12.4 USE OF ZERNIKE CIRCLE POLYNOMIALS FOR THE ANALYSIS OF A HEXAGONAL WAVEFRONT

12.4.1 Zernike Circle Coefficients in Terms of Hexagonal Coefficients

Now, we consider a hexagonal aberration function $W(x, y)$ across a unit hexagon shown in Figure 7-7, and demonstrate the pitfalls of using Zernike circle polynomials for its expansion. Estimating the aberration function with J hexagonal polynomials $H_j(x, y)$ given in Chapter 7, we may write

$$\hat{W}(x, y) = \sum_{j=1}^{J} a_j H_j(x, y) \quad , \tag{12-56}$$

where the orthonormal hexagonal expansion coefficients are given by

$$a_j = \frac{2}{3\sqrt{3}} \int_{\text{hexagon}} W(x, y) H_j \, dx \, dy \quad . \tag{12-57}$$

The mean and the mean values of the estimated aberration function are given by Eqs. (12-4) and (12-6).

An 11×11 conversion matrix $\boldsymbol{M}$ for obtaining the hexagonal polynomials in terms of the Zernike circle polynomials is given in Table 12-6, as obtained from Table 7-1. Its transpose and inverse matrices are given in Tables 12-7 and 12-8, respectively. If only the first 4 polynomials are used in the expansion, then the $\hat{b}_j$ coefficients according to Eq. (12-13) are given by

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \\ \hat{b}_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \sqrt{5/43} \\ 0 & \sqrt{6/5} & 0 & 0 \\ 0 & 0 & \sqrt{6/5} & 0 \\ 0 & 0 & 0 & 2\sqrt{15/43} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_1 + \sqrt{5/43}a_4 \\ \sqrt{6/5}a_2 \\ \sqrt{6/5}a_3 \\ 2\sqrt{15/43}a_4 \end{pmatrix} \quad , \tag{12-58}$$

or

$$\hat{b}_1 = a_1 + \sqrt{5/43}a_4 \quad , \tag{12-59a}$$

$$\hat{b}_2 = \sqrt{6/5}a_2 \quad , \tag{12-59b}$$

$$\hat{b}_3 = \sqrt{6/5}a_3 \quad , \tag{12-59c}$$

and

$$\hat{b}_4 = 2\sqrt{15/43}a_4 \quad . \tag{12-59d}$$

It is evident that the piston coefficient $\hat{b}_1$ is not equal to a_1 and, therefore, does not

Table 12-6. Conversion matrix M for obtaining the Zernike coefficients $\hat{b}_j$ from the orthonormal hexagonal coefficients a_j, as in Eq. (12-12).

1	0	0	0	0	0	0	0	0	0	0
0	$\sqrt{6/5}$	0	0	0	0	0	0	0	0	0
0	0	$\sqrt{6/5}$	0	0	0	0	0	0	0	0
$\sqrt{5/43}$	0	0	$2\sqrt{15/43}$	0	0	0	0	0	0	0
0	0	0	0	$\sqrt{10/7}$	0	0	0	0	0	0
0	0	0	0	0	$\sqrt{10/7}$	0	0	0	0	0
0	0	$16\sqrt{\frac{14}{11055}}$	0	0	0	$10\sqrt{\frac{35}{2211}}$	0	0	0	0
0	$16\sqrt{\frac{14}{11055}}$	0	0	0	0	0	$10\sqrt{\frac{35}{2211}}$	0	0	0
0	0	0	0	0	0	0	0	$\frac{2}{3}\sqrt{5}$	0	0
0	0	0	0	0	0	0	0	0	$2\sqrt{\frac{35}{103}}$	0
$\frac{521}{\sqrt{1072205}}$	0	0	$88\sqrt{\frac{15}{214441}}$	0	0	0	0	0	0	$14\sqrt{\frac{43}{4987}}$

Table 12-7. Transpose matrix M^T for use in Eq. (12-13).

1	0	0	$\sqrt{5/43}$	0	0	0	0	0	0	$\frac{521}{\sqrt{1072205}}$
0	$\sqrt{6/5}$	0	0	0	0	0	$16\sqrt{\frac{14}{11055}}$	0	0	0
0	0	$\sqrt{6/5}$	0	0	0	$16\sqrt{\frac{14}{11055}}$	0	0	0	0
0	0	$2\sqrt{15/43}$	0	0	0	0	0	0	$88\sqrt{\frac{15}{214441}}$	0
0	0	0	0	$\sqrt{10/7}$	0	0	0	0	0	0
0	0	0	0	0	$\sqrt{10/7}$	0	0	0	0	0
0	0	0	0	0	0	$10\sqrt{\frac{35}{2211}}$	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	$\frac{2}{3}\sqrt{5}$	0	0
0	0	0	0	0	0	0	0	0	$2\sqrt{\frac{35}{103}}$	0
0	0	0	0	0	0	0	0	0	0	$14\sqrt{\frac{43}{4987}}$

Table 12-8. Analytical matrix M^{-1} for obtaining the Zernike coefficients a_j from the orthonormal hexagonal coefficients $\hat{b}_j$.

1	0	0	0	0	0	0	0	0	0	0
0	$\sqrt{5/6}$	0	0	0	0	0	0	0	0	0
0	0	$\sqrt{5/6}$	0	0	0	0	0	0	0	0
$-1/2\sqrt{3}$	0	0	$\sqrt{43/15}/2$	0	0	0	0	0	0	0
0	0	0	0	$\sqrt{7/10}$	0	0	0	0	0	0
0	0	0	0	0	$\sqrt{7/10}$	0	0	0	0	0
0	0	$-8/5\sqrt{15}$	0	0	0	$\sqrt{\frac{2211}{35}}/10$	0	0	0	0
0	$-8/5\sqrt{15}$	0	0	0	0	0	$\sqrt{\frac{2211}{35}}/10$	0	0	0
0	0	0	0	0	0	0	0	$3/2\sqrt{5}$	0	0
0	0	0	0	0	0	0	0	0	$\sqrt{\frac{103}{35}}/2$	0
$-1/2\sqrt{5}$	0	0	$-22/7\sqrt{43}$	0	0	0	0	0	0	$\sqrt{\frac{4987}{43}}/14$

represent the mean value of the aberration function. The coefficients $\hat{b}_2$, $\hat{b}_3$, and $\hat{b}_4$ represent the tip, tilt, and defocus circle coefficients.

To see how these coefficients change with the number of polynomials used in the expansion, we consider an expansion using 11 polynomials. The coefficients, obtained from Eq. (12-13), are given by

$$\hat{b}_1 = a_1 + \sqrt{5/43}a_4 + \left(521/\sqrt{1072205}\right)a_{11} \quad , \tag{12-60a}$$

$$\hat{b}_2 = \sqrt{6/5}a_2 + 16\sqrt{14/11055}a_8 \quad , \tag{12-60b}$$

$$\hat{b}_3 = \sqrt{6/5}a_3 + 16\sqrt{14/11055}a_7 \quad , \tag{12-60c}$$

$$\hat{b}_4 = 2\sqrt{15/43}a_4 + 88\sqrt{15/214441}a_{11} \quad , \tag{12-60d}$$

$$\hat{b}_5 = \sqrt{10/7}a_5 \quad , \tag{12-60e}$$

$$\hat{b}_6 = \sqrt{10/7}a_6 \quad , \tag{12-60f}$$

$$\hat{b}_7 = 10\sqrt{35/2211}a_7 \quad , \tag{12-60g}$$

$$\hat{b}_8 = 10\sqrt{35/2211}a_8 \quad , \tag{12-60h}$$

$$\hat{b}_9 = (2/3)\sqrt{5}a_9 \quad , \tag{12-60i}$$

$$\hat{b}_{10} = 2\sqrt{35/103}a_{10} \quad , \tag{12-60j}$$

and

$$\hat{b}_{11} = 14\sqrt{43/4987}a_{11} \quad . \tag{12-60k}$$

It is clear that all of the first four coefficients change, and $\hat{b}_j = M_{jj}a_j$ for $5 \le j \le 11$. For astigmatism (H_5 and H_6), coma (H_7 and H_8), and spherical aberration (H_{11}), the $\hat{b}_j$ coefficient is larger than the corresponding hexagonal coefficient by a factor of $\sqrt{10/7} \approx 1.20, 10\sqrt{35/2211} \approx 1.26$, and $14\sqrt{43/4987} \approx 1.30$, respectively. The astigmatism coefficients $\hat{b}_5$ and $\hat{b}_6$ change if a 15-polynomial expansion is considered. For example, $\hat{b}_5$ then contains contributions from a_{13} and a_{15}, as well. The tip and tilt coefficients $\hat{b}_2$ and $\hat{b}_3$ change further if polynomials H_{16} and H_{17} are included in the expansion. Moreover, H_{16} also contributes to the coma coefficient $\hat{b}_8$, and H_{17} similarly contributes to the coma coefficient $\hat{b}_7$. The piston and defocus coefficients $\hat{b}_1$ and $\hat{b}_4$ do not change until the secondary spherical aberration polynomial H_{22} is included with its coefficient a_{22}. Its inclusion also affects the primary spherical aberration coefficient $\hat{b}_{11}$. Thus, it is easy to see which, when, and by how much the $\hat{b}_j$ coefficients change, depending on the number of polynomials used in the expansion.

12.4.2 Interferometer Setting Errors

The estimated wavefront obtained by using only the first four polynomials represents the best-fit parabolic approximation of the aberration function in a least-squares sense. In terms of the Zernike polynomials, it can be written as

$$\hat{W}(x, y) = \hat{b}_1 Z_1 + \hat{b}_2 Z_2 + \hat{b}_3 Z_3 + \hat{b}_4 Z_4 \tag{12-61a}$$

$$= \hat{b}_1 + 2\hat{b}_2 x + 2\hat{b}_3 y + \sqrt{3}\hat{b}_4 \left(2\rho^2 - 1\right) \quad . \tag{12-61b}$$

Similarly, it can be written in terms of the orthonormal hexagonal polynomials as

$$\hat{W}(x, y) = a_1 H_1 + a_2 H_2 + a_3 H_3 + a_4 H_4 \tag{12-62a}$$

$$= a_1 + 2\sqrt{6/5}a_2 x + 2\sqrt{6/5}a_3 y + a_4\left[\sqrt{5/43} + 6\sqrt{5/43}\left(2\rho^2 - 1\right)\right] \quad . \tag{12-62b}$$

Comparing the right-hand sides of Eqs. (12-61b) and (12-62b) and utilizing Eqs. (12-59a–d), it is seen that the coefficients of x, y, and $x^2 + y^2$, representing the tip, tilt, and defocus values obtained from the Zernike coefficients, are the same as those obtained from the hexagonal coefficients. The estimated piston from the Zernike expansion of Eq.

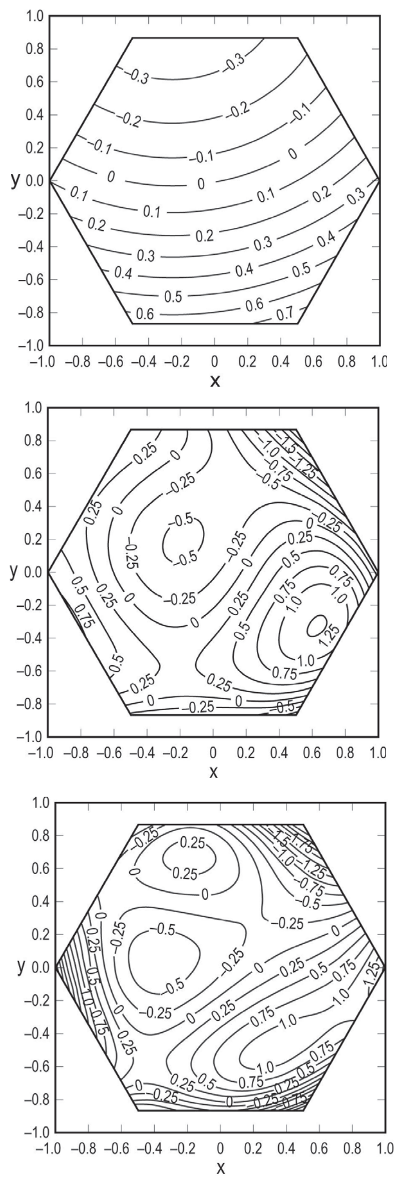
(12-61b) is $\hat{b}_1 - \sqrt{3}\hat{b}_4$. Substituting for $b_1$and b_4 from Eqs. (12-59a–d), we find that it is the same as $a_1 - 5\sqrt{5/43}a_4$ from the hexagonal expansion of Eq. (12-62b). Accordingly, the aberration function obtained by subtracting the piston, tip, tilt, and defocus values from the measured aberration function is independent of the nature of the polynomials used in the expansion, regardless of the domain of the function or the shape of the pupil, so long as the nonorthogonal expansion is in terms of only the first four circle polynomials. The difference function is what is provided to the optician to zero out from the surface under fabrication by polishing. In an interferometer, they represent the lateral and longitudinal errors in the location of a point source illuminating an optical surface under test from its center of curvature. These four terms are generally removed from the aberration function and the remaining function is given to the optician to zero out from the optical surface by polishing.

12.4.3 Numerical Example

As a numerical example, we consider a hexagonal aberration function defined by 15 hexagonal coefficients a_j given in Table 12-9. The mean value of the aberration function is given by $a_1 = 0.0842$. The first 4, the first 11, or all of the 15 coefficients represent the coefficients of a 4-, 11-, or 15-polynomial expansion. The corresponding circle coefficients $\hat{b}_j$ obtained from Eqs. (12-59), (12-60), or in general Eq. (12-13) are also given in Table 12-9. We note that the value of the piston coefficient $\hat{b}_1$ changes as the number of polynomials increases from 4 to 11. Neither equals a_1; and, therefore, they do not represent the mean value. Similarly, the tip, tilt, and defocus coefficients $\hat{b}_2$, $\hat{b}_3$, and $\hat{b}_4$ change. When the number of polynomials increases from 11 to 15, only the astigmatism coefficients $\hat{b}_5$ and $\hat{b}_6$ change, as expected from our discussion in Section 12.4.1. The other coefficients would have changed if higher-order terms were present. The Zernike coefficients b_j obtained from Eq. (17) are also listed in Table 12-9. Their values do not change as the number of polynomials used in the expansion changes. They are different from the corresponding Zernike coefficients $\hat{b}_j$ obtained from Eq. (12-13).

The standard deviation of an aberration function is given by Eq. (12-6) in terms of the hexagonal coefficients. As the number of hexagonal polynomials increases from 4 to 11 to 15, the standard deviation approaches its true value of 0.6068, as indicated in Table 12-10. If Eq. (12-6) is applied to the Zernike coefficients $\hat{b}_j$ or b_j, incorrect values of sigma are obtained. They are also listed in Table 12-10. Once again, whereas a hexagonal coefficient (other than piston) represents the standard deviation of the corresponding polynomial term in the expansion, a Zernike coefficient $\hat{b}_j$ or b_j does not.

The contour plots of the aberration function fitted with 4, 11, and 15 hexagonal polynomials are shown in Figure 12-9. The same plots are obtained with the corresponding properly calculated Zernike coefficients $\hat{b}_j$, illustrating an identical fit. However, different plots are obtained with the improperly calculated Zernike coefficients b_j, as shown in Figure 12-10. If we remove the first four a_j, $\hat{b}_j$, or $\hat{b}_j$ coefficients of piston, tip, tilt, and defocus representing the interferometer setting errors from the aberration function estimated by 11 or 15 polynomials, we obtain the residual aberration function whose contour plots are shown in Figures 12-11, 12-12, and 12-13, respectively. Comparing these figures, it is evident that the residual functions represented in Figures 12-12 and 12-13 are incorrect. Only Figure 12-11 represents the correct residual function. The difference of the residual aberration functions representing the error functions in using the Zernike polynomials and thereby removing the incorrect interferometer setting errors are shown in Figures 12-14 and 12-15. Thus, the contours in these figures represent the difference of the contours in Figures 12-12 and 12-13 from those in Figure 12-11, respectively.



(a)

(b)

(c)

Figure 12-9. Contour plots of a hexagonal aberration function fit with (a) 4, (b) 11, and (c) 15 hexagonal polynomials or circle polynomials with coefficients $\hat{b}_j$.