Math 151C (Advanced Calculus)
Discussion of the Parallelogram Law Problem

Proposition 1. *Suppose that V is a complex normed vector space such that the norm satisfies the parallelogram law, i.e.*

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

for all $u, v \in V$. Then there exists a Hermitian inner product of V which induces the norm.

Proving this result can be done in two major steps: (1) find a candidate function for the inner product, then (2) show that this function is an inner product with all of the desired properties.

Finding a candidate inner product

The goal is to find a function

$$(\cdot,\cdot) : V \times V \longrightarrow \mathbb{C}$$

which satisfies all of the properties of the inner product (linearity in the first term, conjugate linearity in the second term, etc). One approach to this is to begin by assuming that such a function exists, then using its properties to deduce its form. So, suppose that such an inner product exists.

If it exists, then

$$\begin{aligned}
(u,v) &= (u+v,v) - (v,v) && \text{(linearity in the first term)}\\
&= (u+v,u+v) - (u+v,u) - (v,v) && \text{(conjugate linearity in the second term)}\\
&= (u+v,u+v) - (u,u) - (v,u) - (v,v) && \text{(linearity in the first term)}\\
&= \|u+v\|^2 - \|u\|^2 - (v,u) - \|v\|^2. && \text{(the inner product induces the norm)}
\end{aligned}$$

The intuition here is that the inner product (if it exists) induces the norm. Since the properties of the norm are understood, it is desirable to replace instances of inner products with norms whenever possible.

A Hermitian inner product is conjugate symmetric, which means that $(v,u) = \overline{(u,v)}$. Moreover, since $z + \overline{z} = 2\Re(z)$ for any $z \in \mathbb{C}$, it follows that

$$2\Re(u,v) = (u,v) + \overline{(u,v)} = (u,v) + (v,u) = \|u+v\|^2 - \|u\|^2 - \|v\|^2.$$

Therefore

$$\Re(u,v) = \frac{1}{2}\left(\|u+v\|^2 - \|u\|^2 - \|v\|^2\right). \tag{1}$$

Similarly,

$$(u,iv) = \|u+iv\|^2 - \|u\|^2 - (iv,u) - \|v\|^2.$$

Note also that, as the inner product is linear in the first term and conjugate linear in the second term,

$$(u, iv) + (iv, u) = -i(u, v) + i(v, u) = i[-(u, v) + \overline{(u, v)}] = -i2\Im(u, v),$$

since $z - \overline{z} = 2i\Im(z)$ for any $z \in \mathbb{C}$. Therefore

$$-i2\Im(u, iv) = (u, iv) + (iv, u) = \|u + iv\|^2 - \|u\|^2 - \|iv\|^2.$$

In other words,

$$\Im(u, v) = \frac{i}{2}\left(\|u + iv\|^2 - \|u\|^2 - \|v\|^2\right). \tag{2}$$

combining the identities at (1) and (2), if the norm is induced by a Hermitian inner product, then the inner product must be given by

$$(u, v) = \frac{\|u + v\|^2 - \|u\|^2 - \|v\|^2}{2} + i\frac{\|u + iv\|^2 - \|u\|^2 - \|iv\|^2}{2}. \tag{3}$$

As the inner product satisfies the parallelogram law, this may be simplified to

$$(u, v) = \frac{\|u + v\|^2 - \|u - v\|^2}{4} + i\frac{\|u + iv\|^2 - \|u - iv\|^2}{4}. \tag{4}$$

It is worth mentioning that the result in (4) is well-known. It is a property which is satisfied by all Hermitian inner products, called the *polarization identity*.

Showing that this is an inner product

In order to show that the function in (4) is, in fact, an inner product, it is necessary to verify that

(i) $(u, u) = \|u\|^2$—verifying this property automatically gives $(u, u) \geq 0$ for all $u \in V$ with equality if and only if $u = 0$;

(ii) $(u_1 + u_2, v) = (u_1, v) + (u_2, v)$ for all $u_1, u_2, v \in V$;

(iii) $(\lambda u, v) = \lambda(u, v)$ for all $u, v \in V$ and $\lambda \in \mathbb{C}$; and

(iv) $\overline{(u, v)} = (v, u)$ for all $u, v \in V$.

Note that it is unnecessary to prove conjugate linearity in the second term—this property follows from the properties given above. That is, the conjugate symmetry of an inner product (property (iv), combined with the linearity in the fist term (properties (ii) and (iii)), gives conjugate linearity in the second term.

Proof of (i). Let $u \in V$ be arbitrary. A norm must satisfy the scaling property $\|\lambda u\| = |\lambda|\|u\|$ for any $u \in V$ and $\lambda \in \mathbb{C}$. Hence

$$\|u - iu\| = |i|\|u - iu\| = \|i(u - iu)\| = \|u + iu\|.$$

Therefore, being a bit pedantic,

$$
\begin{aligned}
(u,u) &= \frac{\|u+u\|^2 - \|u-u\|^2}{4} + i\frac{\|u+iu\|^2 - \|u-iu\|^2}{4} \\
&= \frac{4\|u\|^2 - \|0\|^2}{4} + i\frac{0}{4} \\
&= \|u\|^2,
\end{aligned}
$$

which verifies property (i). □

Proof of (*ii*). Let $u_1, u_2, v \in V$ be arbitrary. By defintion,

$$
(u_1 + u_2, v) = \frac{\|u_1 + u_2 + v\|^2 - \|u_1 + u_2 - v\|^2}{4} + i\frac{\|u_1 + u_2 + iv\|^2 - \|u_1 + u_2 - iv\|^2}{4}.
$$

The goal is to show that this is equal to $(u_1, v) + (u_2, v)$, hence the next steps come from trying to get the terms of this sum to appear. For example, the term $\|u_1 + v\|^2$ appears in the inner product (u_1, v). Working with the first term in the first fraction above, the parallelogram law gives

$$
\|u_1 + u_2 + v\|^2 = 2\|u_1 + v\|^2 + 2\|u_2\|^2 - \|u_1 - u_2 + v\|^2.
$$

Hence the term $\|u_1 + v\|^2$ can be made to appear, at the cost of some other terms. Similarly,

$$
\|u_1 + u_2 + v\|^2 = 2\|u_2 + v\|^2 + 2\|u_1\|^2 - \|-u_1 + u_2 + v\|^2.
$$

Averaging these two expressions renders the identity

$$
\begin{aligned}
&\|u_1 + u_2 + v\|^2 \\
&\quad = \|u_1 + v\|^2 + \|u_2 + v\|^2 + \|u_1\|^2 + \|u_2\|^2 - \|u_1 - u_2 + v\|^2 - \|-u_1 + u_2 + v\|^2.
\end{aligned}
$$

It is not immediately obvious why this should be useful, but bear with me. Replacing v with $-v$ and taking advantage of the fact that $\|-u\| = \|u\|$ (this is used in the last two terms),

$$
\begin{aligned}
&\|u_1 + u_2 - v\|^2 \\
&\quad = \|u_1 - v\|^2 + \|u_2 - v\|^2 + \|u_1\|^2 + \|u_2\|^2 - \|u_1 - u_2 - v\|^2 - \|-u_1 + u_2 - v\|^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\|u_1 + u_2 + v\|^2 - \|u_1 + u_2 - v\|^2}{4} &= \frac{\|u_1 + v\|^2 + \|u_2 + v\|^2 - \|u_1 - v\|^2 - \|u_2 - v\|^2}{4} \\
&= \frac{\|u_1 + v\|^2 - \|u_1 - v\|^2}{4} + \frac{\|u_2 + v\|^2 - \|u_2 - v\|^2}{4}.
\end{aligned}
$$

Similar computations make it possible to rewrite $\|u_1 + u_2 \pm iv\|^2$, ultimately giving

$$\begin{aligned}
(u_1 + u_2, v) &= \frac{\|u_1 + u_2 + v\|^2 - \|u_1 + u_2 - v\|^2}{4} + i\frac{\|u_1 + u_2 + iv\|^2 - \|u_1 + u_2 - iv\|^2}{4} \\
&= \frac{\|u_1 + v\|^2 - \|u_1 - v\|^2}{4} + \frac{\|u_2 + v\|^2 - \|u_2 - v\|^2}{4} \\
&\qquad + i\frac{\|u_1 + iv\|^2 - \|u_1 - iv\|^2}{4} + i\frac{\|u_2 + iv\|^2 - \|u_2 - iv\|^2}{4} \\
&= \left(\frac{\|u_1 + v\|^2 - \|u_1 - v\|^2}{4} + i\frac{\|u_1 + iv\|^2 - \|u_1 - iv\|^2}{4}\right) \\
&\qquad \left(\frac{\|u_2 + v\|^2 - \|u_2 - v\|^2}{4} + i\frac{\|u_2 + iv\|^2 - \|u_2 - iv\|^2}{4}\right) \\
&= (u_1, v) + (u_2, v).
\end{aligned}$$

Hence this purported inner product is additive in the first term. □

Proof of (iii). Let $u, v \in V$ be arbitrary. First, note that

$$\begin{aligned}
(-u, v) &= \frac{\| - u + v\|^2 - \| - u - v\|^2}{4} + i\frac{\| - u + iv\|^2 - \| - u - iv\|^2}{4} \\
&= \frac{\|u - v\|^2 - \|u + v\|^2}{4} + i\frac{\|u - iv\|^2 - \|u + iv\|^2}{4} \\
&= (-1)\left(\frac{-\|u - v\|^2 + \|u + v\|^2}{4} + i\frac{-\|u - iv\|^2 + \|u + iv\|^2}{4}\right) \\
&= -(u, v).
\end{aligned}$$

Combining this with Property (ii) and an induction argument, it may be shown that

$$(nu, v) = n(u, v)$$

for any integer number n. Now suppose that $q \in \mathbb{Q}$. By definition, there are $m, n \in \mathbb{Z}$ such that $q = m/n$. It then follows from the previous results that

$$n(qu, v) = n\left(\frac{m}{n}u, v\right) = (mu, v) = m(u, v).$$

Dividing through by n gives

$$(qu, v) = q(uv).$$

Next, suppose that $x \in \mathbb{R}$. Choose some sequence $(q_j)_{j\in\mathbb{N}}$ so that $q_j \to x$. The function $x \mapsto (xu, v)$ is continuous with respect to the topology induced by $\| \cdot \|$, as it is the composition of several sums and scalar multiplications. It therefore follows that

$$(xu, v) = \left(\lim_{j\to\infty} q_j u, v\right) = \lim_{j\to\infty}(q_j u, v) = \lim_{j\to\infty} q_j(u, v) = x(u, v).$$

Recall that $\|\pm iw\| = \|w\|$ for any $w \in V$. Hence

$$\begin{aligned}
(iu, v) &= \frac{\|iu + v\|^2 - \|iu - v\|^2}{4} + i\frac{\|iu + iv\|^2 - \|iu - iv\|^2}{4} \\
&= \frac{\|u - iv\|^2 - \|u + iv\|^2}{4} + i\frac{\|u + v\|^2 - \|u - v\|^2}{4} \\
&= i\left(-i\frac{\|u - iv\|^2 - \|u + iv\|^2}{4} + \frac{\|u + v\|^2 - \|u - v\|^2}{4}\right) \\
&= i\left(\frac{\|u + v\|^2 - \|u - v\|^2}{4} + i\frac{\|u + iv\|^2 - \|u - iv\|^2}{4}\right) \\
&= i(u, v).
\end{aligned}$$

Finally, if $\lambda \in \mathbb{C}$, then there are $x, y \in \mathbb{R}$ so that $\lambda = x + iy$. Using the above derived results,

$$(\lambda u, v) = ((x + iy)u, v) = (xu, v) + (iyu, v) = x(u, v) + iy(u, v) = (x + iy)(u, v) = \lambda(u, v).$$

Thus, finally, Property (iii) has been verified. □

Proof of (*iv*). Let $u, v \in V$ be arbitrary. Then

$$\begin{aligned}
\overline{(u, v)} &= \frac{\|u + v\|^2 - \|u - v\|^2}{4} - i\frac{\|u + iv\|^2 - \|u - iv\|^2}{4} \\
&= \frac{\|u + v\|^2 - \|u - v\|^2}{4} - i\frac{\|iu - v\|^2 - \|iu + v\|^2}{4} \\
&= \frac{\|v + u\|^2 - \|v - u\|^2}{4} + i\frac{\|v + iu\|^2 - \|v - iu\|^2}{4} \\
&= (v, u).
\end{aligned}$$

This verifies Property (iv), which completes the proof that $(\cdot, \cdot)$ is an inner product. □