$f : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

for each $j = 1, \dots, n$, there is some M_j such that

$$|D_j f(x)| < M_j \quad \text{for all } x \in E$$

Prove that f is continuous.

Proof: Fix $x \in E$, $\varepsilon > 0$. Set $\delta < \frac{\varepsilon}{n \max\{M_1, \dots, M_n\}}$.

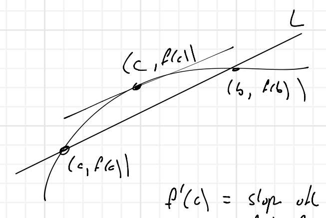
If $y \in B(x, \delta) \cap E$ (i.e. $\|x - y\| < \delta$), then

$$
\begin{aligned}
|f(y) - f(x)| &\le |f(y_1, y_2, \dots, y_n) - f(y_1, y_2, \dots, y_{n-1}, x_n)| \\
&\quad + |f(y_1, y_2, \dots, y_{n-1}, x_n) - f(y_1, y_2, \dots, y_{n-2}, x_{n-1}, x_n)| \\
&\quad + \cdots \\
&\quad + |f(y_1, x_2, \dots, x_{n-1}, x_n) - f(x_1, x_2, \dots, x_{n-1}, x_n)| \\
&\le \left.\begin{aligned} &M_n |y_n - x_n| \\ &+ M_{n-1} |y_{n-1} - x_{n-1}| \\ &+ \cdots \\ &+ M_1 |y_1 - x_1| \end{aligned}\right\} \text{ by MVT in one dimension} \\
&\le (M_1 + M_2 + \cdots + M_n)\delta \\
&\le n \max\{M_1, M_2, \dots, M_n\} \delta \\
&< \varepsilon. \quad \blacksquare
\end{aligned}
$$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$
\begin{aligned}
|f(y) - f(x)| &= |f(y_1, y_2) - f(x_1, x_2)| \\
&= |f(y_1, y_2) - f(y_1, x_2) \\
&\qquad + f(y_1, x_2) - f(x_1, x_2)| \\
&\le |f(y_1, y_2) - f(y_1, x_2)| + |f(y_1, x_2) - f(x_1, x_2)| \\
&\le M_2 |y_2 - x_2| + M_1 |y_1 - x_1| \\
&< (M_1 + M_2)\delta \\
&< \varepsilon
\end{aligned}
$$

$\rightsquigarrow$ choose $\delta < \frac{\varepsilon}{2 \max\{M_1, M_2\}}$



$|D_2 f(x)| \le M_2 \quad \forall x \in E$

$$
\begin{aligned}
\Rightarrow |f(y_1, y_2) - f(y_1, x_2)| &= |D_2 f(y_1, c)| \, |(y_2 - x_2)| \\
&\le M_2 |y_2 - x_2|
\end{aligned}
$$

$f'(c) = \text{slope of } L = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow f'(c)(b - a) = f(b) - f(a)$

By induction: suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Define $f_{n-1} : \mathbb{R}^{(n-1)+1} \longrightarrow \mathbb{R}$

$$(\vec{x}, x_n) \longmapsto f(x_1, x_2, \dots, x_{n-1}, x_n)$$

$\uparrow$

$(x_1, \dots, x_{n-1})$