$\alpha \in \mathbb{R}$ $\qquad x \in (-1,1)$

Show that $(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$ ⊛

$$\left.\begin{aligned}(x^\alpha)' &= \alpha x^{\alpha-1}\\ (x^{\alpha-1})' &= (\alpha-1)x^{\alpha-2}\\ &\vdots\end{aligned}\right\} ???$$

Proof: Let $f(x) = 1+\sum_{n=1}^{\infty}\cdots$. Let a_n denote the n^{th} term of the power series ⊛. Observe that

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}\left|\frac{\cancel{\alpha(\alpha-1)\cdots}(\alpha-n)}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{\alpha(\alpha-1)\cdots(\alpha-n+1)}} \cdot \frac{x^{\cancel{n+1}}}{\cancel{x^n}}\right|$$

$$= \limsup_{n\to\infty} \frac{|\alpha-n|}{n+1}|x| = |x|.$$

Note that $|x|<1$ whenever $x\in(-1,1)$, so by the Ratio Test (Th 3.34) the series converges whenever $x\in(-1,1)$.

Observe that

$$(1+x)f'(x) = \frac{d}{dx}\left(1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n\right)(1+x)$$

$$= \left(\sum_{n=1}^{\infty}\frac{d}{dx}\left(\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n\right)\right)(1+x) \qquad \text{by Corollary to Th 8.1}$$

$$= \left(\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^{n-1}\right)(1+x)$$

$$= \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^{n-1} + \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^{n}$$

$$= \sum_{n=0}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)}{n!}x^{n} + \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^{n}$$

$$= \alpha + \sum_{n=1}^{\infty}\left(\frac{\boxed{\alpha(\alpha-1)\cdots(\alpha-n+1)}(\alpha-n)}{n!} + \frac{n\boxed{\alpha(\alpha-1)\cdots(\alpha-n+1)}}{n(n-1)!}\right)x^n$$

$$= \alpha + \sum_{n=1}^{\infty}\left(\frac{[\alpha(\alpha-1)\cdots(\alpha-n+1)][(\alpha-n)+n]}{n!}\right)x^n$$

$$= \alpha + \alpha\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$

$$= \alpha\left(1+\sum\cdots\right)$$

$$= \alpha f(x).$$

Therefore $(1+x)f'(x) = \alpha f(x)$ for all $x\in(-1,1)$.

Let $y=f(x)$. Then this equation is

$$(1+x)\frac{dy}{dx} = \alpha y \implies \frac{1}{y}dy = \frac{\alpha}{1+x}dx$$

$$\implies \int\frac{1}{y}dy = \int\frac{\alpha}{1+x}dx$$

$$\implies \log|y| = \alpha\log(1+x)+C = \log\left((1+x)^\alpha\right)+C$$

$$\implies y = k(1+x)^\alpha.$$

Since $f(0)=1$, it follows that $y\big|_{x=0} = f(0) = 1 = k(1+0)^\alpha = k$. Thus $k=1$. Therefore

$$f(x) = (1+x)^\alpha.$$ ∎

Th 8.18 (b): $\Gamma(n+1) = n!$ $\forall n\in\mathbb{N}$

Th 8.19 (a): If $f:(0,\infty)\to(0,\infty)$ and
(a) $f(x+1) = x f(x)$
(b) $f(1) = 1$
(c) $\log f$ is convex
Then $f(x) = \Gamma(x)$

$$\alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \qquad \forall \alpha>0$$

By Induction: $\Gamma(\alpha+n) = (\alpha+n-1)\Gamma(\alpha+n-1)$ by 8.18(a)
$= (\alpha+n-1)(\alpha+n-2)\Gamma(\alpha+n-2)$
$= \cdots$
$= (\alpha+n-1)\cdots(\alpha)\Gamma(\alpha)$