Syntactic Monoids in a Category

Jiří Adámek[1], Stefan Milius[*,2], and Henning Urbat[1]

1 **Institut für Theoretische Informatik**
Technische Universität Braunschweig, Germany
2 **Lehrstuhl für Theoretische Informatik**
Friedrich-Alexander Universität Erlangen-Nürnberg, Germany

Abstract

The syntactic monoid of a language is generalized to the level of a symmetric monoidal closed category $\mathcal{D}$. This allows for a uniform treatment of several notions of syntactic algebras known in the literature, including the syntactic monoids of Rabin and Scott ($\mathcal{D}$ = sets), the syntactic semirings of Polák ($\mathcal{D}$ = semilattices), and the syntactic associative algebras of Reutenauer ($\mathcal{D}$ = vector spaces). Assuming that $\mathcal{D}$ is a commutative variety of algebras, we prove that the syntactic $\mathcal{D}$-monoid of a language L can be constructed as a quotient of a free $\mathcal{D}$-monoid modulo the syntactic congruence of L, and that it is isomorphic to the transition $\mathcal{D}$-monoid of the minimal automaton for L in $\mathcal{D}$. Furthermore, in the case where the variety $\mathcal{D}$ is locally finite, we characterize the regular languages as precisely the languages with finite syntactic $\mathcal{D}$-monoids.

1998 ACM Subject Classification F.4.3 Formal Languages

Keywords and phrases Syntactic monoid, transition monoid, algebraic automata theory, duality, coalgebra, algebra, symmetric monoidal closed category, commutative variety

1 Introduction

One of the successes of the theory of coalgebras is that ideas from automata theory can be developed at a level of abstraction where they apply uniformly to many different types of systems. In fact, classical deterministic automata are a standard example of coalgebras for an endofunctor. And that automata theory can be studied with coalgebraic methods rests on the observation that formal languages form the final coalgebra.

The present paper contributes to a new category-theoretic view of *algebraic* automata theory. In this theory one starts with an elegant machine-independent notion of language recognition: a language $L \subseteq X^*$ is recognized by a monoid morphism $e : X^* \to M$ if it is the preimage under e of some subset of M. Regular languages are then characterized as precisely the languages recognized by finite monoids. A key concept, introduced by Rabin and Scott [20] (and earlier in unpublished work of Myhill), is the *syntactic monoid* of a language L. It serves as a canonical algebraic recognizer of L, namely the smallest X-generated monoid recognizing L. Two standard ways to construct the syntactic monoid are:

(1) as a quotient of the free monoid X^* modulo the *syntactic congruence* of L, which is a two-sided version of the well-known Myhill-Nerode equivalence, and
(2) as the *transition monoid* of the minimal automaton for L.

In addition to syntactic monoids there are several related notions of syntactic algebras for (weighted) languages in the literature, most prominently the syntactic idempotent semirings of Polák [19] and the syntactic associative algebras of Reutenauer [21], both of which admit

[*] Stefan Milius acknowledges support by the Deutsche Forschungsgemeinschaft (DFG) under project MI 717/5-1

Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

constructions similar to (1) and (2). A crucial observation is that monoids, idempotent semirings and associative algebras are precisely the monoid objects in the categories of sets, semilattices and vector spaces, respectively. Moreover, these three categories are symmetric monoidal closed w.r.t. their usual tensor product.

The main goal of our paper is thus to develop a theory of algebraic recognition in a general symmetric monoidal closed category $\mathcal{D} = (\mathcal{D}, \otimes, I)$. Following Goguen [12], a *language* in $\mathcal{D}$ is a morphism $L : X^{\circledast} \to Y$ where X is a fixed object of inputs, Y is a fixed object of outputs, and $X^{\circledast}$ denotes the free $\mathcal{D}$-monoid on X. And a $\mathcal{D}$*-automaton* is given by the picture below: it consists of an object of states Q, a morphism i representing the initial state, an output morphism f, and a transition morphism δ which may be presented in its curried form $\lambda\delta$.

$$\begin{tikzcd}
 & X \otimes Q \arrow[d, "\delta"] & \\
I \arrow[r, "i"] & Q \arrow[r, "f"] \arrow[d, "\lambda\delta"] & Y \\
 & {[X, Q]} &
\end{tikzcd} \tag{1}$$

This means that an automaton is at the same time an *algebra* $I + X \otimes Q \xrightarrow{[i,\delta]} Q$ for the functor $FQ = I + X \otimes Q$, and a *coalgebra* $Q \xrightarrow{\langle f, \lambda\delta \rangle} Y \times [X, Q]$ for the functor $TQ = Y \times [X, Q]$. It turns out that much of the classical (co-)algebraic theory of automata in the category of sets extends to this level of generality. Thus Goguen [12] demonstrated that the initial algebra for F coincides with the free $\mathcal{D}$-monoid $X^{\circledast}$, and that every language is accepted by a unique minimal $\mathcal{D}$-automaton. We will add to this picture the observation that the final coalgebra for T is carried by the object of languages $[X^{\circledast}, Y]$, see Proposition 2.21.

In Section 3 we introduce the central concept of our paper, the *syntactic* $\mathcal{D}$*-monoid* of a language $L : X^{\circledast} \to Y$, which by definition is the smallest X-generated $\mathcal{D}$-monoid recognizing L. Assuming that $\mathcal{D}$ is a commutative variety of algebras, we will show that the above constructions (1) and (2) for the classical syntactic monoid adapt to our general setting: the syntactic $\mathcal{D}$-monoid is (1) the quotient of $X^{\circledast}$ modulo the syntactic congruence of L (Theorem 3.14), and (2) the transition $\mathcal{D}$-monoid of the minimal $\mathcal{D}$-automaton for L (Theorem 4.6). As special instances we recover the syntactic monoids of Rabin and Scott ($\mathcal{D}$ = sets), the syntactic semirings of Polák ($\mathcal{D}$ = semilattices) and the syntactic associative algebras of Reutenauer ($\mathcal{D}$ = vector spaces). Furthermore, our categorical setting yields new types of syntactic algebras "for free". For example, we will identify monoids with zero as the algebraic structures representing partial automata (the case $\mathcal{D}$ = pointed sets), which leads to the *syntactic monoid with zero* for a given language. Similarly, by taking as $\mathcal{D}$ the variety of algebras with an involutive unary operation we obtain *syntactic involution monoids*.

Most of the results of our paper apply to arbitrary languages. In Section 5 we will investigate $\mathcal{D}$*-regular languages*, that is, languages accepted by $\mathcal{D}$-automata with a finitely presentable object of states. Under suitable assumptions on $\mathcal{D}$, we will prove that a language is $\mathcal{D}$-regular iff its syntactic $\mathcal{D}$-monoid is carried by a finitely presentable object (Theorem 5.4). We will also derive a dual characterization of the syntactic $\mathcal{D}$-monoid which is new even in the "classical" case $\mathcal{D}$ = sets: if $\mathcal{D}$ is a locally finite variety, and if moreover some other locally finite variety $\mathcal{C}$ is dual to $\mathcal{D}$ on the level of finite objects, the syntactic $\mathcal{D}$-monoid of L dualizes to the local variety of languages in $\mathcal{C}$ generated by the reversed language of L.

Due to space limitations most proofs are omitted or sketched. See [1] for an extended version of this paper.

Related work. Our paper gives a uniform treatment of various notions of syntactic algebras known in the literature [19, 20, 21]. Another categorical approach to (classical) syntactic monoids appears in the work of Ballester-Bolinches, Cosme-Llopez and Rutten [5]. These authors consider automata in the category of sets specified by *equations* or dually by *coequations*, which leads to a construction of the automaton underlying the syntactic monoid of a language. The fact that it forms the transition monoid of a minimal automaton is also interpreted in that setting. In the present paper we take a more general and conceptual approach by studying algebraic recognition in a symmetric monoidal closed category $\mathcal{D}$. One important source of inspiration for our categorical setting was the work of Goguen [12].

In the recent papers [2, 4] we presented a categorical view of *varieties of languages*, another central topic of algebraic automata theory. Building on the duality-based approach of Gehrke, Grigorieff and Pin [11], we generalized Eilenberg's variety theorem and its local version to the level of an abstract (pre-)duality between algebraic categories. The idea to replace monoids by monoid objects in a commutative variety $\mathcal{D}$ originates in this work.

When revising this paper we were made aware of the ongoing work of Bojanczyk [8]. He considers, in lieu of commutative varieties, categories of Eilenberg-Moore algebras for an arbitrary monad on sorted sets, and defines syntactic congruences in this more general setting. Our Theorem 3.14 is a special case of [8, Theorem 3.1].

2 Preliminaries

Throughout this paper we work with deterministic automata in a commutative variety $\mathcal{D}$ of algebras. Recall that a *variety of algebras* is an equational class of algebras over a finitary signature. It is called *commutative* (or *entropic*) if, for any two objects A and B of $\mathcal{D}$, the set $\mathcal{D}(A, B)$ of all homomorphisms from A to B carries a subobject $[A, B] \rightarrowtail B^{|A|}$ of the product of $|A|$ copies of B. Commutative varieties are precisely the categories of Eilenberg-Moore algebras for a commutative finitary monad on the category of sets, see [13, 16]. We fix an object X (of inputs) and an object Y (of outputs) in $\mathcal{D}$.

▸ **Example 2.1.** **1.** **Set** is a commutative variety with $[A, B] = B^A$.

2. A *pointed set* $(A, \bot)$ is a set A together with a chosen point $\bot \in A$. The category $\mathbf{Set}_\bot$ of pointed sets and point-preserving functions is a commutative variety. The point of $[(A, \bot_A), (B, \bot_B)]$ is the constant function with value $\bot_B$.

3. An *involution algebra* is a set with an involutive unary operation $x \mapsto \widetilde{x}$, i.e. $\widetilde{\widetilde{x}} = x$. We call $\widetilde{x}$ the *complement* of x. Morphisms are functions f with $f(\widetilde{x}) = \widetilde{f(x)}$. The variety **Inv** of involution algebras is commutative. Indeed, the set $[A, B]$ of all homomorphisms is an involution algebra with pointwise complementation: $\widetilde{f}$ sends x to $\widetilde{f(x)}$.

4. All other examples we treat in our paper are varieties of modules over a semiring. Given a semiring $\mathbb{S}$ (with 0 and 1) we denote by $\mathbf{Mod}(\mathbb{S})$ the category of all $\mathbb{S}$-modules and module homomorphisms (i.e. $\mathbb{S}$-linear maps). Three interesting special cases of $\mathbf{Mod}(\mathbb{S})$ are:

   a. $\mathbb{S} = \{0, 1\}$, the boolean semiring with $1 + 1 = 1$: the category $\mathbf{JSL}_0$ of join-semilattices with 0, and homomorphisms preserving joins and 0;

   b. $\mathbb{S} = \mathbb{Z}$: the category **Ab** of abelian groups and group homomorphisms;

   c. $\mathbb{S} = \mathbb{K}$ (a field): the category $\mathbf{Vec}(\mathbb{K})$ of vector spaces over $\mathbb{K}$ and linear maps.

▸ **Notation 2.2.** We denote by $\Psi : \mathbf{Set} \to \mathcal{D}$ the left adjoint to the forgetful functor $|{-}| : \mathcal{D} \to \mathbf{Set}$. Thus ΨX_0 is the free object of $\mathcal{D}$ on the set X_0.

▸ **Example 2.3.** **1.** We have $\Psi X_0 = X_0$ for $\mathcal{D} = \mathbf{Set}$ and $\Psi X_0 = X_0 + \{\bot\}$ for $\mathcal{D} = \mathbf{Set}_\bot$.

2. For $\mathcal{D} = \mathbf{Inv}$ the free involution algebra on X_0 is $\Psi X_0 = X_0 + \widetilde{X_0}$ where $\widetilde{X_0}$ is a copy of X_0 (whose elements are denoted $\widetilde{x}$ for $x \in X_0$). The involution swaps the copies of X_0, and the universal arrow $X_0 \to X_0 + \widetilde{X_0}$ is the left coproduct injection.
3. For $\mathcal{D} = \mathbf{Mod}(\mathbb{S})$ the free module ΨX_0 is the submodule of $\mathbb{S}^{X_0}$ on all functions $X_0 \to \mathbb{S}$ with finite support. Equivalently, ΨX_0 consists of formal linear combinations $\sum_{i=1}^{n} s_i x_i$ with $s_i \in \mathbb{S}$ and $x_i \in X_0$. In particular, $\Psi X_0 = \mathcal{P}_f X_0$ (finite subsets of X_0) for $\mathcal{D} = \mathbf{JSL}_0$, and ΨX_0 is the vector space with basis X_0 for $\mathcal{D} = \mathbf{Vec}(\mathbb{K})$.

▸ **Definition 2.4.** Given objects A, B and C of $\mathcal{D}$, a *bimorphism* from A, B to C is a function $f : |A| \times |B| \to |C|$ such that the maps $f(a,-) : |B| \to |C|$ and $f(-,b) : |A| \to |C|$ carry morphisms of $\mathcal{D}$ for every $a \in |A|$ and $b \in |B|$. A *tensor product* of A and B is a universal bimorphism $t : |A| \times |B| \to |A \otimes B|$, which means that for every bimorphism $f : |A| \times |B| \to |C|$ there is a unique morphism $f' : A \otimes B \to C$ in $\mathcal{D}$ with $f' \cdot t = f$.

▸ **Theorem 2.5** (Banaschweski and Nelson [6]). *Every commutative variety $\mathcal{D}$ has tensor products, making $\mathcal{D} = (\mathcal{D}, \otimes, I)$ with $I = \Psi 1$ a symmetric monoidal closed category. That is, we have the following bijective correspondence of morphisms, natural in $A, B, C \in \mathcal{D}$:*

$$\begin{array}{rl} f: & A \otimes B \to C \\ \hline \lambda f: & A \to [B, C] \end{array}$$

▸ **Remark 2.6.** Recall that a *monoid* (M, m, i) in a monoidal category $(\mathcal{D}, \otimes, I)$ (with tensor product $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and tensor unit $I \in \mathcal{D}$) is an object M equipped with a multiplication $m : M \otimes M \to M$ and unit $i : I \to M$ satisfying the usual associative and unit laws. Due to $\otimes$ and $I = \Psi 1$ representing bimorphisms, this categorical definition is equivalent to the following algebraic one in our setting: a $\mathcal{D}$-*monoid* is a triple $(M, \bullet, i)$ where M is an object of $\mathcal{D}$ and $(|M|, \bullet, i)$ is a monoid in **Set** with $\bullet : |M| \times |M| \to |M|$ a bimorphism of $\mathcal{D}$. A *morphism* $h : (M, \bullet, i) \to (M', \bullet', i')$ of $\mathcal{D}$-monoids is a morphism $h : M \to M'$ of $\mathcal{D}$ such that $|h| : |M| \to |M'|$ is a monoid morphism in **Set**. We denote by $\mathbf{Mon}(\mathcal{D})$ the category of $\mathcal{D}$-monoids and their homomorphisms. In the following we will freely work with $\mathcal{D}$-monoids in both categorical and algebraic disguise.

▸ **Example 2.7.** 1. In **Set** the tensor product is the cartesian product, $I = \{*\}$, and **Set**-monoids are ordinary monoids.
2. In $\mathbf{Set}_\perp$ we have $I = \{\perp, *\}$, and the tensor product of pointed sets $(A, \perp_A)$ and $(B, \perp_A)$ is $A \otimes B = (A \backslash \{\perp_A\}) \times (B \backslash \{\perp_B\}) + \{\perp\}$. $\mathbf{Set}_\perp$-monoids are precisely monoids with zero. Indeed, given a $\mathbf{Set}_\perp$-monoid structure on $(A, \perp)$ we have $x \bullet \perp = \perp = \perp \bullet x$ for all x because $\bullet$ is a bimorphism, i.e. $\perp$ is a zero element. Morphisms of $\mathbf{Mon}(\mathbf{Set}_\perp)$ are zero-preserving monoid morphisms.
3. An **Inv**-monoid (also called an *involution monoid*) is a monoid equipped with an involution $x \mapsto \widetilde{x}$ such that $x \bullet \widetilde{y} = \widetilde{x} \bullet y = \widetilde{x \bullet y}$. For example, for any set A the power set $\mathcal{P}A$ naturally carries the structure of an involution monoid: the involution takes complements, $\widetilde{S} = A \backslash S$, and the monoid multiplication is the symmetric difference $S \oplus T = (S \backslash T) \cup (T \backslash S)$.
4. $\mathbf{JSL}_0$-monoids are precisely idempotent semirings (with 0 and 1). Indeed, a $\mathbf{JSL}_0$-monoid on a semilattice (i.e. a commutative idempotent monoid) $(D, +, 0)$ is given by a unit 1 and a monoid multiplication that, being a bimorphism, distributes over $+$ and 0.
5. More generally, a $\mathbf{Mod}(\mathbb{S})$-monoid is precisely an associative algebra over $\mathbb{S}$: it consists of an $\mathbb{S}$-module together with a unit 1 and a monoid multiplication that distributes over $+$ and 0 and moreover preserves scalar multiplication in both components.

▸ **Notation 2.8.** We denote by $X^{\otimes n}$ $(n < \omega)$ the n-fold tensor power of X, recursively defined by $X^{\otimes 0} = I$ and $X^{\otimes (n+1)} = X \otimes X^{\otimes n}$.

▸ **Proposition 2.9** (see Mac Lane [15]). *The forgetful functor* $\mathbf{Mon}(\mathcal{D}) \to \mathcal{D}$ *has a left adjoint assigning to every object* X *the free* $\mathcal{D}$*-monoid* $X^\circledast = \coprod_{n<\omega} X^{\otimes n}$. *The monoid structure* $(X^\circledast, m_X, i_X)$ *is given by the coproduct injection* $i_X : I = X^{\otimes 0} \to X^\circledast$ *and* $m_X : X^\circledast \otimes X^\circledast \to X^\circledast$, *where* $X^\circledast \otimes X^\circledast = \coprod_{n,k<\omega} X^{\otimes n} \otimes X^{\otimes k}$ *and* m_X *has as its* (n,k)*-component the* $(n+k)$*-th coproduct injection. The universal arrow* $\eta_X : X \to X^\circledast$ *is the first coproduct injection.*

▸ **Proposition 2.10.** *The free* $\mathcal{D}$*-monoid on* $X = \Psi X_0$ *is* $X^\circledast = \Psi X_0^*$. *Its monoid multiplication extends the concatenation of words in* X_0^*, *and its unit is the empty word* ε.

▸ **Example 2.11.**
1. In **Set** we have $X^\circledast = X^*$. In $\mathbf{Set}_\perp$ with $X = \Psi X_0 = X_0 + \{\perp\}$ we get $X^\circledast = X_0^* + \{\perp\}$. The product $x \bullet y$ is concatenation for $x, y \in X_0^*$, and otherwise $\perp$.
2. In **Inv** with $X = \Psi X_0 = X_0 + \widetilde{X_0}$ we have $X^\circledast = X_0^* + \widetilde{X_0^*}$. The multiplication restricted to X_0^* is concatenation, and is otherwise determined by $\tilde{u} \bullet v = \widetilde{uv} = u \bullet \tilde{v}$ for $u, v \in X_0^*$.
3. In $\mathbf{JSL}_0$ with $X = \Psi X_0 = \mathcal{P}_f X_0$ we have $X^\circledast = \mathcal{P}_f X_0^*$, the semiring of all finite languages over X_0. Its addition is union and its multiplication is the concatentation of languages.
4. More generally, in $\mathbf{Mod}(\mathbb{S})$ with $X = \Psi X_0$ we get $X^\circledast = \Psi X_0^* = \mathbb{S}[X_0]$, the module of all finite $\mathbb{S}$-weighted languages over the alphabet X_0. Hence the elements of $\mathbb{S}[X_0]$ are functions $c : X_0^* \to \mathbb{S}$ with finite support, which may be expressed as polynomials $\sum_{i=1}^n c(w_i) w_i$ with $w_i \in X_0^*$ and $c(w_i) \in \mathbb{S}$. The $\mathbb{S}$-algebraic structure of $\mathbb{S}[X_0]$ is given by the usual addition, scalar multiplication and product of polynomials.

▸ **Definition 2.12** (Goguen [12]). A $\mathcal{D}$-*automaton* (Q, δ, i, f) consists of an object Q (of states) and morphisms $\delta : X \otimes Q \to Q$, $i : I \to Q$ and $f : Q \to Y$; see Diagram (1). An *automata homomorphism* $h : (Q, \delta, i, f) \to (Q', \delta', i', f')$ is a morphism $h : Q \to Q'$ preserving transitions as well as initial states and outputs, i.e. making the following diagrams commute:

$$\begin{tikzcd}
X \otimes Q \arrow[r, "\delta"] \arrow[d, "X\otimes h"'] & Q \arrow[d, "h"] \\
X \otimes Q' \arrow[r, "\delta'"'] & Q'
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "i"] \arrow[dr, "i'"'] & Q \arrow[r, "f"] \arrow[d, "h"] & Y \\
& Q' \arrow[ur, "f'"'] &
\end{tikzcd}$$

The above definition makes sense in any monoidal category $\mathcal{D}$. In our setting, since $I = \Psi 1$, the morphism i chooses an *initial state* in $|Q|$. Moreover, if $X = \Psi X_0$ for some set X_0 (of inputs), the morphism δ amounts to a choice of endomorphisms $\delta_a : Q \to Q$ for $a \in X_0$, representing transitions. This follows from the bijections

$$\begin{array}{ll}
\Psi X_0 \otimes Q \to Q & \text{in } \mathcal{D} \\ \hline
\Psi X_0 \to [Q, Q] & \text{in } \mathcal{D} \\ \hline
X_0 \to \mathcal{D}(Q, Q) & \text{in } \mathbf{Set}
\end{array}$$

▸ **Example 2.13.**
1. The classical deterministic automata are the case $\mathcal{D} = \mathbf{Set}$ and $Y = \{0, 1\}$. Here $f : Q \to \{0, 1\}$ defines the set $F = f^{-1}[1] \subseteq Q$ of final states. For general Y we get deterministic Moore automata with outputs in Y.
2. The setting $\mathcal{D} = \mathbf{Set}_\perp$ with $X = X_0 + \{\perp\}$ and $Y = \{\perp, 1\}$ gives *partial* deterministic automata. Indeed, the state object $(Q, \perp)$ has transitions $\delta_a : (Q, \perp) \to (Q, \perp)$ for $a \in X_0$ preserving $\perp$, that is, $\perp$ is a sink state. Equivalently, we may consider δ_a as a partial transition map on the state set $Q \backslash \{\perp\}$. The morphism $f : (Q, \perp) \to \{\perp, 1\}$ again determines a set of final states $F = f^{-1}[1]$ (in particular, $\perp$ is non-final). And the morphism $i : \{\perp, *\} \to (Q, \perp)$ determines a partial initial state: either $i(*)$ lies in $Q \backslash \{\perp\}$, or no initial state is defined.

3. In $\mathcal{D} = \mathbf{Inv}$ let us choose $X = X_0 + \widetilde{X_0}$ and $Y = \{0, 1\}$ with $\widetilde{0} = 1$. An **Inv**-automaton is a deterministic automaton with complementary states $x \mapsto \widetilde{x}$ such that (i) for every transition $p \xrightarrow{a} q$ there is a complementary transition $\widetilde{p} \xrightarrow{a} \widetilde{q}$ and (ii) a state q is final iff $\widetilde{q}$ is non-final.
4. For $\mathcal{D} = \mathbf{JSL}_0$ with $X = \mathcal{P}_f X_0$ and $Y = \{0, 1\}$ (the two-chain) an automaton consists of a semilattice Q of states, transitions $\delta_a : Q \to Q$ for $a \in X_0$ preserving finite joins (including 0), an initial state $i \in Q$ and a homomorphism $f : Q \to \{0, 1\}$ which defines a *prime upset* $F = f^{-1}[1] \subseteq Q$ of final states. The latter means that a finite join of states is final iff one of the states is. In particular, 0 is non-final.
5. More generally, automata in $\mathcal{D} = \mathbf{Mod}(\mathbb{S})$ with $X = \Psi X_0$ and $Y = \mathbb{S}$ are $\mathbb{S}$-*weighted automata*. Such an automaton consists of an $\mathbb{S}$-module Q of states, linear transitions $\delta_a : Q \to Q$ for $a \in X_0$, an initial state $i \in Q$ and a linear output map $f : Q \to \mathbb{S}$.

▸ **Remark 2.14.** 1. An *algebra* for an endofunctor F of $\mathcal{D}$ is a pair (Q, α) of an object Q and a morphism $\alpha : FQ \to Q$. A *homomorphism* $h : (Q, \alpha) \to (Q', \alpha')$ of F-algebras is a morphism $h : Q \to Q'$ with $h \cdot \alpha = \alpha' \cdot Fh$. Throughout this paper we work with the endofunctor $FQ = I + X \otimes Q$; its algebras are denoted as triples (Q, δ, i) with $\delta : X \otimes Q \to Q$ and $i : I \to Q$. Hence $\mathcal{D}$-automata are precisely F-algebras equipped with an output morphism $f : Q \to Y$. Moreover, automata homomorphisms are precisely F-algebra homomorphisms preserving outputs.

2. Analogously, a *coalgebra* for an endofunctor T of $\mathcal{D}$ is a pair (Q, γ) of an object Q and a morphism $\gamma : Q \to TQ$. Throughout this paper we work with the endofunctor $TQ = Y \times [X, Q]$; its coalgebras are denoted as triples (Q, τ, f) with $\tau : Q \to [X, Q]$ and $f : Q \to Y$. Hence $\mathcal{D}$-automata are precisely *pointed* T-coalgebras, i.e. T-coalgebras equipped with a morphism $i : I \to Q$. Indeed, given a pointed coalgebra $I \xrightarrow{i} Q \xrightarrow{\langle f, \tau \rangle} Y \times [X, Q]$, the morphism $Q \xrightarrow{\tau} [X, Q]$ is the curried form of a morphism $Q \otimes X \xrightarrow{\cong} X \otimes Q \xrightarrow{\delta} Q$. Automata homomorphisms are T-coalgebra homomorphisms preserving initial states.

▸ **Definition 2.15.** Given a $\mathcal{D}$-monoid (M, m, i) and a morphism $e : X \to M$ of $\mathcal{D}$, the *F-algebra associated to M and e* has carrier M and structure

$$[i, \delta] = (I + X \otimes M \xrightarrow{I + e \otimes M} I + M \otimes M \xrightarrow{[i,m]} M).$$

In particular, the F-algebra associated to the free monoid $X^{\circledast}$ (and its universal arrow η_X) is

$$[i_X, \delta_X] = (I + X \otimes X^{\circledast} \xrightarrow{I + \eta_X \otimes X^{\circledast}} I + X^{\circledast} \otimes X^{\circledast} \xrightarrow{[i_X, m_X]} X^{\circledast}).$$

▸ **Example 2.16.** In **Set** every monoid M together with an “input” map $e : X \to M$ determines an F-algebra with initial state i and transitions $\delta_a = - \bullet e(a)$ for all $a \in X$. The F-algebra associated to X^* is the usual automaton of words: its initial state is ε and the transitions are given by $w \xrightarrow{a} wa$ for $a \in X$.

▸ **Proposition 2.17** (Goguen [12]). *For any symmetric monoidal closed category $\mathcal{D}$ with countable coproducts, $X^{\circledast}$ is the initial algebra for F.*

▸ **Remark 2.18.** Given any F-algebra (Q, δ, i) the unique F-algebra homomorphism $e_Q : X^{\circledast} \to Q$ is constructed as follows: extend the morphism $\lambda\delta : X \to [Q, Q]$ to a $\mathcal{D}$-monoid morphism $(\lambda\delta)^+ : X^{\circledast} \to [Q, Q]$. Then

$$e_Q = (X^{\circledast} \cong X^{\circledast} \otimes I \xrightarrow{(\lambda\delta)^+ \otimes i} [Q, Q] \otimes Q \xrightarrow{\mathsf{ev}} Q), \tag{2}$$

where ev is the ‘evaluation morphism’, i.e. the counit of the adjunction $- \otimes Q \dashv [Q, -]$.

Notation 2.19. $\delta^{\circledast} : X^{\circledast} \otimes Q \to Q$ denotes the uncurried form of $(\lambda\delta)^+ : X^{\circledast} \to [Q, Q]$.

Remark 2.20. Recall from Rutten [22] that the final coalgebra for the functor $TQ = \{0,1\} \times Q^X$ on **Set** is the coalgebra $\mathcal{P}X^* \cong [X^*, \{0,1\}]$ of all languages over X. Given any coalgebra Q, the unique coalgebra homomorphism from Q to $\mathcal{P}\Sigma^*$ assigns to every state q the language accepted by q (as an initial state). These observations generalize to our present setting. The object $[X^{\circledast}, Y]$ of $\mathcal{D}$ carries the following T-coalgebra structure: its transition morphism $\tau_{[X^{\circledast},Y]} : [X^{\circledast}, Y] \to [X, [X^{\circledast}, Y]]$ is the two-fold curryfication of

$$[X^{\circledast}, Y] \otimes X \otimes X^{\circledast} \xrightarrow{[X^{\circledast},Y]\otimes\eta_X\otimes X^{\circledast}} [X^{\circledast}, Y] \otimes X^{\circledast} \otimes X^{\circledast} \xrightarrow{[X^{\circledast},Y]\otimes m_X} [X^{\circledast}, Y] \otimes X^{\circledast} \xrightarrow{\mathsf{ev}} Y,$$

and its output morphism $f_{[X^{\circledast},Y]} : [X^{\circledast}, Y] \to Y$ is

$$f_{[X^{\circledast},Y]} = ([X^{\circledast}, Y] \cong [X^{\circledast}, Y] \otimes I \xrightarrow{[X^{\circledast},Y]\otimes i_X} [X^{\circledast}, Y] \otimes X^{\circledast} \xrightarrow{\mathsf{ev}} Y).$$

Proposition 2.21. *$[X^{\circledast}, Y]$ is the final coalgebra for T.*

Proof sketch. Given any coalgebra (Q, τ, f), let $\delta : X \otimes Q \to Q$ be the uncurried version of $\tau : Q \to [X, Q]$, see Remark 2.14. Then the unique coalgebra homomorphism into $[X^{\circledast}, Y]$ is $\lambda h : Q \to [X^{\circledast}, Y]$, where $h = (Q \otimes X^{\circledast} \cong X^{\circledast} \otimes Q \xrightarrow{\delta^{\circledast}} Q \xrightarrow{f} Y)$. ◂

Definition 2.22 (Goguen [12])**.** A *language* in $\mathcal{D}$ is a morphism $L : X^{\circledast} \to Y$.

Note that if $X = \Psi X_0$ (and hence $X^{\circledast} = \Psi X_0^*$) for some set X_0, one can identify a language $L : X^{\circledast} = \Psi X_0^* \to Y$ in $\mathcal{D}$ with its adjoint transpose $\widetilde{L} : X_0^* \to |Y|$, via the adjunction $\Psi \dashv |{-}| : \mathcal{D} \to \mathbf{Set}$. In the case where $|Y|$ is a two-element set, $\widetilde{L}$ is the characteristic function of a “classical” language $L_0 \subseteq X_0^*$.

Example 2.23.

1. In $\mathcal{D} = \mathbf{Set}$ (with $X^{\circledast} = X^*$ and $Y = \{0,1\}$) one represents $L_0 \subseteq X^*$ by its characteristic function $L : X^* \to \{0,1\}$.
2. In $\mathcal{D} = \mathbf{Set}_\bot$ (with $X = X_0 + \{\bot\}$, $X^{\circledast} = X_0^* + \{\bot\}$ and $Y = \{\bot, 1\}$) one represents $L_0 \subseteq X_0^*$ by its extended characteristic function $L : X_0^* + \{\bot\} \to \{\bot, 1\}$ where $L(\bot) = \bot$.
3. In $\mathcal{D} = \mathbf{Inv}$ (with $X = X_0 + \widetilde{X_0}$, $X^{\circledast} = X_0^* + \widetilde{X_0^*}$ and $Y = \{0,1\}$) one represents $L_0 \subseteq X_0^*$ by $L : X_0^* + \widetilde{X_0^*} \to \{0,1\}$ where $L(w) = 1$ iff $w \in L_0$ and $L(\widetilde{w}) = 1$ iff $w \notin L_0$ for all words $w \in X_0^*$.
4. In $\mathcal{D} = \mathbf{JSL}_0$ (with $X = \mathcal{P}_f X_0$, $X^{\circledast} = \mathcal{P}_f X_0^*$ and $Y = \{0,1\}$) one represents $L_0 \subseteq X_0^*$ by $L : \mathcal{P}_f X_0^* \to \{0,1\}$ where $L(U) = 1$ iff $U \cap L_0 \neq \varnothing$.
5. In $\mathcal{D} = \mathbf{Mod}(\mathbb{S})$ (with $X = \Psi X_0$, $X^{\circledast} = \mathbb{S}[X_0]$ and $Y = \mathbb{S}$) an $\mathbb{S}$-weighted language $L_0 : X_0^* \to \mathbb{S}$ is represented by its free extension to a module homomorphism

$$L : \mathbb{S}[X_0^*] \to \mathbb{S}, \quad L\left(\sum_{i=1}^{n} c(w_i) w_i\right) = \sum_{i=1}^{n} c(w_i) L_0(w_i).$$

Definition 2.24 (Goguen [12])**.** The language *accepted* by a $\mathcal{D}$-automaton (Q, δ, i, f) is $L_Q = (X^{\circledast} \xrightarrow{e_Q} Q \xrightarrow{f} Y)$, where e_Q is the F-algebra homomorphism of Remark 2.18.

Example 2.25.

1. In $\mathcal{D} = \mathbf{Set}$ with $Y = \{0,1\}$, the homomorphism $e_Q : X^* \to Q$ assigns to every word w the state it computes in Q, i.e. the state the automaton reaches on input w. Thus $L_Q(w) = 1$ iff Q terminates in a final state on input w, which is precisely the standard definition of the accepted language of an automaton. For general Y, the function $L_Q : X^* \to Y$ is the behavior of the Moore automaton Q, i.e. $L_Q(w)$ is the output of the last state in the computation of w.

2. For $\mathcal{D} = \mathbf{Set}_\perp$ with $X = X_0 + \{\perp\}$ and $Y = \{\perp, 1\}$, we have $e_Q : X_0^* + \{\perp\} \to (Q, \perp)$ sending $\perp$ to $\perp$, and sending a word in X_0^* to the state it computes (if any), and to $\perp$ otherwise. Hence $L_Q : X_0^* + \{\perp\} \to \{\perp, 1\}$ defines (via the preimage of 1) the usual language accepted by a partial automaton.
3. In $\mathcal{D} = \mathbf{Inv}$ with $X = X_0 + \widetilde{X_0}$ and $Y = \{0, 1\}$, the map $L_Q : X_0^* + \widetilde{X_0^*} \to \{0, 1\}$ sends $w \in X_0^*$ to 1 iff w computes a final state, and it sends $\widetilde{w} \in \widetilde{X_0^*}$ to 1 iff w computes a non-final state.
4. In $\mathcal{D} = \mathbf{JSL}_0$ with $X = \mathcal{P}_f X_0$ and $Y = \{0, 1\}$, the map $L_Q : \mathcal{P} X_0^* \to \{0, 1\}$ assigns to $U \in \mathcal{P}_f X_0^*$ the value 1 iff the computation of at least one word in U ends in a final state.
5. In $\mathcal{D} = \mathbf{Mod}(\mathbb{S})$ with $X = \Psi X_0$ and $Y = \mathbb{S}$, the map $L_Q : \mathbb{S}[X_0^*] \to \mathbb{S}$ assigns to $\sum_{i=1}^n c(w_i) w_i$ the value $\sum_{i=1}^n c(w_i) y_i$, where y_i is the output of the state Q reaches on input w_i. Taking $Q = \mathbb{S}^n$ for some natural number n yields a classical n-state weighted automaton, and in this case one can show that the restriction of L_Q to X_0^* is is the usual language of a weighted automaton.

▸ **Remark 2.26.** By Remark 2.14 every $\mathcal{D}$-automaton (Q, δ, i, f) is an F-algebra as well as a T-coalgebra. Our above definition of L_Q was purely algebraic. The corresponding coalgebraic definition uses the unique coalgebra homomorphism $c_Q : Q \to [X^\circledast, Y]$ into the final T-coalgebra and precomposes with $i : I \to Q$ to get a morphism $c_Q \cdot i : I \to [X^\circledast, Y]$ (choosing a language, i.e. an element of $[X^\circledast, Y]$). Unsurprisingly, the results are equal:

▸ **Proposition 2.27.** *The language $L_Q : X^\circledast \to Y$ of an automaton (Q, δ, i, f) is the uncurried form of the morphism $c_Q \cdot i : I \to [X^\circledast, Y]$.*

3 Algebraic Recognition and Syntactic $\mathcal{D}$-Monoids

In classical algebraic automata theory one considers recognition of languages by (ordinary) monoids in lieu of automata. One key concept is the *syntactic monoid* which is characterized as the smallest monoid recognizing a given language. There are also related concepts of canonical algebraic recognizers in the literature, e.g. the syntactic idempotent semiring and the syntactic associative algebra. In this section we will give a uniform account of algebraic language recognition in our categorical setting. Our main result is the definition and construction of a minimal algebraic recognizer, the *syntactic $\mathcal{D}$-monoid* of a language.

▸ **Definition 3.1.** A $\mathcal{D}$-monoid morphism $e : X^\circledast \to M$ *recognizes* the language $L : X^\circledast \to Y$ if there exists a morphism $f : M \to Y$ of $\mathcal{D}$ with $L = f \cdot e$.

▸ **Example 3.2.** We use the notation of Example 2.23.

1. $\mathcal{D} = \mathbf{Set}$ with $X^\circledast = X^*$ and $Y = \{0, 1\}$: given a monoid M, a function $f : M \to \{0, 1\}$ defines a subset $F = f^{-1}[1] \subseteq M$. Hence a monoid morphism $e : X^* \to M$ recognizes L via f (i.e. $L = f \cdot e$) iff $L_0 = e^{-1}[F]$. This is the classical notion of recognition of a language $L_0 \subseteq X^*$ by a monoid, see e.g. Pin [18].
2. $\mathcal{D} = \mathbf{Set}_\perp$ with $X = X_0 + \{\perp\}$, $X^\circledast = X_0^* + \{\perp\}$ and $Y = \{\perp, 1\}$: given a monoid with zero M, a $\mathbf{Set}_\perp$-morphism $f : M \to \{\perp, 1\}$ defines a subset $F = f^{-1}[1]$ of $M \backslash \{0\}$. A zero-preserving monoid morphism $e : X_0^* + \{\perp\} \to M$ recognizes L via f iff $L_0 = e^{-1}[F]$.
3. $\mathcal{D} = \mathbf{Inv}$ with $X = X_0 + \widetilde{X_0}$, $X^\circledast = X_0^* + \widetilde{X_0^*}$ and $Y = \{0, 1\}$: for an involution monoid M to give a morphism $f : M \to \{0, 1\}$ means to give a subset $F = f^{-1}[1] \subseteq M$ satisfying $m \in F$ iff $\widetilde{m} \notin F$. Then L is recognized by $e : X_0^* + \widetilde{X_0^*} \to M$ via f iff $L_0 = X_0^* \cap e^{-1}[F]$.

4. $\mathcal{D} = \mathbf{JSL}_0$ with $X = \mathcal{P}_f X_0$, $X^{\circledast} = \mathcal{P}_f X_0^*$ and $Y = \{0, 1\}$: for an idempotent semiring M a morphism $f : M \to Y$ defines a prime upset $F = f^{-1}[1]$, see Example 2.13. Hence L is recognized by a semiring homomorphism $e : \mathcal{P}_f X_0^* \to M$ via f iff $L_0 = X_0^* \cap e^{-1}[F]$. Here we identify X_0^* with the set of all singleton languages $\{w\}$, $w \in X_0^*$. This is the concept of language recognition introduced by Polák [19] (except that he puts $F = f^{-1}[0]$, so 0 and 1 must be swapped, as well as F and $M \backslash F$).
5. $\mathcal{D} = \mathbf{Mod}(\$)$ with $X = \Psi X_0$, $X^{\circledast} = \$[X_0]$ and $Y = \$$: given an associative algebra M, the language L is recognized by $e : \$[X_0] \to M$ via $f : M \to \$$ iff $L = f \cdot e$. For the case where the semiring $\$$ is a ring, this notion of recognition is due to Reutenauer [21].

▸ **Remark 3.3.** 1. Since $\mathcal{D}$ and $\mathbf{Mon}(\mathcal{D})$ are varieties, we have the usual factorization system of regular epimorphisms (= surjective homomorphisms) and monomorphisms (= injective homomorphisms). *Quotients* and *subobjects* are understood w.r.t. this system.

2. By an *X-generated $\mathcal{D}$-monoid* we mean a quotient $e : X^{\circledast} \twoheadrightarrow M$ in $\mathbf{Mon}(\mathcal{D})$. For two such quotients $e_i : X^{\circledast} \twoheadrightarrow M_i$, $i = 1, 2$, we say, as usual, that e_1 is *smaller or equal to* e_2 (notation: $e_1 \leqslant e_2$) if e_1 factorizes through e_2. Note that if $X = \Psi X_0$, the free $\mathcal{D}$-monoid $X^{\circledast} = \Psi X_0^*$ on X is also the free $\mathcal{D}$-monoid on the set X_0 (w.r.t. the forgetful functor $\mathbf{Mon}(\mathcal{D}) \to \mathbf{Set}$), see Proposition 2.10.In this case, to give a quotient $e : X^{\circledast} \twoheadrightarrow M$ is equivalent to giving a set of generators for the $\mathcal{D}$-monoid M indexed by X_0 – which is why M may also be called an *X_0-generated $\mathcal{D}$-monoid*.

3. Let $e : X^{\circledast} \twoheadrightarrow M$ be an X-generated $\mathcal{D}$-monoid with unit $i : I \to M$ and multiplication $m : M \otimes M \to M$. Recall that $\eta_X : X \to X^{\circledast}$ denotes the universal morphism of the free $\mathcal{D}$-monoid on X and consider the F-algebra associated to M and $X \xrightarrow{\eta_X} X^{\circledast} \xrightarrow{e} M$ (see Definition 2.15). Thus, together with a given $f : M \to Y$ an X-generated $\mathcal{D}$-monoid induces an automaton (M, δ, i, f) called the *derived* automaton.

▸ **Lemma 3.4.** *The language recognized by an X-generated $\mathcal{D}$-monoid $e : X^{\circledast} \twoheadrightarrow M$ via $f : M \to Y$ is the language accepted by its derived automaton.*

We are now ready to give an abstract account of syntactic algebras in our setting. In classical algebraic automata theory the syntactic monoid of a language is characterized as the smallest monoid recognizing that language. We will use this property as our definition of the syntactic $\mathcal{D}$-monoid.

▸ **Definition 3.5.** The *syntactic $\mathcal{D}$-monoid* of language $L : X^{\circledast} \to Y$, denoted by $\mathsf{Syn}(L)$, is the smallest X-generated monoid recognizing L.

In more detail, the syntactic $\mathcal{D}$-monoid is an X-generated $\mathcal{D}$-monoid $e_L : X^{\circledast} \twoheadrightarrow \mathsf{Syn}(L)$ together with a morphism $f_L : \mathsf{Syn}(L) \to Y$ of $\mathcal{D}$ such that (i) e_L recognizes L via f_L, and (ii) for every X-generated $\mathcal{D}$-monoid $e : X^{\circledast} \twoheadrightarrow M$ recognizing L via $f : M \to Y$ we have $e_L \leqslant e$, that is, the left-hand triangle below commutes for some $\mathcal{D}$-monoid morphism h:

$$\begin{tikzcd}
X^{\circledast} \arrow[r, two heads, "e"] \arrow[dr, two heads, "e_L"'] & M \arrow[r, "f"] \arrow[d, two heads, "h"] & Y \\
& \mathsf{Syn}(L) \arrow[ur, "f_L"'] &
\end{tikzcd}$$

Note that the right-hand triangle also commutes since e is epimorphic and $f \cdot e = L = f_L \cdot e_L$. The universal property determines $\mathsf{Syn}(L)$, e_L and f_L uniquely up to isomorphism. A construction of $\mathsf{Syn}(L)$ is given below (Construction 3.13). We first consider a special case:

▸ **Example 3.6.** In $\mathcal{D} = \mathbf{Set}$ with $Y = \{0, 1\}$, the syntactic monoid of a language $L \subseteq X^*$ can be constructed as the quotient of X^* modulo the *syntactic congruence*, see e.g. [18]:

$$\mathsf{Syn}(L) = X^*/{\sim}, \qquad \text{where } u \sim v \text{ iff for all } x, y \in X^*\text{: } xuy \in L \iff xvy \in L.$$

We aim to generalize this construction to our categorical setting. First note the following

▸ **Lemma 3.7.** *Let $\mathcal{D}$ be any symmetric monoidal closed category with countable coproducts. Then the forgetful functor $\mathbf{Mon}(\mathcal{D}) \to \mathcal{D}$ preserves reflexive coequalizers.*

▸ **Notation 3.8.** Let (M, m, i) be a $\mathcal{D}$-monoid and $x : I \to M$. We write $x \bullet -$ and $- \bullet x$ for the following morphisms, respectively:

$$M \cong I \otimes M \xrightarrow{x \otimes M} M \otimes M \xrightarrow{m} M \qquad \text{and} \qquad M \cong M \otimes I \xrightarrow{M \otimes x} M \otimes M \xrightarrow{m} M.$$

Recall that in our setting, where $\mathcal{D}$ is a commutative variety, we have $I = \Psi 1$ and so the morphism x is the adjoint transpose of an element of M (see Remark 2.6). In the following we shall often write $x \bullet y$, identifying $x, y : I \to M$ with their corresponding elements of M.

▸ **Definition 3.9.** The *syntactic congruence* of a language $L : X^{\circledast} \to Y$ is the following relation on the underlying set of $X^{\circledast}$:

$$E = \{(u, v) \in X^{\circledast} \times X^{\circledast} \mid \forall x, y \in X^{\circledast} : L(x \bullet u \bullet y) = L(x \bullet v \bullet y)\}$$

The projection maps are denoted by $l, r : E \to X^{\circledast}$.

▸ **Lemma 3.10.** *The set E carries a canonical $\mathcal{D}$-algebraic structure making it a $\mathcal{D}$-object.*

Proof sketch. Just observe that $E = \bigcap E_{x,y}$ where for fixed $x, y \in X^{\circledast}$ the object $E_{x,y}$ is the kernel of the $\mathcal{D}$-morphism $X^{\circledast} \xrightarrow{x \bullet -} X^{\circledast} \xrightarrow{- \bullet y} X^{\circledast} \xrightarrow{L} Y$. ◂

That the name syntactic *congruence* makes sense follows from Lemma 3.11 below. First recall that a $\mathcal{D}$*-monoid congruence* on a given $\mathcal{D}$-monoid M is an equivalence relation in $\mathbf{Mon}(\mathcal{D})$, that is, a jointly monic pair $c_1, c_2 : C \to M$ of $\mathcal{D}$-monoid morphisms (equivalently a $\mathcal{D}$-submonoid $\langle c_1, c_2 \rangle : C \rightarrowtail M \times M$) which is reflexive, symmetric and transitive. Congruences on M are ordered as subobjects of $M \times M$, i.e. via inclusion.

▸ **Lemma 3.11.** *E is a $\mathcal{D}$-monoid congruence on $X^{\circledast}$.*

We can give an alternative, more conceptual, description of E:

▸ **Lemma 3.12.** *Let $l_0, r_0 : K \to X^{\circledast}$ be the kernel pair of $L : X^{\circledast} \to Y$ in $\mathcal{D}$. Then $l, r : E \to X^{\circledast}$ is the* largest *$\mathcal{D}$-monoid congruence contained in K.*

▸ **Construction 3.13.** Let $L : X^{\circledast} \to Y$ be a language and $l, r : E \to X^{\circledast}$ its syntactic congruence. We construct the $\mathcal{D}$-monoid $\mathsf{Syn}(L)$ as the coequalizer of l and r in $\mathbf{Mon}(\mathcal{D})$:

$$\begin{tikzcd} E \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & X^{\circledast} \arrow[r, two heads, "e_L"] & \mathsf{Syn}(L). \end{tikzcd}$$

We need to show that $\mathsf{Syn}(L)$ has the universal property of Definition 3.5, which first requires to define the morphism $f_L : \mathsf{Syn}(L) \to Y$ with $L = f_L \cdot e_L$. To this end consider the diagram below, where l_0, r_0 is the kernel pair of L and m witnesses that E is contained in K, i.e. $l = l_0 \cdot m$ and $r = r_0 \cdot m$ (see Lemma 3.12).

$$\begin{tikzcd} K \arrow[r, shift left, "l_0"] \arrow[r, shift right, "r_0"'] & X^{\circledast} \arrow[r, "L"] \arrow[dr, two heads, "e_L"'] & Y \\ E \arrow[u, tail, "m"] \arrow[ur, shift left, "l"] \arrow[ur, shift right, "r"'] & & \mathsf{Syn}(L) \arrow[u, dashed, "f_L"'] \end{tikzcd}$$

By Lemma 3.7 the morphism e_L is also a coequalizer of l and r in $\mathcal{D}$. Since $L \cdot l = L \cdot r$ by the above diagram, this yields a unique $f_L : \mathsf{Syn}(L) \to Y$ with $L = f_L \cdot e_L$. In other words, $\mathsf{Syn}(L)$ recognizes L via f_L.

▸ **Theorem 3.14.** $\mathsf{Syn}(L)$ *together with* e_L *and* f_L *forms the syntactic* $\mathcal{D}$*-monoid of* L*.*

Proof sketch. This follows from the correspondence between kernel pairs and regular quotients: since $l, r : E \to X^{\circledast}$ is the largest congruence contained in the kernel pair of L by Lemma 3.12, the coequalizer e_L of l, r is the smallest quotient of $X^{\circledast}$ recognizing L. ◂

▸ **Remark 3.15.** Our proof of Theorem 3.14 is quite conceptual and works in a general symmetric monoidal closed category $\mathcal{D}$ with enough structure. On this level of generality one would use Lemma 3.12 to *define* the syntactic congruence E as the largest $\mathcal{D}$-monoid congruence contained in the kernel of $L : X^{\circledast} \to Y$. However, it is unclear whether such a congruence exists in this generality and so its existence might have to be taken as an assumption. Hence we restricted ourselves to the setting of a commutative variety $\mathcal{D}$.

▸ **Example 3.16.** Using the notation of Example 2.23 we obtain the following concrete syntactic algebras:

1. In $\mathbf{Set}_\perp$ with $X = X_0 + \{\perp\}$ and $Y = \{\perp, 1\}$ the *syntactic monoid with zero* of a language $L_0 \subseteq X_0^*$ is $(X_0^* + \{\perp\})/{\sim}$ where, for all $u, v \in X_0^* + \{\perp\}$,

$$u \sim v \quad \text{iff} \quad \text{for all } x, y \in X_0^* : xuy \in L_0 \Leftrightarrow xvy \in L_0.$$

The zero element is the congruence class of $\perp$.

2. In $\mathbf{Inv}$ with $X = X_0 + \widetilde{X_0}$ and $Y = \{0, 1\}$ the *syntactic involution monoid* of a language $L_0 \subseteq X_0^*$ is the quotient of $X_0 + \widetilde{X_0}^*$ modulo the congruence $\sim$ defined for words $u, v \in X_0^*$ as follows:

$$\text{(i)}\ \ u \sim v \quad \text{iff} \quad \widetilde{u} \sim \widetilde{v} \quad \text{iff} \quad \text{for all } x, y \in X_0^* : xuy \in L_0 \iff xvy \in L_0;$$
$$\text{(ii)}\ \ u \sim \widetilde{v} \quad \text{iff} \quad \widetilde{u} \sim v \quad \text{iff} \quad \text{for all } x, y \in X_0^* : xuy \in L_0 \iff xvy \notin L_0.$$

3. In $\mathbf{Mod}(\mathbb{S})$ with $X = \Psi X_0$ and $Y = \mathbb{S}$ the *syntactic associative* $\mathbb{S}$*-algebra* of a weighted language $L_0 : X_0^* \to \mathbb{S}$ is the quotient of $\mathbb{S}[X_0]$ modulo the congruence defined for $U, V \in \mathbb{S}[X_0]$ as follows:

$$U \sim V \quad \text{iff} \quad \text{for all } x, y \in X_0^* : L(xUy) = L(xVy) \tag{3}$$

Indeed, since $L : \mathbb{S}[X_0] \to \mathbb{S}$ is linear, (3) implies $L(PUQ) = L(PVQ)$ for all $P, Q \in \mathbb{S}[X_0]$, which is the syntactic congruence of Definition 3.9.

4. In particular, for $\mathcal{D} = \mathbf{JSL}_0$ with $X = \mathcal{P}_f X_0$ and $Y = \{0, 1\}$, we get the *syntactic (idempotent) semiring* of a language $L_0 \subseteq X_0^*$ introduced by Polák [19]: it is the quotient $\mathcal{P}_f X_0^*/{\sim}$ where for $U, V \in \mathcal{P}_f X_0^*$ we have

$$U \sim V \quad \text{iff} \quad \text{for all } x, y \in X_0^* : (xUy) \cap L_0 \neq \varnothing \iff xVy \cap L_0 \neq \varnothing.$$

5. For $\mathcal{D} = \mathbf{Vec}(\mathbb{K})$ with $X = \Psi X_0$ and $Y = \mathbb{K}$, the *syntactic* $\mathbb{K}$*-algebra* of a $\mathbb{K}$-weighted language $L_0 : X_0^* \to \mathbb{K}$ is the quotient $\mathbb{K}[X_0]/I$ of the $\mathbb{K}$-algebra of finite weighted languages modulo the ideal

$$I = \{V \in \mathbb{K}[X_0] \mid \text{for all } x, y \in X_0^* : L(xVy) = 0\}.$$

Indeed, the congruence this ideal I generates ($U \sim V$ iff $U - V \in I$) is precisely (3). Syntactic $\mathbb{K}$-algebras were studied by Reutenauer [21].

6. Analogously, for $\mathcal{D} = \mathbf{Ab}$ with $X = \Psi X_0$ and $Y = \mathbb{Z}$, the *syntactic ring* of a $\mathbb{Z}$-weighted language $L_0 : X_0^* \to \mathbb{Z}$ is the quotient $\mathbb{Z}[X_0]/I$, where I is the ideal of all $V \in \mathbb{Z}[X_0]$ with $L(xVy) = 0$ for all $x, y \in X_0^*$.

4 Transition $\mathcal{D}$-Monoids

Here we present another construction of the syntactic $\mathcal{D}$-monoid of a language: it is the transition $\mathcal{D}$-monoid of the minimal $\mathcal{D}$-automaton for this language. Recall that for any object Q of a closed monoidal category $\mathcal{D}$, the object $[Q,Q]$ forms a $\mathcal{D}$-monoid w.r.t. composition.

▸ **Definition 4.1.** The *transition $\mathcal{D}$-monoid* $\mathsf{T}(Q)$ of an F-algebra (Q,δ,i) is the image of the $\mathcal{D}$-monoid morphism $(\lambda\delta)^+ : X^{\circledast} \to [Q,Q]$ extending $\lambda\delta : X \to [Q,Q]$:

$$\begin{tikzcd}
X^{\circledast} \arrow[rr, "(\lambda\delta)^+"] \arrow[dr, two heads, "e_{\mathsf{T}(Q)}"'] & & {[Q,Q]} \\
& \mathsf{T}(Q) \arrow[ur, tail, "m_{\mathsf{T}(Q)}"'] &
\end{tikzcd}$$

▸ **Example 4.2.**

1. In **Set** the transition monoid of an F-algebra Q (i.e. an automaton without final states) is the monoid of all extended transition maps $\delta_w = \delta_{a_n} \cdot \cdots \cdot \delta_{a_1} : Q \to Q$ for $w = a_1 \cdots a_n \in X^*$, with unit $\mathsf{id}_Q = \delta_\varepsilon$ and composition as multiplication.
2. In $\mathbf{Set}_\bot$ with $X = X_0 + \{\bot\}$ (the setting for partial automata) this is completely analogous, except that we add the constant endomap of Q with value $\bot$.
3. In **Inv** with $X = X_0 + \widetilde{X_0}$ we get the involution monoid of all δ_w and $\widetilde{\delta_w}$. Again the unit is δ_ε, and the multiplication is determined by composition plus the equations $x\widetilde{y} = \widetilde{xy} = \widetilde{x}y$.
4. In $\mathbf{JSL}_0$ with $X = \mathcal{P}_f X_0$ the transition semiring consists of all finite joins of extended transitions, i.e. all semilattice homomorphisms of the form $\delta_{w_1} \vee \cdots \vee \delta_{w_n}$ for $\{w_1,\ldots,w_n\} \in \mathcal{P}_f X_0^*$. The transition semiring was introduced by Polák [19].
5. In $\mathbf{Mod}(\mathbb{S})$ with $X = \Psi X_0$ the associative transition algebra consists of all linear maps of the form $\sum_{i=1}^n s_i \delta_{w_i}$ with $s_i \in \mathbb{S}$ and $w_i \in X_0^*$.

Recall from Definition 2.12 that a $\mathcal{D}$-automaton is an F-algebra Q together with an output morphism $f : Q \to Y$. Hence we can speak of the transition $\mathcal{D}$-monoid of a $\mathcal{D}$-automaton.

▸ **Proposition 4.3.** *The language accepted by a $\mathcal{D}$-automaton (Q,δ,f,i) is recognized by the $\mathcal{D}$-monoid morphism $e_{\mathsf{T}(Q)} : X^{\circledast} \twoheadrightarrow \mathsf{T}(Q)$.*

Proof sketch. The desired morphism $f_{\mathsf{T}(Q)} : \mathsf{T}(Q) \to Y$ with $L_Q = f_{\mathsf{T}(Q)} \cdot e_{\mathsf{T}(Q)}$ is

$$f_{\mathsf{T}(Q)} = (\mathsf{T}(Q) \xrightarrow{m_{\mathsf{T}(Q)}} [Q,Q] \cong [Q,Q] \otimes I \xrightarrow{[Q,Q]\otimes i} [Q,Q] \otimes Q \xrightarrow{\mathsf{ev}} Q \xrightarrow{f} Y).$$ ◂

▸ **Definition 4.4.** A $\mathcal{D}$-automaton (Q,δ,i,f) is called *minimal* iff it is

(a) *reachable*: the unique F-algebra homomorphism $X^{\circledast} \to Q$ is surjective;
(b) *simple*: the unique T-coalgebra homomorphism $Q \to [X^{\circledast}, Y]$ is injective.

▸ **Theorem 4.5** (Goguen [12])**.** *Every language $L : X^{\circledast} \to Y$ is accepted by a minimal $\mathcal{D}$-automaton* $\mathsf{Min}(L)$*, unique up to isomorphism. Given any reachable automaton Q accepting L, there is a unique surjective automata homomorphism from Q into* $\mathsf{Min}(L)$*.*

This leads to the announced construction of syntactic $\mathcal{D}$-monoids via transition $\mathcal{D}$-monoids. The case $\mathcal{D} = \mathbf{Set}$ is a standard result of algebraic automata theory (see e.g. Pin [18]), and the case $\mathcal{D} = \mathbf{JSL}_0$ is due to Polák [19].

▸ **Theorem 4.6.** *The syntactic $\mathcal{D}$-monoid of a language $L : X^{\circledast} \to Y$ is isomorphic to the transition $\mathcal{D}$-monoid of its minimal $\mathcal{D}$-automaton:*

$$\mathsf{Syn}(L) \cong \mathsf{T}(\mathsf{Min}(L)).$$

Proof sketch. Using reachability and simplicity of $\mathsf{Min}(L)$, one proves that the quotients $e_L : X^{\circledast} \twoheadrightarrow \mathsf{Syn}(L)$ and $e_{\mathsf{T}(\mathsf{Min}(L))} : X^{\circledast} \twoheadrightarrow \mathsf{T}(\mathsf{Min}(L))$ have the same kernel pair, namely the syntactic congruence of L. This implies the statement of the theorem. ◂

5 $\mathcal{D}$-Regular Languages

Our results so far apply to arbitrary languages in $\mathcal{D}$. In the present section we focus on *regular languages*, which in $\mathcal{D} = \mathbf{Set}$ are the languages accepted by finite automata, or equivalently the languages recognized by finite monoids. For arbitrary $\mathcal{D}$ the role of finite sets is taken over by finitely presentable objects. Recall that an object D of $\mathcal{D}$ is *finitely presentable* if the hom-functor $\mathcal{D}(D, -) : \mathcal{D} \to \mathbf{Set}$ preserves filtered colimits. Equivalently, D is an algebra presentable with finitely many generators and relations.

▸ **Definition 5.1.** A language $L : X^{\circledast} \to Y$ is called $\mathcal{D}$-*regular* if it is accepted by some $\mathcal{D}$-automaton with a finitely presentable object of states.

To work with this definition, we need the following

▸ **Assumptions 5.2.** We assume that the full subcategory $\mathcal{D}_f$ of finitely presentable objects of $\mathcal{D}$ is closed under subobjects, strong quotients and finite products.

▸ **Example 5.3.** **1.** Recall that a variety is *locally finite* if all finitely presentable algebras (equivalently all finitely generated free algebras) are finite. Every locally finite variety satisfies the above assumptions. This includes our examples $\mathbf{Set}$, $\mathbf{Set}_\perp$, $\mathbf{Inv}$ and $\mathbf{JSL}_0$.

2. A semiring $\mathbb{S}$ is called *Noetherian* if all submodules of finitely generated $\mathbb{S}$-modules are finitely generated. In this case, as shown in [10], the category $\mathbf{Mod}(\mathbb{S})$ satisfies our assumptions. Every field is Noetherian, as is every finitely generated commutative ring, so $\mathbf{Vec}(\mathbb{K})$ and $\mathbf{Ab} = \mathbf{Mod}(\mathbb{Z})$ are special instances.

▸ **Theorem 5.4.** *For any language $L : X^{\circledast} \to Y$ the following statements are equivalent:*

(a) L is $\mathcal{D}$-regular.
(b) The minimal $\mathcal{D}$-automaton $\mathsf{Min}(L)$ *has finitely presentable carrier.*
(c) L is recognized by some $\mathcal{D}$-monoid with finitely presentable carrier.
(d) The syntactic $\mathcal{D}$-monoid $\mathsf{Syn}(L)$ *has finitely presentable carrier.*

Proof sketch. This follows immediately from the universal properties of $\mathsf{Syn}(L)$ and $\mathsf{Min}(L)$ and the assumed closure properties of $\mathcal{D}_f$. ◂

Just as the collection of all languages is internalized by the final coalgebra $[X^{\circledast}, Y]$, see Proposition 2.21, we can internalize the regular languages by means of the *rational coalgebra*.

▸ **Definition 5.5.** The *rational coalgebra* ϱT for T is the colimit (taken in the category of T-coalgebras and homomorphisms) of all T-coalgebras with finitely presentable carrier.

▸ **Proposition 5.6.** *There is a one-to-one correspondence between $\mathcal{D}$-regular languages and elements $I \to \varrho T$ of the rational coalgebra.*

We conclude this section with an interesting dual perspective on syntactic monoids, based on our previous work [2, 4]. For lack of space we restrict to the case $\mathcal{D} = \mathbf{Set}$. This category is *predual* to the category $\mathbf{BA}$ of boolean algebras in the sense that the full subcategories of finite sets and finite boolean algebras are dually equivalent. Indeed, this is a restriction of the well-known Stone duality: the dual equivalence functor assigns to a finite boolean algebra B the set $\mathsf{At}(B)$ of its atoms, and to a boolean homomorphism $h : A \to B$ the map $\mathsf{At}(h) : \mathsf{At}(B) \to \mathsf{At}(A)$ sending $b \in \mathsf{At}(B)$ to the unique atom $a \in \mathsf{At}(A)$ with $ha \geqslant b$.

How do the concepts we investigated in $\mathbf{Set}$ – languages, automata and monoids – dualize to $\mathbf{BA}$? Observe that $\mathsf{Reg}(X)$, the boolean algebra of regular languages over the alphabet

X, can be viewed as a deterministic automaton: its final states are the regular languages containing the empty word, and the transitions are given by $L \xrightarrow{a} a^{-1}L$ for $a \in X$, where $a^{-1}L = \{w \in X^* : aw \in L\}$ is the *left derivative* of L w.r.t. the letter a. (Similarly, the *right derivative* of L w.r.t. a is $La^{-1} = \{w \in X^* : wa \in L\}$.) This makes $\mathsf{Reg}(X)$ a coalgebra for the endofunctor $\overline{T} = \{0,1\} \times \mathsf{Id}^X$ on **BA**. Since the two-chain $\{0,1\}$ is dual to the singleton set 1, finite coalgebras for $\overline{T}$ dualize to finite *algebras* for the functor $F = 1 + X \times \mathsf{Id} \cong 1 + \coprod_X \mathsf{Id}$ on **Set**. Based on this, we proved in [2] that further (i) finite $\overline{T}$-subcoalgebras of $\mathsf{Reg}(X)$ dualize to finite quotient algebras of the initial F-algebra X^*, and (ii) finite *local varieties of languages* (i.e. finite $\overline{T}$-subcoalgebras of $\mathsf{Reg}(X)$ closed under right derivatives) dualize to those F-algebras associated to X-generated monoids, see Definition 2.15. For a regular language $L \subseteq X^*$ the F-algebras associated to the minimal automaton $\mathsf{Min}(L)$ and the syntactic monoid $\mathsf{Syn}(L)$ are finite. Their dual $\overline{T}$-coalgebras are characterized as follows:

▸ **Theorem 5.7.** *Let $L \subseteq X^*$ be a regular language, and L^{rev} its reversed language.*

(a) $\mathsf{Min}(L)$ *is dual to the smallest subcoalgebra of* $\mathsf{Reg}(X)$ *containing* L^{rev}.
(b) $\mathsf{Syn}(L)$ *is dual to the smallest local variety of languages containing* L^{rev}.

Part (a) of this theorem adds to the recently developed dual view of minimal automata, see [7] and also [17, 3]. All the above considerations generalize from **BA**/**Set** to arbitrary pairs $\mathcal{C}/\mathcal{D}$ of predual locally finite varieties of algebras. Examples include the self-predual varieties $\mathcal{C} = \mathcal{D} = \mathbf{JSL}_0$ and $\mathcal{C} = \mathcal{D} = \mathbf{Vec}(\mathbb{K})$ for a finite field $\mathbb{K}$.

6 Conclusions and Future Work

We proposed the first steps of a categorical theory of algebraic language recognition. Despite our assumption that $\mathcal{D}$ is a commutative variety, the bulk of our definitions, constructions and proofs works in any symmetric monoidal closed category with enough structure. However, the construction of the syntactic monoid via the syntactic congruence, and the proof that it coincides with a transition monoid, required the concrete algebraic setting. It remains an open problem to develop a genuinely abstract framework for our theory. In particular, such a generalized setting should provide the means for incorporating *ordered* algebras, e.g. the syntactic ordered monoids of Pin [18]. We expect this can be achieved by working with (order-)enriched categories, where the coequalizer in our construction of the syntactic monoid is replaced by a coinserter. A more general theory of recognition might also open the door to treating algebraic recognizers for additional types of behaviors, including Wilke algebras [23] (representing ω-languages) and forest algebras [9] (representing tree and forest languages).

One of the leading themes of algebraic automata theory is the classification of languages in terms of their syntactic algebras. For instance, by Schützenberger's theorem a language is star-free iff its syntactic monoid is aperiodic. We hope that our conceptual view of syntactic monoids (notably their dual characterization in Theorem 5.7) can contribute to a duality-based approach to such results, leading to generalizations and new proof techniques.

References

1 Jiří Adámek, Stefan Milius, , and Henning Urbat. Syntactic monoids in a category. Extended version. `http://arxiv.org/abs/1504.02694`, 2015.

2 Jiří Adámek, Stefan Milius, Robert S. R. Myers, and Henning Urbat. Generalized Eilenberg Theorem I: Local Varieties of Languages. In Anca Muscholl, editor, *Proc. Foundations of Software Science and Computation Structures (FoSSaCS)*, volume 8412 of *Lecture Notes Comput. Sci.*, pages 366–380. Springer, 2014.

3 Jiří Adámek, Stefan Milius, Robert S. R. Myers, and Henning Urbat. On continuous nondeterminism and state minimality. In Bart Jacobs, Alexandra Silva, and Sam Staton, editors, *Proc. Mathematical Foundations of Programming Science (MFPS XXX)*, volume 308 of *Electron. Notes Theor. Comput. Sci.*, pages 3–23. Elsevier, 2014.

4 Jiří Adámek, Stefan Milius, Robert S. R. Myers, and Henning Urbat. Varieties of Languages in a Category. Accepted for LICS 2015. `http://arxiv.org/abs/1501.05180`, 2015.

5 A. Ballester-Bolinches, E. Cosme-Llopez, and J.J.M.M. Rutten. The dual equivalence of equations and coequations for automata. Technical report, CWI, 2014.

6 Bernhard Banaschweski and Evelyn Nelson. Tensor products and bimorphisms. *Canad. Math. Bull.*, 19:385–402, 1976.

7 Nick Bezhanishvili, Clemens Kupke, and Prakash Panangaden. Minimization via duality. In Luke Ong and Ruy de Queiroz, editors, *Logic, Language, Information and Computation*, volume 7456 of *Lecture Notes in Computer Science*, pages 191–205. Springer Berlin Heidelberg, 2012.

8 Mikołaj Bojánczyk. Recognisable languages over monads. Preprint: `http://arxiv.org/abs/1502.04898`, 2015.

9 Mikołaj Bojánczyk and Igor Walukiewicz. Forest algebras. In *Automata and Logic: History and Perspectives*, pages 107–132, 2006.

10 Marcello M. Bonsangue, Stefan Milius, and Alexandra Silva. Sound and complete axiomatizations of coalgebraic language equivalence. *ACM Trans. Comput. Log.*, 14(1:7), 2013.

11 Mai Gehrke, Serge Grigorieff, and Jean-Éric Pin. Duality and equational theory of regular languages. In *Proc. ICALP 2008, Part II*, volume 5126 of *Lecture Notes Comput. Sci.*, pages 246–257. Springer, 2008.

12 Joseph A. Goguen. Discrete-time machines in closed monoidal categories. I. *J. Comput. Syst. Sci.*, 10(1):1–43, 1975.

13 Anders Kock. Monads on symmetric monoidal closed categories. *Arch. Math.*, 21:1–10, 1970.

14 Stephen Lack. On the monadicity of finitary monads. *J. Pure Appl. Algebra*, 140:65–73, 1999.

15 Saunders Mac Lane. *Categories for the working mathematician.* Springer, 2nd edition, 1998.

16 Fred Linton. Autonomous equational categories. *J. Math. Mech.*, 15:637–642, 166.

17 Robert S. R. Myers, Jiří Adámek, Stefan Milius, and Henning Urbat. Canonical nondeterministic automata. In Marcello M. Bonsangue, editor, *Proc. Coalgebraic Methods in Computer Science (CMCS'14)*, volume 8446 of *Lecture Notes Comput. Sci.*, pages 189–210. Springer, 2014.

18 Jean-Éric Pin. Mathematical foundations of automata theory. available at `http://www.liafa.jussieu.fr/~jep/PDF/MPRI/MPRI.pdf`, January 2015.

19 Libor Polák. Syntactic semiring of a language. In Jiří Sgall, Aleš Pultr, and Petr Kolman, editors, *Proc. International Symposium on Mathematical Foundations of Computer Science (MFCS)*, volume 2136 of *Lecture Notes Comput. Sci.*, pages 611–620. Springer, 2001.

20 Michael O. Rabin and Dana S. Scott. Finite automata and their decision problems. *IBM J. Res. Dev.*, 3(2):114–125, April 1959.

21 Christophe Reutenauer. Séries formelles et algèbres syntactiques. *J. Algebra*, 66:448–483, 1980.

22 Jan J. M. M. Rutten. Universal coalgebra: a theory of systems. *Theoret. Comput. Sci.*, 249(1):3–80, 2000.

23 Thomas Wilke. An Eilenberg Theorem for Infinity-Languages. In *Proc. ICALP 91*, pages 588–599. Springer, 1991.

A Proofs

This Appendix contains all proofs and additional details we omitted due to space limitations.

Details for Example 2.1.4.

A *semiring* $\mathcal{S} = (S, +, \cdot, 0, 1)$ consists of a commutative monoid $(S, +, 0)$ and a monoid $(S, \cdot, 1)$ such that

$$0x = x0 = 0, \quad x(y+z) = xy + xz \quad \text{and} \quad (x+y)z = xy + yz.$$

A *module* over a semiring S is a commutative monoid $(M, +, 0)$ together with a scalar multiplication $\cdot : S \times M \to M$ such that the following laws hold:

$$\begin{aligned} (r+s)a &= ra + sa, & r(a+b) &= ra + rb, & (rs)a &= r(sa), \\ 0a &= 0, & 1a &= 1, & r0 &= 0. \end{aligned}$$

Proof of Proposition 2.10

A constructive proof can be found in [2]. Here we give a more conceptual argument, using the universal property of the tensor product. Observe that the functor $\Psi : \mathbf{Set} \to \mathcal{D}$ is strongly monoidal, i.e., it preserves the unit and tensor product up to natural isomorphism. Indeed, we have $\Psi 1 = I$ by definition. To see that $\Psi(A \times B) \cong \Psi A \otimes \Psi B$ for all sets A and B, consider the following bijections (natural in D):

$$\begin{aligned} \mathcal{D}(\Psi(A \times B), D) &\cong \mathbf{Set}(A \times B, |D|) \\ &\cong \mathbf{Set}(A, |D|^B) \\ &\cong \mathcal{D}(\Psi A, D^B) \\ &\cong \mathcal{D}(\Psi A, [\Psi B, D]) \\ &\cong \mathcal{D}(\Psi A \otimes \Psi B, D). \end{aligned}$$

This implies $\Psi(A \times B) \cong \Psi A \otimes \Psi B$ by the Yoneda lemma. Using the fact that Ψ preserves coproducts, being a left adjoint, we conclude

$$X^{\circledast} \cong \coprod_{n<\omega} X^{\otimes n} \cong \coprod_{n<\omega} \Psi X_0^n \cong \Psi(\coprod_{n<\omega} X_0^n) = \Psi X_0^*.$$

Alternatively one can show that the right adjoint $|-| : \mathcal{D} \to \mathbf{Set}$ is a monoidal functor. This implies that Ψ preserves free monoids.

Details for Remark 2.18

First we recall the $\mathcal{D}$-monoid structure on $[Q, Q]$. Let $\iota'_Q : Q \to I \otimes Q$ be the left unit isomorphism. Then the unit $j : I \to [Q, Q]$ and multiplication $m : [Q, Q] \otimes [Q, Q] \to [Q, Q]$ are the unique morphisms making the following diagram commutative, respectively:

$$\begin{tikzcd} {[Q,Q] \otimes Q} \arrow[r, "\mathsf{ev}"] & Q \\ I \otimes Q \arrow[u, "j \otimes Q"] \arrow[ur, "\iota'_Q"'] & \end{tikzcd} \qquad \begin{tikzcd} {[Q,Q] \otimes Q} \arrow[r, "\mathsf{ev}"] & Q \\ {[Q,Q] \otimes [Q,Q] \otimes Q} \arrow[u, "m \otimes Q"] \arrow[r, "{[Q,Q] \otimes \mathsf{ev}}"'] & {[Q,Q] \otimes Q} \arrow[u, "\mathsf{ev}"'] \end{tikzcd}$$

Now all we have to show is that the morphism in (2) is an F-algebra homomorphism. It then follows that it is the unique one e_Q. Indeed, the diagram below commutes:

$$\begin{tikzcd}
X \otimes X^{\circledast} \arrow[r, "\eta_X \otimes X^{\circledast}"] \arrow[d, "X \otimes \iota_{X^{\circledast}}"] & X^{\circledast} \otimes X^{\circledast} \arrow[r, "m_X"] \arrow[d, "X^{\circledast} \otimes \iota_{X^{\circledast}}"] & X^{\circledast} \arrow[d, "\iota_{X^{\circledast}}"] & I \arrow[l, "i_X"'] \arrow[d, "\iota_I"'] \arrow[dddl, bend left, "i"] \\
X \otimes X^{\circledast} \otimes I \arrow[r, "\eta_X \otimes X^{\circledast}"] \arrow[d, "X \otimes (\lambda\delta)^+ \otimes i"] & X^{\circledast} \otimes X^{\circledast} \otimes I \arrow[r, "m_X \otimes I"] \arrow[d, "(\lambda\delta)^+ \otimes (\lambda\delta)^+ \otimes i"] & X^{\circledast} \otimes I \arrow[d, "(\lambda\delta)^+ \otimes i"] & I \otimes I \arrow[l, "i_X \otimes I"'] \arrow[d, "I \otimes i"'] \\
X \otimes [Q,Q] \otimes Q \arrow[r, "\lambda\delta \otimes [Q,Q] \otimes Q"] \arrow[d, "X \otimes \mathsf{ev}"] & {[Q,Q] \otimes [Q,Q] \otimes Q} \arrow[r, "m \otimes Q"] \arrow[d, "{[Q,Q] \otimes \mathsf{ev}}"'] & {[Q,Q] \otimes Q} \arrow[d, "\mathsf{ev}"] & I \otimes Q \arrow[l, "j \otimes Q"'] \arrow[dl, "\iota'_Q"] \\
X \otimes Q \arrow[r, "\lambda\delta \otimes Q"] \arrow[rr, bend right, "\delta"'] & {[Q,Q] \otimes Q} \arrow[r, "\mathsf{ev}"] & Q &
\end{tikzcd}$$

Proof of Proposition 2.21

Given any coalgebra (Q, τ, f), consider the morphism $\delta = (X \otimes Q \xrightarrow{\cong} Q \otimes X \xrightarrow{\beta} Q)$ where β is the uncurried form of $\tau : Q \to [X, Q]$, and denote by $\delta^{\circledast} : X^{\circledast} \otimes Q \to Q$ the extension of δ as in Remark 2.18. We claim that the unique coalgebra homomorphism into $[X^{\circledast}, Y]$ is $\lambda h : Q \to [X^{\circledast}, Y]$, where

$$h = (Q \otimes X^{\circledast} \cong X^{\circledast} \otimes Q \xrightarrow{\delta^{\circledast}} Q \xrightarrow{f} Y).$$

Let us first prove that h is indeed a coalgebra homomorphism. Preservation of outputs is shown by the following commutative diagram:

$$\begin{tikzcd}
Q \arrow[rr, equal] \arrow[dr, "\cong"] \arrow[ddr, "\cong"'] \arrow[ddd, "\lambda h"'] & & Q \arrow[r, "f"] & Y \\
& I \otimes Q \arrow[r, "i_X \otimes Q"] & X^{\circledast} \otimes Q \arrow[u, "\delta^{\circledast}"'] & \\
& Q \otimes I \arrow[r, "Q \otimes i_X"'] \arrow[u, "\cong"] \arrow[d, "\lambda h \otimes I"] & Q \otimes X^{\circledast} \arrow[u, "\cong"'] \arrow[ur, "h"] \arrow[dr, "\lambda h \otimes X^{\circledast}"] & \\
{[X^{\circledast}, Y]} \arrow[r, "\cong"'] & {[X^{\circledast}, Y] \otimes I} \arrow[rr, "{[X^{\circledast}, Y] \otimes i_X}"'] & & {[X^{\circledast}, Y] \otimes X^{\circledast}} \arrow[uuu, "\mathsf{ev}"']
\end{tikzcd}$$

For preservation of transitions it suffices to show that the following diagram commutes, where $\overline{\tau} : [X^{\circledast}, Y] \otimes X \to [X^{\circledast}, Y]$ is the uncurried coalgebra structure of $[X^{\circledast}, Y]$:

$$\begin{tikzcd}
Q \otimes X \arrow[r, "\beta"] \arrow[d, "\lambda h \otimes X"'] & Q \arrow[d, "\lambda h"] \\
{[X^{\circledast}, Y] \otimes X} \arrow[r, "\overline{\tau}"'] & {[X^{\circledast}, Y]}
\end{tikzcd}$$

But the above diagram is precisely the curried version of the following one (where we omit writing $\otimes$ for space reasons):

$$\begin{tikzcd}
QXX^{\circledast} \arrow[rrrrr, "\beta X^{\circledast}"] \arrow[ddd, "\lambda h X X^{\circledast}"'] \arrow[ddr, "Q\eta_X X^{\circledast}"'] & & & & & QX^{\circledast} \arrow[dl, "\cong"'] \arrow[ddd, "h"] \\
 & & (*) & & X^{\circledast}Q \arrow[d, "\delta^{\circledast}"] & \\
 & QX^{\circledast}X^{\circledast} \arrow[r, "Qm_X"] \arrow[d, "\lambda h X^{\circledast} X^{\circledast}"] & QX^{\circledast} \arrow[r, "\cong"] \arrow[d, "\lambda h X^{\circledast}"] \arrow[drrr, "h"] & X^{\circledast}Q \arrow[r, "\delta^{\circledast}"] & Q \arrow[dr, "f"] & \\
{[X^{\circledast}, Y]XX^{\circledast}} \arrow[r, "{[X^{\circledast},Y]\eta_X X^{\circledast}}"] & {[X^{\circledast}, Y]X^{\circledast}X^{\circledast}} \arrow[r, "{[X^{\circledast},Y]m_X}"] & {[X^{\circledast}, Y]X^{\circledast}} \arrow[rrr, "\mathsf{ev}"'] & & & Y
\end{tikzcd}$$

The part $(*)$ follows easily from the definition of $\delta^{\circledast}$, and the other parts are clear. Now suppose that any coalgebra homomorphism of $\lambda h : Q \to [X^{\circledast}, Y]$ is given. We show that $h : Q \otimes X^{\circledast} \to Y$ is determined by the composites $(Q \otimes X^{\otimes n} \xrightarrow{Q\otimes i_n} Q \otimes X^{\circledast} \xrightarrow{h} Y)$, $n < \omega$, where $i_n : X^{\otimes n} \to X^{\circledast}$ is the n-th coproduct injection. This proves the uniqueness of λh: since $\otimes$ preserves coproducts, the morphisms $(Q \otimes i_n)_{n<\omega}$ form a coproduct cocone. For $n = 0$, the claim is proved by the diagram

$$\begin{tikzcd}
Q \otimes I \arrow[rrr, "\cong"] \arrow[dr, "\lambda h \otimes I"] \arrow[ddd, "Q\otimes i_0"'] & & & Q \arrow[dl, "\lambda h"'] \arrow[ddd, "f"] \\
 & {[X^{\circledast}, Y] \otimes I} \arrow[r, shift left, "\cong"] \arrow[d, "{[X^{\circledast},Y]\otimes i_0}"'] & {[X^{\circledast}, Y]} \arrow[l, shift left, "\cong"] \arrow[ddr, "f_{[X^{\circledast},Y]}"'] & \\
 & {[X^{\circledast}, Y] \otimes X^{\circledast}} \arrow[drr, "\mathsf{ev}"'] & & \\
Q \otimes X^{\circledast} \arrow[ur, "\lambda h \otimes X^{\circledast}"] \arrow[rrr, "h"'] & & & Y
\end{tikzcd}$$

And the following diagram shows that $h \cdot (Q \otimes i_{n+1})$ is determined by $h \cdot (Q \otimes i_n)$ (again we omit $\otimes$, in particular we write X^n for $X^{\otimes n}$):

$$\begin{tikzcd}
QXX^n \arrow[rrr, "Qi_{n+1}"] \arrow[dr, "\lambda h X X^n"] \arrow[dddd, "\beta X^n"'] & & & QX^{\circledast} \arrow[r, "h"] \arrow[d, "\lambda h X^{\circledast}"] & Y \\
 & {[X^{\circledast}, Y]XX^n} \arrow[rr, "{[X^{\circledast},Y]i_{n+1}}"] \arrow[dr, "{[X^{\circledast},Y]Xi_n}"] \arrow[dd, "\overline{\tau} X^n"'] & & {[X^{\circledast}, Y]X^{\circledast}} \arrow[ur, "\mathsf{ev}"] & \\
 & & {[X^{\circledast}, Y]XX^{\circledast}} \arrow[r, "{[X^{\circledast},Y]\eta_X X^{\circledast}}"] \arrow[dr, "\overline{\tau} X^{\circledast}"'] & {[X^{\circledast}, Y]X^{\circledast}X^{\circledast}} \arrow[u, "{[X^{\circledast},Y]m_X}"'] & \\
 & {[X^{\circledast}, Y]X^n} \arrow[rr, "{[X^{\circledast},Y]i_n}"'] & & {[X^{\circledast}, Y]X^{\circledast}} \arrow[uuur, "\mathsf{ev}"'] & (*) \\
QX^n \arrow[ur, "\lambda h X^n"'] \arrow[rrrr, "Qi_n"'] & & & & QX^{\circledast} \arrow[ul, "\lambda h X^{\circledast}"'] \arrow[uuuu, "h"']
\end{tikzcd}$$

Note that part $(*)$ commutes by the definition of the coalgebra structure on $[X^{\circledast}, Y]$ and all other parts are easy to see.

Proof of Proposition 2.27

Recall from the proof of Proposition 2.21 that the uncurried version of c_Q is the morphism

$$Q \otimes X^{\circledast} \cong X^{\circledast} \otimes Q \xrightarrow{\delta^{\circledast}} Q \xrightarrow{f} Y.$$

Hence $c_Q \cdot i : I \to [X^{\circledast}, Y]$ determines the language

$$X^{\circledast} \cong X^{\circledast} \otimes I \xrightarrow{X^{\circledast} \otimes i} X^{\circledast} \otimes Q \xrightarrow{\delta^{\circledast}} Q \xrightarrow{f} Y,$$

and this is precisely L_Q, as shown by the diagram below (where r_X is the F-algebra structure of $X^{\circledast}$):

$$\begin{tikzcd}
X^{\circledast} \arrow[r, "\cong"] \arrow[rrr, bend left, "\mathsf{id}"] & X^{\circledast} \otimes I \arrow[r, "X^{\circledast} \otimes i_X"] \arrow[dr, swap, "X^{\circledast} \otimes i"] & X^{\circledast} \otimes X^{\circledast} \arrow[r, "r_X^{\circledast}"] \arrow[d, "X^{\circledast} \otimes e_Q"] & X^{\circledast} \arrow[r, "L_Q"] \arrow[d, "e_Q"] & Y \\
 & & X^{\circledast} \otimes Q \arrow[r, swap, "\delta^{\circledast}"] & Q \arrow[ur, swap, "f"] &
\end{tikzcd}$$

This completes the proof.

Proof of Lemma 3.4

Let $e : X^{\circledast} \twoheadrightarrow M$ be an X-generated $\mathcal{D}$-monoid and let $f : M \to Y$. Then this recognizes the language $L = f \cdot e$. We are done once we prove that e is the unique F-algebra morphism from $X^{\circledast}$ to the F-algebra associated to M and $e \cdot \eta_X$ (cf. Remark 3.3). Recall from Proposition 2.17 that the initial F-algebra is the F-algebra associated to the free $\mathcal{D}$-monoid $X^{\circledast}$ and η_X. Then the following diagram clearly commutes since e is a $\mathcal{D}$-monoid morphism:

$$\begin{tikzcd}
FX^{\circledast} = I + X \otimes X^{\circledast} \arrow[r, "I + \eta_X \otimes X^{\circledast}"] \arrow[d, swap, "Fe = I + X \otimes e"] & I + X^{\circledast} \otimes X^{\circledast} \arrow[r, "{[i_X, m_X]}"] \arrow[d, swap, "I + e \otimes e"] & X^{\circledast} \arrow[d, "e"] \\
FM = I + X \otimes M \arrow[r, swap, "I + (e \cdot \eta_X) \otimes M"] & I + M \otimes M \arrow[r, swap, "{[i, m]}"] & M
\end{tikzcd}$$

This completes the proof.

Proof of Lemma 3.7

First of all we know from Lack [14, Theorem 2] that the forgetful functor $U : \mathbf{Mon}(\mathcal{D}) \to \mathcal{D}$ is monadic, and by Proposition 2.9 the monad is $\mathsf{Id}^{\circledast} = \coprod_{n<\omega} \mathsf{Id}^{\otimes n}$. It suffices to show that $\mathsf{Id}^{\circledast}$ preserves reflexive coequalizers: it then follows that U preserves (in fact, creates) them. For that it is sufficient to prove that each $\mathsf{Id}^{\otimes n}$ preserves reflexive coequalizers. This follows from (the proof of) [14, Lemma 1] using that in our setting both $X \otimes -$ and $- \otimes X$ preserve all colimits (being left-adjoints).

Proof of Lemma 3.10

Observe that $E = \bigcap E_{x,y}$ where for fixed $x, y \in X^{\circledast}$, $E_{x,y}$ is the kernel of the $\mathcal{D}$-morphism

$$X^{\circledast} \xrightarrow{x\bullet -} X^{\circledast} \xrightarrow{-\bullet y} X^{\circledast} \xrightarrow{L} Y\,. \tag{4}$$

Since all limits in $\mathcal{D}$ (and in particular, kernels and intersections) are created by the forgetful functor $|{-}| : \mathcal{D} \to \mathbf{Set}$ we see that E has a canonical structure of a $\mathcal{D}$-object as desired.

Proof of Lemma 3.11

It suffices to prove that pairs in E are closed under the monoid operation of $X^{\circledast}$. Clearly $(\varepsilon, \varepsilon) \in E$, where $\varepsilon : 1 \to X^{\circledast}$ is the adjoint transpose of the unit $I = \Psi 1 \to X^{\circledast}$ of the free $\mathcal{D}$-monoid on X. Given (u, v) and (u', v') in E we show that $(u \bullet u', v \bullet v')$ is in E, too. Indeed, we have for all $x, y \in X^{\circledast}$ that

$$L(x \bullet u \bullet u' \bullet y) = L(x \bullet v \bullet u' \bullet y) = L(x \bullet v \bullet v' \bullet y).$$

Proof of Lemma 3.12

To see that E from Definition 3.9 satisfies this property let $l', r' : E' \to X^{\circledast}$ be any $\mathcal{D}$-monoid congruence contained in K via $m' : E' \rightarrowtail K$ with $l_0 \cdot m' = l'$ and $r_0 \cdot m' = r'$. Since l' is a $\mathcal{D}$-monoid morphism it is easy to see that for every $x : I \to X^{\circledast}$ the following square commutes (note that $(x, x) \in E'$ since E' is reflexive):

$$\begin{tikzcd}
E' \arrow[r, "(x,x)\bullet -"] \arrow[d, "l'"'] & E' \arrow[d, "l'"] \\
X^{\circledast} \arrow[r, "x\bullet -"'] & X^{\circledast}
\end{tikzcd}$$

and similarly for r' in lieu of l' and/or $- \bullet y$ in lieu of $x \bullet -$. It follows that the morphism (4) in the proof of Lemma 3.10 merges l' and r':

$$\begin{aligned}
L \cdot (- \bullet y) \cdot (x \bullet -) \cdot l' &= L \cdot l' \cdot (- \bullet (y, y) \cdot ((x, x) \bullet -) \\
&= L \cdot l_0 \cdot m' \cdot (- \bullet (y, y)) \cdot ((x, x) \bullet -) \\
&= L \cdot r_0 \cdot m' \cdot (- \bullet (y, y)) \cdot ((x, x) \bullet -) \\
&= L \cdot r' \cdot (- \bullet (y, y)) \cdot ((x, x) \bullet -) \\
&= L \cdot (- \bullet y) \cdot (x \bullet -) \cdot r'
\end{aligned}$$

Hence E' is contained in any $E_{x,y}$, and therefore it is contained in their intersection E.

Proof of Theorem 3.14

Suppose that we have an X-generated $\mathcal{D}$-monoid $e : X^{\circledast} \twoheadrightarrow M$ and a morphism $f : M \to Y$ recognizing L, i.e. such that $L = f \cdot e$. Let $l_0, r_0 : K \to X^{\circledast}$ be the kernel pair of L as in Lemma 3.12 and take the kernel pair $l_M, r_M : K_M \to X^{\circledast}$ of e. Now clearly we have

$$L \cdot l_M = f \cdot e \cdot l_M = f \cdot e \cdot r_M = L \cdot r_M.$$

Hence, there is a unique $n : K_M \to K$ such that $l_0 \cdot n = l_M$ and $r_0 \cdot n = r_M$. It follows that n is monomorphic and so K_M is a $\mathcal{D}$-monoid congruence contained in K. Consequently

K_M is contained in E (the largest $\mathcal{D}$-monoid congruence contained K by Lemma 3.12) via $o : K_M \rightarrowtail E$ with $l_M = l \cdot o$ and $r_M = r \cdot o$. Then we obtain that

$$e_L \cdot l_M = e_L \cdot l \cdot o = e_L \cdot r \cdot o = e_L \cdot r_M.$$

Thus, using that $e : X^{\circledast} \to M$ is the coequalizer of its kernel pair l_M, r_M we obtain a unique $h : M \to \mathsf{Syn}(L)$ with $e_L = h \cdot e$, as desired.

Proof of Proposition 4.3

Let (Q, δ, i, f) be a $\mathcal{D}$-automaton. By definition it accepts the language $L_Q = (X^{\circledast} \xrightarrow{e_Q} Q \xrightarrow{f} Y)$ where e_Q is the unique F-algebra morphism. Consider the morphism that evaluates any endomorphism of Q at the initial state:

$$\mathsf{ev}_i = ([Q,Q] \cong [Q,Q] \otimes I \xrightarrow{[Q,Q]\otimes i} [Q,Q] \otimes Q \xrightarrow{\mathsf{ev}} Q).$$

Now let

$$f_{\mathsf{T}(Q)} = (\mathsf{T}(Q) \xrightarrow{m_{\mathsf{T}(Q)}} [Q,Q] \xrightarrow{\mathsf{ev}_i} Q \xrightarrow{f} Y).$$

With this morphism $\mathsf{T}(Q)$ recognizes L; indeed, using the canonical isomorphism $\iota_Z : Z \to Z \otimes I$ we compute:

$$\begin{aligned}
L_Q &= f \cdot e_Q \\
&= f \cdot \mathsf{ev} \cdot ((\lambda\delta)^+ \otimes i) \cdot \iota_{X^{\circledast}} && \text{(see Remark 2.18)} \\
&= f \cdot \mathsf{ev}_i \cdot \iota_{[Q,Q]}^{-1} \cdot ((\lambda\delta)^+ \otimes I) \cdot \iota_{X^{\circledast}} && \text{(def. of } \mathsf{ev}_i) \\
&= f \cdot \mathsf{ev}_i \cdot (\lambda\delta)^+ && \text{(naturality of } \iota) \\
&= f \cdot \mathsf{ev}_i \cdot m_{\mathsf{T}(Q)} \cdot e_{\mathsf{T}(Q)} && \text{(see Definition 4.1)} \\
&= f_{\mathsf{T}(Q)} \cdot e_{\mathsf{T}(Q)} && \text{(def. of } f_{\mathsf{T}(Q)})
\end{aligned}$$

This completes the proof.

Proof of Theorem 4.6

Let $\mathsf{Min}(L) = (Q, \delta, i, f)$, and write $\delta_x : Q \to Q$ for $e_{\mathsf{T}(Q)}(x)$ $(x \in X^{\circledast})$. Note that $\delta_{x \bullet y} = \delta_y \cdot \delta_x$ for all $x, y \in X^{\circledast}$ since $e_{\mathsf{T}(Q)}$ is a $\mathcal{D}$-monoid morphism. Observe also that the unique F-algebra homomorphism $e_Q : X^{\circledast} \to Q$ assigns to $x \in X^{\circledast}$ the element $\delta_x \cdot i : I \to Q$, and the unique T-coalgebra homomorphism $m_Q : Q \to [X^{\circledast}, Y]$ assigns to a state $q : I \to Q$ the language $x \mapsto f \cdot \delta_x \cdot q$. It suffices to show that the kernel of $e_{\mathsf{T}(Q)}$ is the syntactic congruence of L, that is, for all $u, v \in X^{\circledast}$ one has

$$\delta_u = \delta_v \quad \text{iff} \quad \forall x, y \in X^{\circledast} : L(x \bullet u \bullet y) = L(x \bullet v \bullet y).$$

To see this, we reason as follows:

$$\begin{aligned}
\delta_u = \delta_v &\Leftrightarrow \forall x : \delta_u \cdot e_Q(x) = \delta_v \cdot e_Q(x) && (e_Q \text{ surjective}) \\
&\Leftrightarrow \forall x : \delta_u \cdot \delta_x \cdot i = \delta_v \cdot \delta_x \cdot i && (\text{def. } e_Q) \\
&\Leftrightarrow \forall x : m_Q \cdot \delta_u \cdot \delta_x \cdot i = m_Q \cdot \delta_v \cdot \delta_x \cdot i && (m_Q \text{ injective}) \\
&\Leftrightarrow \forall x, y : f \cdot \delta_y \cdot \delta_u \cdot \delta_x \cdot i = f \cdot \delta_y \cdot \delta_v \cdot \delta_x \cdot i && (\text{def. } m_Q) \\
&\Leftrightarrow \forall x, y : f \cdot \delta_{x \bullet u \bullet y} \cdot i = f \cdot \delta_{x \bullet v \bullet y} \cdot i && (\text{def. } \delta_{(-)}) \\
&\Leftrightarrow \forall x, y : f \cdot e_Q(x \bullet u \bullet y) = f \cdot e_Q(x \bullet v \bullet z) && (\text{def. } e_Q) \\
&\Leftrightarrow \forall x, y : L(x \bullet u \bullet y) = L(x \bullet v \bullet y) && (L = L_Q)
\end{aligned}$$

Proof of Theorem 5.4

Remark.

1. The functor $FQ = I + X \otimes Q$ preserves strong epimorphisms because the left adjoint $X \otimes -$ does and strong epimorphisms are closed under coproducts. Therefore every F-algebra homomorphism factorizes into a surjective homomorphism (carried by a strong epimorphism in $\mathcal{D}$) and an injective one (carried by a monomorphism in $\mathcal{D}$). By the *reachable part* Q_r of an automaton (Q, δ, f, i) we mean the image of the initial F-algebra homomorphism, i.e., $e_Q = (X^{\circledast} \xrightarrow{e_r} Q_r \xrightarrow{m_r} Q)$. Putting $f_r := f \cdot m_r : Q_r \to Y$, the F-algebra Q_r becomes an automaton, and m_r an automata homomorphism. ◂
2. In the following automata, (co-)algebras and monoids with finitely presentable carrier are referred to as *fp-automata*, *fp-(co-)algebras* and *fp-monoids*, respectively.

Now for the proof of the theorem. (a)⇔(b) follows from Theorem 4.5 and the closure of $\mathcal{D}_f$ under subobjects and strong quotients. Similarly, $(c) \Leftrightarrow (d)$ follows from the universal property of the syntactic monoid (see Definition 3.5) and again closure of $\mathcal{D}_f$ under subobjects and strong quotients. (c)⇒(a) is a consequence of Lemma 3.4. To prove (a)⇒(c), let Q be any fp-automaton accepting L. Then by Proposition 4.3 the transition monoid $\mathsf{T}(Q) \rightarrowtail [Q, Q]$ recognizes L, so by closure of $\mathcal{D}_f$ under subobjects it suffices to show that $[Q, Q]$ is a finitely presentable object of $\mathcal{D}$. Assuming that Q has n generators as an algebra of $\mathcal{D}$, the map $[Q, Q] \to Q^n$ defined by restriction to the set of generators is an injective $\mathcal{D}$-morphism. Since $\mathcal{D}_f$ is closed under subobjects and finite products, it follows that $[Q, Q]$ is finitely presentable.

Proof of Proposition 5.6

We describe mutually inverse maps

$$(I \xrightarrow{x} \varrho T) \mapsto (X^{\circledast} \xrightarrow{L_x} Y) \qquad \text{and} \qquad (X^{\circledast} \xrightarrow{L} Y) \mapsto (I \xrightarrow{x_L} \varrho T)$$

between the elements of ϱT and the $\mathcal{D}$-regular languages. Let $h_Q : Q \to \varrho T$ be the injections of the colimit ϱT, where $Q = (Q, \delta_Q, f_Q)$ ranges over all fp-coalgebras. Note that this colimit is filtered since $\mathcal{D}_f$ is closed under finite colimits. Moreover, since colimits of coalgebras are formed in the underlying category, the morphisms h_Q also form a colimit cocone in $\mathcal{D}$. Given an element $I \xrightarrow{x} \varrho T$ of the rational coalgebra we define a $\mathcal{D}$-regular language $L_x : X^{\circledast} \to Y$ as follows: since $I = \Psi 1$ is finitely presentable, there exists an fp-coalgebra Q and a morphism $i_Q : I \to Q$ such that $x = h_Q \cdot i_Q$. For the F-algebra (Q, δ_Q, i_Q) we have the unique F-algebra homomorphism $e_Q : X^{\circledast} \to Q$, and we put $L_x := f_Q \cdot e_Q$.

We need to show that L_x is well-defined, that is, for any other factorization $x = h_{Q'} \cdot i_{Q'}$ we have $f_Q \cdot e_Q = f_{Q'} \cdot e_{Q'}$. Given such a factorization, since the h_Q form a filtered colimit, there exists an fp-coalgebra $Q'' = (Q'', \delta_{Q''}, f_{Q''})$ and coalgebra homomorphisms $h_{QQ'} : Q \to Q'$ and $h_{Q'Q''} : Q' \to Q''$ with $h_{QQ'} \cdot i_Q = h_{Q'Q''} \cdot i_{Q'} =: i_{Q''}$. Then for the F-algebra $(Q'', \delta_{Q''}, i_{Q''})$ we have the unique homomorphism $e_{Q''} : X^{\circledast} \to Q''$. Moreover, $h_{QQ'}$ and $h_{Q'Q''}$ are also homomorphisms of F-algebras. If follows that

$$\begin{aligned} f_Q \cdot e_Q &= f_{Q''} \cdot h_{QQ''} \cdot e_Q && (h_{QQ''} \text{ coalgebra homomorphism}) \\ &= f_{Q''} \cdot e_{Q''} && (h_{QQ''}\ F\text{-algebra hom., } X^{\circledast} \text{ initial}) \end{aligned}$$

and analogously $f_{Q'} \cdot e_{Q'} = f_{Q''} \cdot e_{Q''}$. Hence $f_Q \cdot e_Q = f_{Q'} \cdot e_{Q'}$, as claimed.

Conversely, let $L : X^{\circledast} \to Y$ be a $\mathcal{D}$-regular language. Then there exists an fp-automaton (Q, δ_Q, i_Q, f_Q) with $L = f_Q \cdot e_Q$. Put $x_L := h_Q \cdot i_Q : I \to \varrho T$. To prove the well-definedness of x_L, consider the automata homomorphisms

$$\begin{tikzcd} Q & Q_r \arrow[l, tail, "m"'] \arrow[r, dashed, two heads, "e"] & \mathsf{Min}(L) \end{tikzcd}$$

of Theorem 4.5. Then

$$\begin{aligned} h_Q \cdot i_Q &= h_Q \cdot m \cdot i_{Q_r} && (m \text{ algebra hom.}) \\ &= h_{Q_r} \cdot i_{Q_r} && ((h_Q) \text{ cocone and } m \text{ coalgebra hom.}) \\ &= h_{\mathsf{Min}(L)} \cdot e \cdot i_{Q_r} && ((h_Q) \text{ cocone and } e \text{ coalgebra hom.}) \\ &= h_{\mathsf{Min}(L)} \cdot i_{\mathsf{Min}(L)} && (e \text{ algebra hom.}) \end{aligned}$$

Hence $x_L = h_Q \cdot i_Q$ is independent of the choice of Q. It now follows immediately from the definitions that $x \mapsto L_x$ and $L \mapsto x_L$ are mutually inverse and hence define the desired bijective correspondence.

B Dual Characterization of Syntactic Monoids

Here we give a more detailed account of the dual view of syntactic monoids indicated in Section 5. This section are largely based on results from our papers [2, 4] where a categorical generalization of Eilenberg's variety theorem was proved. We work with the following

▸ **Assumptions B.1.** From now on $\mathcal{D}$ is a locally finite entropic variety whose epimorphisms are surjective. Moreover, we assume that there is another locally finite variety $\mathcal{C}$ such that the full subcategories $\mathcal{C}_f$ and $\mathcal{D}_f$ of finite algebras are dually equivalent. (Two such varieties $\mathcal{C}$ and $\mathcal{D}$ are called *predual*.)

The action of the equivalence functor $\mathcal{C}_f^{op} \xrightarrow{\simeq} \mathcal{D}_f$ on objects and morphisms is written $Q \mapsto \widehat{Q}$ and $f \mapsto \widehat{f}$. Letting $\overline{I} \in \mathcal{C}_f$ denote the free one-generated object of $\mathcal{C}$ we choose the output object $Y \in \mathcal{D}_f$ to be the dual object of $\overline{I}$. Moreover, let $\overline{Y} \in \mathcal{C}_f$ be the dual object of $I \in \mathcal{D}_f$, the free one-generated object of $\mathcal{D}$. Finally, we put $X = \Psi X_0$ for a finite alphabet X_0. Note that the underlying sets of $\overline{Y}$ and Y are isomorphic:

$$|\overline{Y}| \cong \mathcal{C}(\overline{I}, \overline{Y}) \cong \mathcal{D}(I, Y) \cong |Y|.$$

To simplify the presentation, we will assume in the following that $\overline{Y}$ and Y have a two-element underlying set $\{0, 1\}$. This is, however, inessential – see Remark B.14 at the end of this section.

▸ **Example B.2.** The categories $\mathcal{C}$ and $\mathcal{D}$ in the table below satisfy our assumptions.

$\mathcal{C}$	$\mathcal{D}$
$\mathbf{BA}$	$\mathbf{Set}$
$\mathbf{BR}$	$\mathbf{Set}_\perp$
$\mathbf{JSL}_0$	$\mathbf{JSL}_0$
$\mathbf{Vec}(\mathbb{Z}_2)$	$\mathbf{Vec}(\mathbb{Z}_2)$

The case $\mathbf{BA}/\mathbf{Set}$ is a restriction of Stone duality: the dual equivalence functor $\mathbf{BA}_f^{op} \xrightarrow{\simeq} \mathbf{Set}_f$ assigns to a finite boolean algebra B the set $\mathsf{At}(B)$ of its atoms, and to a homomorphism $h : A \to B$ the map $\mathsf{At}(h) : \mathsf{At}(B) \to \mathsf{At}(A)$ sending $b \in \mathsf{At}(B)$ to the unique atom $a \in \mathsf{At}(A)$

with $ha \geqslant b$. Using a similar Stone-type duality, we proved in [4] that the the category $\mathbf{BR}$ of non-unital boolean rings (i.e., rings without 1 satisfying the equation $x \cdot x = x$) is predual to $\mathbf{Set}_\perp$. The other two examples above correspond to the well-known self-duality of finite semilattices and finite-dimensional vector spaces, respectively. We refer to [4] for details.

On $\mathcal{C}$ we consider the endofunctor $\overline{T}Q = \overline{Y} \times Q^{X_0}$. Its coalgebras are precisely deterministic automata in $\mathcal{C}$ without an initial state, represented as triples (Q, γ_a, f) with transition morphisms $\gamma_a : Q \to Q$ $(a \in X_0)$ and an output morphism $f : Q \to \overline{Y}$.

▸ **Example B.3.** In $\mathcal{C} = \mathbf{BA}$ a $\overline{T}$-coalgebra is a deterministic automaton with a boolean algebra Q of states, boolean transitions morphisms γ_a, and an output map $f : Q \to \{0,1\}$ which specifies (via the preimage of 1) an ultrafilter $F \subseteq Q$ of final states.

The rational coalgebra $\varrho\overline{T}$ of $\overline{T}$ (i.e., the colimit of all finite $\overline{T}$-coalgebras) has as states the regular languages over X_0. The final state predicate $f : \varrho\overline{T} \to \overline{Y} = \{0,1\}$ sends a language to 1 iff it contains the empty word ε, and the transitions $\gamma_a : \varrho\overline{T} \to \varrho\overline{T}$ for $a \in X_0$ are given by $\gamma_a(L) = a^{-1}L$. Here $a^{-1}L = \{w \in X_0^* : aw \in L\}$ denotes the *left derivative* of L w.r.t. the letter a. Similarly, the *right derivatives* of L are defined by $La^{-1} = \{w \in X_0^* : wa \in L\}$ for $a \in X_0$.

▸ **Example B.4.** In $\mathcal{C} = \mathbf{BA}$ the rational $\overline{T}$-coalgebra is the boolean algebra of all regular languages over X_0 (w.r.t. union, intersection and complement), equipped with the above transitions and final states. Note that the transition map $a^{-1}(-)$ is indeed a boolean homomorphism because left derivatives preserve all boolean operations. Moreover, the final states – viz. the set of all regular languages containing the empty word – form a (principal) ultrafilter.

The coalgebra $\varrho\overline{T}$ is characterized by a universal property: every finite $\overline{T}$-coalgebra has a unique coalgebra homomorphism into it (which sends a state to the language it accepts in the classical sense of automata theory). A finite $\overline{T}$-coalgebra is called a *subcoalgebra* of $\varrho\overline{T}$ if this unique morphism is injective, i.e., distinct states accept distinct languages. In [2] we related finite $\overline{T}$-coalgebras in $\mathcal{C}$ to finite F-algebras in the predual category $\mathcal{D}$. Note that $X = \Psi X_0$ implies $FA = I + X \otimes A \cong I + \coprod_{X_0} A$, so F-algebras F-algebras (A, δ) can be represented as triples (A, δ_a, i) with $\delta_a : A \to A$ $(a \in X_0)$ and $i : I \to A$. They correspond to automata in $\mathcal{D}$ with inputs from the alphabet X_0 and without final states.

▸ **Proposition B.5** (see [2]). *(a) The categories of finite $\overline{T}$-coalgebras and finite F-algebras are dually equivalent. The equivalence maps any finite $\overline{T}$-coalgebra $Q = (Q, \gamma_a, f)$ to its* dual F-algebra $\widehat{Q} = (\widehat{Q}, \widehat{\gamma_a}, \widehat{f})$*:*

$$(\overline{Y} \xleftarrow{f} Q \xrightarrow{\gamma_a} Q) \quad \mapsto \quad (I \xrightarrow{\widehat{f}} \widehat{Q} \xleftarrow{\widehat{\gamma_a}} \widehat{Q}).$$

(b) A finite $\overline{T}$-coalgebra Q is a subcoalgebra of $\varrho\overline{T}$ iff its dual F-algebra $\widehat{Q}$ is a quotient of the initial F-algebra $X^{\circledast}$.

▸ **Example B.6.** For a finite $\overline{T}$-coalgebra (Q, γ_a, f) in $\mathbf{BA}$ the dual F-algebra $\widehat{Q}$ has as states the atoms of Q, and the initial state is the unique atomic final state of Q. Moreover, there is a transition $z \xrightarrow{a} z'$ for $a \in X_0$ in $\widehat{Q}$ iff z' is the unique atom with $\gamma_a(z') \geqslant z$ in Q.

By a *local variety of languages over X_0 in* $\mathcal{C}$ we mean a subcoalgebra V of $\varrho\overline{T}$ closed under right derivatives (i.e. $L \in |V|$ implies $La^{-1} \in |V|$ for all $a \in X_0$). Note that a local variety is also closed under the $\mathcal{C}$-algebraic operations of $\varrho\overline{T}$, being a sub*algebra* of $\varrho\overline{T}$ in $\mathcal{C}$, and under left derivatives, being a sub*coalgebra* of $\varrho\overline{T}$.

▸ **Example B.7.** A local variety of languages in $\mathbf{BA}$ is a set of regular languages over X_0 closed under the boolean operations (union, intersection and complement) as well as left and right derivatives. This concept was introduced by Gehrke, Grigorieff and Pin [11].

In the following proposition recall that every X_0-generated $\mathcal{D}$-monoid defines an F-algebra, see Definition 2.15.

▸ **Proposition B.8** (see [2]). *A finite subcoalgebra V of $\varrho\overline{T}$ is a local variety iff its dual F-algebra $\widehat{V}$ is derived from an X_0-generated $\mathcal{D}$-monoid.*

In other words, given a finite local variety $V \rightarrowtail \varrho\overline{T}$ in $\mathcal{C}$, there exists a unique $\mathcal{D}$-monoid structure on $\widehat{V}$ making the unique (surjective) F-algebra homomorphism $e_{\widehat{V}} : X^{\circledast} \to \widehat{V}$ is $\mathcal{D}$-monoid morphism. Hence the monoid multiplication on $\widehat{V}$ is (well-)defined by $e_{\widehat{V}}(x) \bullet e_{\widehat{V}}(y) := e_{\widehat{V}}(x \bullet y)$ for all $x, y \in X^{\circledast}$, and the unit it the initial state of the F-algebra $\widehat{V}$.

▸ **Remark B.9.** A *pointed* $\overline{T}$-coalgebra is a $\overline{T}$-coalgebra (Q, γ_a, f) equipped with an initial state $i : \overline{I} \to Q$. Observe that every finite pointed $\overline{T}$-coalgebra (Q, γ_a, f, i) dualizes to a finite $\mathcal{D}$-automaton $(\widehat{Q}, \widehat{\gamma}_a, \widehat{f}, \widehat{i})$. The *language* of (Q, δ_a, f, i) is the function

$$L_Q : X_0^* \to |\overline{Y}|, \quad a_1 \dots a_n \mapsto f \cdot \delta_{a_n} \cdot \dots \cdot \delta_{a_1} \cdot i.$$

Letting $m_Q : Q \to \varrho\overline{T}$ denote the unique coalgebra homomorphism, L_Q is precisely the element of $\varrho\overline{T}$ determined by $I \xrightarrow{i} Q \xrightarrow{m_Q} \varrho\overline{T}$. Since $|\overline{Y}| = |Y|$ and $X^{\circledast} = \Psi X_0^*$, the function $L_Q : X_0^* \to |\overline{Y}|$ can be identified with its adjoint transpose $L_Q^@ : X^{\circledast} \to Y$, i.e., with a language in $\mathcal{D}$. The *reversal* of a language $L : X^{\circledast} \to Y$ in $\mathcal{D}$ is $L^{\mathsf{rev}} = L \cdot \mathsf{rev} : X^{\circledast} \to Y$, where $\mathsf{rev} : X^{\circledast} \to X^{\circledast}$ denotes the unique morphism of $\mathcal{D}$ extending the function $X_0^* \to X_0^*$ that reverses words.

▸ **Proposition B.10** (see [4]). *The language accepted by a finite pointed $\overline{T}$-coalgebra is the reversal of the language accepted by its dual $\mathcal{D}$-automaton .*

If a finite X_0-generated $\mathcal{D}$-monoid $e : X^{\circledast} \to M$ recognizes a language $L : X^{\circledast} \to Y$ via $f : M \to Y$, i.e., $L = (\, X^{\circledast} \overset{e}{\twoheadrightarrow} M \xrightarrow{f} Y \,)$, we dually get the morphism $\overline{I} \xrightarrow{i} V \overset{m}{\rightarrowtail} \varrho\overline{T}$ (where V is the local variety dual to M, i is the dual morphism of f and m is the unique coalgebra homomorphism) choosing the element L^{rev} of $\varrho\overline{T}$. Now suppose that L is a regular language, and let V_L be the finite local variety of languages dual to the syntactic $\mathcal{D}$ monoid $\mathsf{Syn}(L)$, see Proposition B.8. The universal property of $\mathsf{Syn}(L)$ in Definition 3.5 then dualizes as follows: V_L is (i) a local variety containing L^{rev}, and (ii) for every local variety $V \rightarrowtail \varrho\overline{T}$ containing L^{rev}, the local variety V_L is contained in V. In other words, V_L is the *smallest* local variety containing L^{rev}.

$$\begin{tikzcd}
\varrho\overline{T} & V \arrow[l, tail] & \overline{I} \arrow[l] \arrow[dl] \\
& V_L \arrow[ul, tail] \arrow[u, dashed, tail] &
\end{tikzcd}$$

In summary, we have proved following dual characterization of syntactic $\mathcal{D}$-monoids:

▸ **Theorem B.11.** *For every regular language L the syntactic $\mathcal{D}$-monoid $\mathsf{Syn}(L)$ is dual to the smallest local variety of languages over X_0 in $\mathcal{C}$ containing L^{rev}.*

▸ **Example B.12.** For $\mathcal{C} = \mathbf{BA}$ and $\mathcal{D} = \mathbf{Set}$ the previous theorem gives the following construction of the syntactic monoid of a regular language $L \subseteq X^*$:

1. Form the smallest local variety of languages $V_L \subseteq \varrho\overline{T}$ containing L^{rev}. Hence V_L is the closure of the (finite) set of all both-sided derivatives $u^{-1}L^{\mathsf{rev}}v^{-1} = \{w \in X^* : uwv \in L^{\mathsf{rev}}\}$ $(u, v \in X^*)$ under union, intersection and complement.
2. Compute the F-algebra $\widehat{V_L}$ dual to the coalgebra V_L. The states of $\widehat{V_L}$ are the atoms of V_L, and the initial state is the unique atom $i \in V_L$ containing the empty word. Given atoms $z, z' \in V_L$ and $a \in X$, there is a transition $z \xrightarrow{a} z'$ in $\widehat{V_L}$ iff z' is the (unique) atom with $z \subseteq a^{-1}z'$.
3. Define a monoid multiplication on $\widehat{V_L}$ as follows: given states $z, z' \in \widehat{V_L}$, choose words $w, w' \in X^*$ with $i \xrightarrow{w} z$ and $i \xrightarrow{w'} z'$ in $\widehat{V_L}$. Then $z \bullet z'$ is the state reached on input ww', i.e., $i \xrightarrow{ww'} z \bullet z'$. The resulting monoid (with multiplication $\bullet$ and unit i) is $\mathsf{Syn}(L)$.

By dropping right derivatives and using the correspondence between finite subcoalgebras of $\varrho\overline{T}$ and finite quotient algebras of $X^{\circledast}$, one also gets the following dual characterization of minimal $\mathcal{D}$-automata:

▸ **Theorem B.13.** *For every regular language L the minimal $\mathcal{D}$-automaton for L is dual to the smallest subcoalgebra of $\varrho\overline{T}$ containing L^{rev}*

▸ **Remark B.14.** Our above assumption that Y and $\overline{Y}$ have two elements is inessential. Without this assumption, the rational coalgebra $\varrho\overline{T}$ is not carried by regular languages, but more generally by *regular behaviors*, i.e, functions $b : X_0^* \to |Y|$ realized by finite Moore automata with output set $|Y|$. The coalgebra structure is given by the output map $b \mapsto b(\varepsilon)$, and transitions $b \xrightarrow{a} a^{-1}b$ for $a \in X_0$, where $a^{-1}b$ is the (generalized) left derivative of b defined by $a^{-1}b(w) = b(aw)$. (Right derivatives are defined analogously.) A *local variety of behaviors over X_0 in $\mathcal{C}$* is a subcoalgebra of $\varrho\overline{T}$ closed under right derivatives. All results of this section hold for this more general setting, see Section 5 of [4] for details. In particular, this allows us to cover the case $\mathcal{C} = \mathcal{D} = \mathbf{Vec}(\mathbb{K})$ for an arbitrary finite field $\mathbb{K}$. In this case Theorem 5.7 states that the syntactic associative algebra of a rational weighted language $L : X_0^* \to \mathbb{K}$ dualizes to the smallest set of rational weighted languages that contains L^{rev} and is closed under scalar multiplication, addition and left and right derivatives.