6) Suppose that $f(x)f(y) = f(x+y)$ $\forall\, x,y \in \mathbb{R}$. Show that if f is assumed to be continuous, then $f(x) = e^{cx}$.

Let $q \in \mathbb{Q}$, $q = r/s$, $r,s \in \mathbb{Z}$.

$$f(q) = f\left(\tfrac{r}{s}\right)$$
$$= f\left(\tfrac{1}{s}\right)^r$$

$$\boxed{f(m) = f(\underbrace{1+1+\cdots+1}_{m \text{ times}}) = f(1)^m \quad \forall m \in \mathbb{N}}$$

$$f(1) = f\left(s \cdot \tfrac{1}{s}\right)$$
$$= f\left(\tfrac{1}{s}\right)^s$$

$$f(0) = f(1-1) = f(1)f(-1)$$

$$\Rightarrow f\left(\tfrac{1}{s}\right) = f(1)^{1/s}$$

$$\Rightarrow f(q) = f\left(\tfrac{1}{s}\right)^r = \left(f(1)^{1/s}\right)^r = f(1)^{r/s}$$

Let $f(1) = c$ be arbitrary. Then for $q = r/s \in \mathbb{Q}$,
$$f(q) = f(1)^{r/s} = c^{r/s} = e^{\log(c)\, r/s} = e^{cq}.$$

[The rationals are dense in $\mathbb{R}$ and f is continuous, thus we must have that
$$f(x) = e^{cx}.$$

(*) On $\mathbb{Q}$, $f(q) = e^{cq}$. Let $g : \mathbb{R} \to \mathbb{R} : x \mapsto e^{cx}$. $f \equiv g$ on $\mathbb{Q}$, g is continuous, and $\mathbb{Q}$ is dense in $\mathbb{R}$. Therefore $f \equiv g$ on $\mathbb{R}$.

Either find the Th in Rudin, or justify this statement.

Ex 4.4

~~8) $|\sin(nx)|$, $|n \sin(x)|$ are even & periodic, STS that~~

~~$|\sin(nx)| \le n|\sin(x)|$ for $x \in [0, \pi)$~~

~~If $x \in [0,\pi)$, then~~
~~$|\sin(nx)| - n|\sin(x)| = \underbrace{\sin(nx) - n\sin(x)}_{f(x)} \le 0$ ← WTS~~

~~Know $f(0) = 0$ thus STS $f'(x) \le 0$ on $[0,\pi)$~~

~~$f'(x) = n\cos(nx) - n\cos(x) = n(\cos(nx) - \cos(x))$~~

~~cos is decreasing on $[0, \pi/n)$ and $nx > x$~~

~~Near 0 $\sin(x) \approx x$~~
~~Always $\sin(x) < x$~~

~~If $x > \frac{\pi}{n}$ $\quad \sin(nx) - n\sin(x)$~~
~~$\Rightarrow nx > \pi$ $\quad \sin(nx) < nx$~~

Induction is Definitely the way to go:

$n = 0, 1$: "Obvious" $\quad |\sin(0 \cdot x)| = \sin(0) = 0 \le 0 \cdot |\sin(x)|$
$|\sin(1 \cdot x)| = \sin(x) \le \sin(x) = 1 \cdot |\sin(x)|$

Suppose that $|\sin(kx)| \le k|\sin(x)|$. (*) Then

$$|\sin((k+1)x)| = |\sin(kx + x)|$$

angle sum formula & triangle inequality

$$\le |\cos(kx)\sin(x)| + |\sin(kx)\cos(x)|$$

$$\le |\sin(x)| + |\sin(kx)| \quad \left\{ \begin{array}{l} |\cos(kx)| \le 1 \\ |\cos(x)| \le 1 \end{array} \right. \forall x$$

$$\le |\sin(x)| + k|\sin(x)|$$

induction hypothesis (*)

$$= (k+1)|\sin(x)|. \quad \blacksquare$$