$$\sum_i \sum_j a_{ij} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) \qquad a_{ij} = \begin{cases} 0 & \text{if } i < j \\ -1 & \text{if } i = j \\ 2^{j-i} & \text{if } i > j \end{cases}$$

$$= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i-1} 2^{j-i} - 1 \right)$$

$$= \sum_{i=1}^{\infty} \left(2^{-i} \sum_{j=1}^{i-1} 2^{j} - 1 \right)$$

$$= \sum_{i=1}^{\infty} \left(2^{-i} \left(2^{i} - 2 \right) - 1 \right)$$

$$= \sum_{i=1}^{\infty} \left(1 - 2^{-i+1} - 1 \right)$$

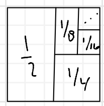
$$= -\sum_{i=1}^{\infty} 2^{1-i}$$

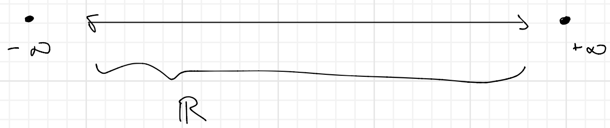
$$= -2 \left(\sum_{i=1}^{\infty} 2^{-i} \right)$$

$$= -2$$

$\blacksquare$

$$S = 2 + \cancel{2^2} + \cdots + \cancel{2^{i-1}}$$
$$2S = \cancel{2^2} + \cancel{2^3} + \cdots + 2^{i}$$
$$(1-2)S = 2 - 2^{i}$$
$$\Rightarrow S = 2^{i} - 2$$





Neighbourhoods of $-\infty$ are intervals $(-\infty, b)$
Neighbourhoods of $+\infty$ are intervals $(a, +\infty)$

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ w/ the appropriate topology

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Theorem: If $a_{ij} \geq 0 \ \forall i, j$ then

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}.$$

Proof: $\boxed{\text{Case:}}$ Suppose that $\sum_i b_i$ converges, where

$$b_i = \sum_{j=1}^{\infty} |a_{ij}| = \sum_{j=1}^{\infty} a_{ij}.$$

By Thm 8.3, done. //

$\boxed{\text{Case}}$ Suppose that $\sum_i b_i$ does <u>not</u> converge.

$$\Rightarrow \sum_i b_i = +\infty$$

Prove by contradiction. Suppose that

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} < +\infty.$$