Stirling:

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} = 1 \qquad\qquad \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \approx 1$$

$$\Rightarrow n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Show $\displaystyle\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\,\Gamma(x)} = 1 \quad \forall c \in \mathbb{R}$

$$\Gamma(x+c) = \Gamma(x+c+1-1) = \Gamma((x+c-1)+1) \approx \left(\frac{x+c-1}{e}\right)^{x+c-1}\sqrt{2\pi(x+c-1)}$$

$$\Gamma(x) = \Gamma(x-1+1) = \Gamma((x-1)+1) \approx \left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}$$

$$\lim_{x\to\infty}\frac{\Gamma(x+c)}{x^c\,\Gamma(x)} = \lim_{x\to\infty}\frac{\left(\frac{x+c-1}{e}\right)^{x+c-1}\sqrt{2\pi(x+c-1)}}{\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}\;x^c}$$

If $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$ for all $u,v \in V$ a cplx vector space, then $\|\cdot\|$ is induced by a Hermitian inner product.

Want to show $\exists$ a Hermitian inner product $(\cdot,\cdot)$ s.t. $\|u\|^2 = (u,u)$.

* $(u+v, w) = (u,w) + (v,w)$ & $(u, v+w) = (u,v) + (u,w)$
* $(au, v) = a(u,v)$ & $(u, av) = \bar{a}(u,v)$
* $\overline{(u,v)} = (v,u)$
* $(u,u) \geq 0$, 0 iff $u = 0$

$$(\cdot,\cdot) : V\times V \longrightarrow \mathbb{C}, \quad (u,v) \longmapsto \tfrac{1}{2}\left(\|u+v\|^2 - \|u\|^2 - \|v\|^2\right)$$

$$(u, \alpha v) = \bar{\alpha}(u,v)$$

$(u,u) = \|u\|^2$

$(au,u) = a\|u\|^2$ & $(u,au) = \bar{a}\|u\|^2$

$$\begin{aligned}(u,v) &= (u+v-v, v) = (u+v,v) - (v,v)\\ &= (u+v, u+v-u) - (v,v)\\ &= (u+v, u+v) - (u+v, u) - (v,v)\\ &= (u+v, u+v) - (u,u) - \underbrace{(v,u)}_{\overline{(u,v)}} - (v,v)\end{aligned}$$

$$\underbrace{(u,v) + \overline{(u,v)}}_{2\,\mathrm{Re}((u,v))} = \|u+v\|^2 - \|u\|^2 - \|v\|^2$$

$$\begin{aligned}(u,-v) &= (u+v-v, v) = (u-v, v) + (+v, v)\\ &= (u-v, v-u+u) + (+v,v)\\ &= (u-v, -(u-v)) + (u-v, u) + (v,v)\\ &= -(u-v, u-v) + (u,u) - (v,u) + (v,v)\\ &= -\|u-v\|^2 + \|u\|^2 - \underbrace{(v,u)}_{\overline{(u,v)}} + \|v\|^2\end{aligned}$$

$$(u,-v) + \overline{(u,v)} = \|u\|^2 + \|v\|^2 - \|u-v\|^2$$

$$\begin{aligned}(u,v) &= (u+v-v, v)\\ &= (u+v, v) - (v,v)\\ &= (u+v, v+u-u) - (v,v)\\ &= (u+v, u+v) + (u+v, -u) - (v,v)\\ &= (u+v, u+v) - (u,u) - (v,u) - (u,u)\\ &= \|u+v\|^2 - \|u\|^2 - \overline{(u,v)} - \|v\|^2\end{aligned}$$

$$\begin{aligned}\Rightarrow 2\,\mathrm{Re}(u,v) &= (u,v) + \overline{(u,v)}\\ &= \|u+v\|^2 - \|u\|^2 - \|v\|^2\end{aligned}$$

$$\begin{aligned}2i\,\mathrm{Im}(u,v) &= (u,v) - \overline{(u,v)}\\ \Rightarrow 2\,\mathrm{Im}(u,v) &= -i(u,v) + i\overline{(u,v)}\\ &= (u, iv) + i(v,u)\\ &= (u, iv) + (iv, u)\end{aligned}$$

$$\begin{aligned}(u,iv) &= (u+iv-iv, iv)\\ &= (u+iv, iv) - (iv, iv)\\ &= (u+iv, iv+u-u) - \|v\|^2\\ &= (u+iv, u+iv) + (u+iv, -u) - \|v\|^2\\ &= (u+iv, u+iv) - (u,u) - (iv,u) - \|v\|^2\\ &= \|u+iv\|^2 - \|u\|^2 - \overline{(u,iv)} - \|v\|^2\end{aligned}$$

$$\begin{aligned}\Rightarrow (u,iv) + \overline{(u,iv)} &= \|u+iv\|^2 - \|u\|^2 - \|v\|^2\\ &= (u,iv) + (iv,u)\\ &= -i(u,v) + i(v,u)\\ &= -i(u,v) + i\overline{(u,v)}\\ &= i\left(-(u,v) + \overline{(u,v)}\right)\\ &= i\left(-2i\,\mathrm{Im}(u,v)\right)\\ &= 2\,\mathrm{Im}(u,v)\end{aligned}$$

That is, $2\,\mathrm{Im}(u,v) = \|u+iv\|^2 - \|u\|^2 - \|v\|^2$

Candidate IP:

$\|iv\|^2$

$$\begin{aligned}(u,v) &= \frac{\|u+v\|^2 - \|u\|^2 - \|v\|^2}{2} + i\,\frac{\|u+iv\|^2 - \|u\|^2 - \|v\|^2}{2}\\ &= \frac{\|u+v\|^2 - \frac{1}{2}\left(\|u+v\|^2 + \|u-v\|^2\right)}{2} + i\,\frac{\|u+iv\|^2 - \frac{1}{2}\left(\|u+iv\|^2 + \|u-iv\|^2\right)}{2}\\ &= \frac{\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2}{4}\end{aligned}$$