$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

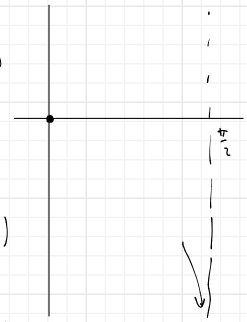
$$f(x) = x^3 - \underbrace{\sin(x)^2}_{\leq x^2} \tan(x) \approx x^2(x - \tan(x))$$

$\sin(x) \leq x \quad \forall x \in [0,\infty)$

$\Rightarrow \sin(x)^2 \leq x^2$

$$f(x) = x^3 - \sin(x)^2 \tan(x) \geq x^3 - x^2 \tan(x)$$

as $x \to \frac{\pi}{2}$, $f(x) \to -\infty$



$$f(x) = x^3 - \sin(x)^2 \tan(x)$$

$$\begin{aligned}
f'(x) &= 3x^2 - \left(2\sin(x)\cos(x)\tan(x) + \sin(x)^2 \sec(x)^2\right) \\
&= 3x^2 - 2\sin(x)^2 - \tan(x)^2 \\
&= 3x^2 - 1 + \cos(2x) - \tan(x)^2
\end{aligned}$$

$$\begin{aligned}
f''(x) &= 6x - 2\sin(2x) - 2\tan(x)\sec(x)^2 \\
&= 6x - 2\sin(2x) - 2\tan(x)(1+\tan(x)^2) \\
&= 6x - 2\sin(2x) - 2\tan(x) - 2\tan(x)^3
\end{aligned}$$

$$\begin{aligned}
f'''(x) &= 6 - 4\cos(2x) - 2\sec(x)^2 - 6\tan(x)^2\sec(x)^2 \\
&= 6 - 4\cos(2x) - (1+\tan(x)^2) - 6\tan(x)^2(1+\tan(x)^2) \\
&= 5 - 4\cos(2x) - 7\tan(x)^2 - 6\tan(x)^4
\end{aligned}$$

$$\begin{aligned}
\cos(2\theta) &= \cos(\theta)^2 - \sin(\theta)^2 \\
&= (1-\sin(\theta)^2) - \sin(\theta)^2 \\
&= 1 - 2\sin(\theta)^2 \\
\Rightarrow \sin(\theta)^2 &= \frac{1-\cos(2\theta)}{2}
\end{aligned}$$

$$\begin{aligned}
f^{(4)}(x) &= 8\sin(2x) - p(\tan(x)) \\
f^{(5)}(x) &= 16\cos(2x) - \text{〃} \\
f^{(6)}(x) &= \underbrace{-32\sin(2x) - \text{〃}}_{}
\end{aligned}$$

p is a polynomial w/ positive coefficients

all of these terms are negative!!!

Solution: Observe that $f(0) = 0$ and $f(x) \to -\infty$ as $x \to \frac{\pi}{2}$. Thus in order to show that $f(x) < 0$ on $(0, \frac{\pi}{2})$, it is sufficient to show that $f'(x) < 0$ on $(0, \frac{\pi}{2})$. But

$f'(x) =$ 〃.

This has the same property as f, so STS $f'' < 0$. Indeed,

$f''(x) =$ 〃

$f^{(3)}(x) =$ 〃

$f^{(4)}(x) =$ 〃

$f^{(5)}(x) =$ 〃

$f^{(6)}(x) =$ 〃.

For $n = 0, \dots, 5$, $f^{(n)}(0) = 0$ and $f^{(n)}(x) \to -\infty$ as $x \to \frac{\pi}{2}$. So to show that $f^{(n)}(x) < 0$, STS $f^{(n+1)}(x) < 0$. $f^{(6)}(x)$ is negative on $(0, \frac{\pi}{2})$. ∎

$\gamma \in C^1$ in $\mathbb{C}$ parameterized by $[a,b]$

$\gamma(t) \neq 0$

Define $\quad \mathrm{Ind}(\gamma) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)}\,dt.$

Show that $\mathrm{Ind}(\gamma) \in \mathbb{Z}$.

Hint: $\exists \varphi$ on $[a,b]$ with $\varphi' = \gamma'/\gamma$, $\varphi(a) = 0$.

Thus $\gamma \exp(-\varphi)$ is constant. Since $\gamma(a) = \gamma(b)$ it follows that $\exp(\varphi(b)) = \exp(\varphi(a)) = 1$. Note that $\varphi(b) = 2\pi i\,\mathrm{Ind}(\gamma)$.

Let

$$\varphi(s) = \int_a^s \frac{\gamma'(t)}{\gamma(t)}\,dt \underset{\text{FTC}}{\Rightarrow} \varphi'(s) = \frac{\gamma'(s)}{\gamma(s)}.$$

$$\begin{aligned}
&\frac{d}{dt}\gamma(t)\exp(-\varphi(t)) \\
&= \gamma'(t)\exp(-\varphi(t)) - \gamma(t)\exp(-\varphi(t))\varphi'(t) \\
&= \gamma'(t)\exp(-\varphi(t)) - \gamma'(t)\exp(-\varphi(t)) \\
&= 0
\end{aligned}$$

Therefore $\gamma(t)\exp(-\varphi(t))$ is constant.

$$\gamma(a)\exp(-\varphi(a)) = \gamma(b)\exp(-\varphi(b))$$

$$\Rightarrow \underbrace{\exp(-\varphi(a))}_{\varphi(a)=0} = \exp(-\varphi(b)) = 1.$$

$e^z = 1$ iff $z = 2\pi i k$, $k \in \mathbb{Z}$

$\Rightarrow \varphi(b) = 2\pi i k$ for some $k \in \mathbb{Z}$

$$\underbrace{\varphi(b) = \int_a^b \frac{\gamma'(t)}{\gamma(t)}\,dt}_{2\pi i\,\mathrm{Ind}(\gamma)} = 2\pi i k$$

$\Rightarrow \mathrm{Ind}(\gamma) = k, \quad k \in \mathbb{Z}$ ∎