In $\mathbb{R}$, $\delta(\{3\},\{7\}) = 4$

$\delta(\{1,7\},\{3,6\})$

$= \sup\{ d(a,\{3,6\}), d(\{1,7\},b) \mid a\in\{1,7\}, b\in\{3,6\}\}$

$= \sup\{2, 1, 2, 1\}$

$= 2$

Take $X =$ unit circle $= \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 = 1\}$

$Y = X \cup \{*\}$

$\uparrow$ $(0,0)$ or $(0,1)$ or ...

(i) $\delta(F(A),F(B)) \leq \mathrm{Lip}(F)\delta(A,B)$,

(ii) $\delta\left(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\right) \leq \sup_{i\in I} \delta(A_i,B_i)$.

(1) If $F: X\to X$ is a contraction, then $F: \mathcal{B}(X)\to\mathcal{B}(X)$ is <u>also</u> a contraction.

$\mathcal{B}(X) =$ set of all nonempty, closed, bounded subsets of X

$\Rightarrow$ Banach fixed point thm then says that there is a set $A\in\mathcal{B}(X)$ such that

$F(A) = A$

<u>Needs</u> $(\mathcal{B}(X),\delta)$ is complete

$F: X \to X$

$x \mapsto F(x)$

singleton points

By an abuse of notation, if $A\subseteq X$, we can write

$F(A) = \{F(x) \mid x\in A\}$

e.g. $F:\mathbb{R}\to\mathbb{R}$

$x\mapsto x^2$

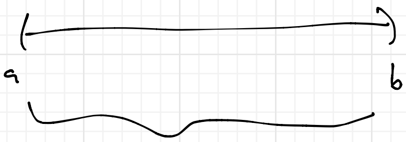
$F((-2,3))$

$= \{F(x) \mid x\in(-2,3)\}$

$= [0,9)$

HW: Convince yourselves that (i) & (ii) on page 719 are true

In $\mathbb{R}$, what should the "measure" of an interval (a,b) be?



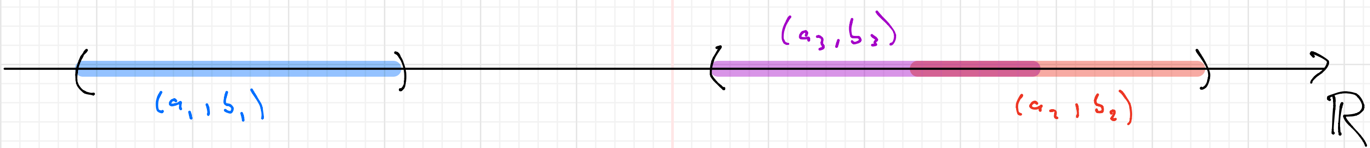
length = measure = $b-a$

$$\mu((a,b)) = b-a$$

$$\mu(\emptyset) = 0 \quad ?$$

since $\mu(\emptyset) = \mu((0,0))$
$= 0-0$
$= 0$

$$\{(a_i, b_i)\} \subseteq \mathcal{P}(\mathbb{R})$$

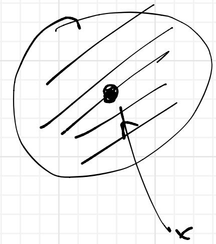


$$\begin{aligned}
&\mu\big((a_1,b_1)\cup(a_2,b_2)\cup(a_3,b_3)\big) \\
&= \mu\big((a_1,b_1)\cup(a_3,b_2)\big) \\
&= \mu((a_1,b_1)) + \mu((a_3,b_2)) \\
&= (b_1-a_1) + (b_2-a_3) \\
&< (b_1-a_1) + (b_2-a_2) + (b_3-a_3) \\
&= \sum_{i=1}^{3} \mu(a_i,b_i)
\end{aligned}$$

HW: Read §2.5 (2-3)
Try to make heads or tails of it

In $\mathbb{R}^2$, a ball is a disk:

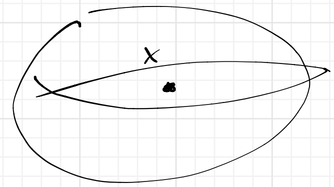
$$\text{volume} = \pi \underbrace{r}_{\text{radius}}^2$$



$\{y \in \mathbb{R} \mid x^2+y^2 \leq r^2\}$

In $\mathbb{R}^3$, a ball is a ball:

$$\text{volume} = \frac{4}{3}\pi r^3$$



In general, in a metric space X, the "volume" of a ball $B(x,r) = \{y \in X \mid x^2+y^2 \leq r^2\}$ is $C r^s$.

READ & UNDERSTAND:

2.6. Hausdorff measure.

(1) Let the real number $k \geq 0$ be fixed. For every $\delta > 0$ and $E \subset X$ we define

$$\mathcal{H}^k_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} \alpha_k 2^{-k} (\operatorname{diam} E_i)^k : E \subset \bigcup_{i=1}^{\infty} E_i, \operatorname{diam} E_i \leq \delta\right\}$$

$$\mathcal{H}^k(E) = \lim_{\delta \to 0} \mathcal{H}^l_\delta(E) = \sup_{\delta \geq 0} \mathcal{H}^k_\delta(E).$$

$\mathcal{H}^k(E)$ is called the *Hausdorff k-dimensional measure of E*. A reference

Try to compute

$$\mathcal{H}^s(\underbrace{(0,1)}_{\text{interval } (0,1) \subseteq \mathbb{R}})$$

for various s.

$$\operatorname{diam}(E) = \sup\{d(x,y) \mid x, y \in E\}$$