Q1 = Rudin 9.8
Q2 = Rudin 9.9
Q3 = Rudin 9.6
Q4 : ???

Q4 (10 points)

Decide whether the following is true or false and prove your conclusion.

Statement: Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be such that for every $x \in \mathbb{R}^m$ and every unit vector $e \in \mathbb{R}^m$, the directional derivative of f at x in the direction e exists. Then f is differentiable.

$f(x,y) = \dfrac{x^3}{x^2+y^2}$?

If f is differentiable, then $D_e f$ exists for all e.

$D_e f(\vec{0})$ $\qquad e = \langle e_1, e_2 \rangle$

$$= \lim_{t\to 0} \frac{f(\vec{0}+te) - f(\vec{0})}{t}$$

$$= \lim_{t\to 0} \frac{\dfrac{t^3 e_1^3}{t^2e_1^2 + t^2e_2^2}}{t}$$

$$= \lim_{t\to 0} \frac{\cancel{t} e_1^3}{\cancel{t}(e_1^2+e_2^2)}$$

$$= \lim_{t\to 0} e_1^3$$

$$= e_1^3$$

$D_1 f(0) = 1$
$D_2 f(0) = 0$

Defn 9.11: f is diff @ x if $\exists A$

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0$$

Th 9.17: $f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i$ $\qquad f : \mathbb{R}^n \to \mathbb{R}^m$
$\{u_i\}$ is std basis for $\mathbb{R}^m$
$\{e_j\}$ is std basis for $\mathbb{R}^n$

$f'(x)e_j = (D_1 f)(x)$ $\qquad \mathbb{R}^2 \to \mathbb{R}$

$f'(x) = (D_1 f(x), D_2 f(x))$

$D_1 f(0) = D_1\left(\dfrac{x^3}{x^2+y^2}\right)\Big|_{(x,y)=(0,0)}$

$= \dfrac{3x^2(x^2+y^2) - x^3(2x)}{(x^2+y^2)^2}$

$=$

OR $\quad D_1 f(0) = \lim_{h\to 0} \dfrac{f(x+te_1) - f(x)}{t}$

IF f is diff @ 0, then $f'(0) = [1, 0]$ by Th 9.17.

In the definition of f', $A = [1, 0]$

$$\lim_{h\to 0} \frac{|f(0+h) - \cancel{f(0)} - Ah|}{|h|} = \lim_{h\to 0} \frac{\left|f(h_1,h_2) - [1,0]\binom{h_1}{h_2}\right|}{|(h_1,h_2)|}$$

$$\lim_{h\to 0} \frac{\left|\dfrac{h_1^3}{h_1^2+h_2^2} - h_1\right|}{|h_1^2+h_2^2|^{\frac{1}{2}}} = \lim_{h\to 0} \frac{|h_1^3 - h_1^3 - h_1h_2^2|}{\sqrt{h_1^2+h_2^2}\,(h_1^2+h_2^2)}$$

$$= \lim_{h\to 0} \frac{|h_1 h_2^2|}{(h_1^2+h_2^2)^{3/2}} = \lim_{h_1=h_2\to 0} \frac{h_1^3}{(2h_1^2)^{3/2}} = \left(\frac{1}{2}\right)^{3/2} \neq 0$$