 5 \rangle$

Is $\alpha < \gamma$? No!

$\boxed{\text{HW}}$ Read Section 2.1. Cook up examples for (3)-(7) or prepare special lists of IDKs.

Prove Prop (8).