Universal Direct Analytic Models for the Minimum of Triangular Fuzzy Numbers

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Abstract. This paper reveals the analytic models for the results of fuzzy arithmetic operations, in particular, minimum of fuzzy sets. Special attention is paid to the synthesis of the universal direct models for minimum of triangular fuzzy numbers with different relations between their parameters. Furthermore, we present the components of the universal library of the resulting direct models for various combinations of the triangular fuzzy numbers masks. Modeling results confirm the efficiency of the proposed soft computing models for real-time fuzzy information processing.

Keywords: fuzzy number, minimum, direct model, library, real time.

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1 Introduction

The development of the efficient methods for big data analysis and dynamic information processing in the real-time is one of the most important tasks of the signal processing, as well as control and decision making in uncertainty [1,2,3]. The big volume of data and high speed of its appearance requires using special mathematical approaches developed in the theory of artificial intelligence and computational optimization [4]. In some cases, the complexity of mathematical formalization of processes and systems in the conditions of uncertainty, it is necessary to advance and develop new mathematical methods and approaches [5]. One of these approaches, flexible to solving real-world problem, is a theory of fuzzy sets and fuzzy logic, initially developed and published by professor Lotfi Zadeh [6] in 1965. Since the introduction of the theory of fuzzy sets, there has been significant attention, in particular, in terms of its practical applications of mathematical methods in all fields of science and technology. The scientists around the world are aware of the fundamental theoretical developments in the theory of fuzzy sets and fuzzy logic [7-10].

The fuzzy set $\underset{\sim}{A}$ that is specified on the basis of the universal set E, is called [6] the set of pairs $(x, \mu_{\underset{\sim}{A}}(x))$, where $x \in E$, $\mu_{\underset{\sim}{A}}(x) \in [0,1]$. Fuzzy sets and fuzzy logic allow solving different tasks in uncertain conditions in the field of decision-making and complex systems control in economics, management, engineering and logistics

[3], in particular, in marine transportation [11], investment [5], finances [12] and other fields. Special attention is paid to data analysis using fuzzy mathematics and soft computing [4,13,14].

In many cases, developing the solution to the problems require fulfilling diverse fuzzy arithmetic operations, such as addition, subtraction, multiplication division, minimum and maximum calculations [14,15,16].

2 Related Works and Problem Statement

Inverse models of resulting membership functions (MFs) for different arithmetic operations with fuzzy numbers based on using α-cuts do not always provide high performance of computing operations and often lead to complications in solving control problems in real time [2,7,17,18]. Thus, the development of universal direct analytic models, that allow formalizing fuzzy arithmetic operations to improve their operating speed and accuracy, is an important direction in the fuzzy information processing and data analysis [19,20]. Scholarly attention to the fuzzy set method in the past decade resulted in publications analyzing the synthesis of inverse and direct analytic models for resulting fuzzy sets of such fuzzy arithmetic operations as fuzzy addition $(+)$ [13,14,19,20], fuzzy subtraction $(-)$ [13,14,21], fuzzy multiplication $(\cdot)$ [13,14,15,16] and fuzzy division $(:)$ [13,14].

The analytic approach, proposed in [13,15,19], allows to form the universal resulting MFs for fuzzy arithmetic operations with triangular and bell-shape fuzzy numbers based only on the initial parameters of the abovementioned fuzzy numbers, for example, based on the parameters $a_1, a_0, a_2, b_1, b_0, b_2$ for the triangular fuzzy numbers [19] $\underset{\sim}{A} = (a_1, a_0, a_2)$ and $\underset{\sim}{B} = (b_1, b_0, b_2)$. Using direct models [13,15,19-21] for corresponding arithmetic operation $(*) \in \{(+), (-), (\cdot), (:)\}$ makes it possible to calculate the value of the resulting MFs in real-time for any output value x of the resulting MF's support:

$$x \in \operatorname{supp}\left(\underset{\sim}{A} (*) \underset{\sim}{B}\right),$$

where $(*)$ is one of the arithmetic operations from the set $\{(+), (-), (\cdot), (:)\}$.

One of the most difficult fuzzy arithmetic operations in terms of its mathematical formalization is an operation of minimum of the fuzzy numbers (FNs-minimum). Using Max-Min or Min-Max convolutions for the FNs-minimum realization [13,14] at times leads to increased complexity and reduced processing speed or to the violation of the convexity and normality properties in the resulting fuzzy sets. Kauffman and Gupta in [13] considered the geometrical approach based on the α-cuts for the calculation of the FNs-minimum for fuzzy numbers with different shapes of MFs.

Computational algorithms for the operations of FNs-minimum on the basis of α-cuts [13,14,21] have high computational complexity, as it is performed in turn for all α-levels with the step of discreteness $\Delta\alpha$, which value significantly affects the accuracy and operating speed of the computational processes [15,20]. Therefore, α-

cuts of the fuzzy set $\underset{\sim}{A} \in R$ (Fig. 1) is ordinary subset $A_\alpha = \{x \mid \mu_{\underset{\sim}{A}}(x) \geq \alpha\}$, $\alpha \in [0,1]$, that contains elements $x \in R$ whose degree of membership to a set $\underset{\sim}{A}$ is not less than α. The subsets A_α та B_α that determine the appropriate α-cuts of fuzzy sets $\underset{\sim}{A}, \underset{\sim}{B} \in R$ can be written as $A_\alpha = [a_1(\alpha),\ a_2(\alpha)]$, $B_\alpha = [b_1(\alpha),\ b_2(\alpha)]$, $\alpha \in [0,1]$.

The shape of the MFs of fuzzy numbers and the relationship between their parameters have significant impact on the synthesis of the direct models of resulting MFs for fuzzy arithmetic operations. Some individual cases require the need to create the special set or the special library of the direct models of resulting MFs depending on different factors (i.e., for triangular fuzzy numbers, on the relationship between parameters $a_1, a_0, a_2, b_1, b_0, b_2$). These special libraries of direct models for resulting MFs are presented in [19-21] for arithmetic operation addition [19,20] and subtraction [21] with different kinds of asymmetrical fuzzy numbers. The usage of these special libraries allows researchers to increase the computational properties of fuzzy arithmetic operations with abovementioned asymmetrical fuzzy numbers.

The aim of this work is to provide the synthesis of the universal analytical models of resulting MFs for the FNs-minimum and create a library of direct models for triangular fuzzy numbers (TrFNs) with different combinations of their parameters (Fig. 1) in order to increase operating speed and to reduce the volume, complexity and accuracy of fuzzy information processing.

3 Synthesis of Analytic Models for Minimum of Fuzzy Numbers

The TrFNs $\underset{\sim}{A} = (a_1, a_0, a_2)$ and $\underset{\sim}{B} = (b_1, b_0, b_2)$ have MFs $\mu_{\underset{\sim}{A}}(x)$ and $\mu_{\underset{\sim}{B}}(x)$ with parameters $\mu_{\underset{\sim}{A}}(a_1) = \mu_{\underset{\sim}{A}}(a_2) = \mu_{\underset{\sim}{B}}(b_1) = \mu_{\underset{\sim}{B}}(b_2) = 0$, $\mu_{\underset{\sim}{A}}(a_0) = \mu_{\underset{\sim}{B}}(b_0) = 1$.

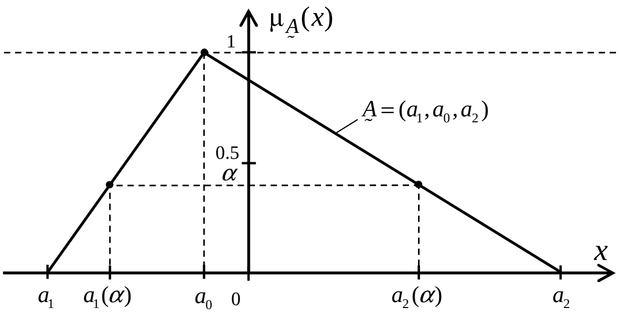

Fig. 1. Triangular Fuzzy Number $\underset{\sim}{A}$, $\underset{\sim}{A} \in R$

The inverse A_α, B_α and direct $\mu_{\underset{\sim}{A}}(x)$, $\mu_{\underset{\sim}{B}}(x)$ models of the TrFNs $\underset{\sim}{A}, \underset{\sim}{B} \in R$ are determined [13-15,19-21] by the corresponding dependencies (1)-(4):

$$A_\alpha = \left[a_1(\alpha), a_2(\alpha)\right] = \left[a_1 + \alpha(a_0 - a_1), a_2 - \alpha(a_2 - a_0)\right], \quad (1)$$

$$\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0, \forall(x \le a_1) \cup (x \ge a_2) \\ F_{left}(x, a_1, a_0), \forall(a_1 < x \le a_0) \\ F_{right}(x, a_0, a_2), \forall(a_0 < x < a_2) \end{cases} = \begin{cases} 0, \forall(x \le a_1) \cup (x \ge a_2) \\ (x - a_1)/(a_0 - a_1), \forall(a_1 < x \le a_0) \\ (a_2 - x)/(a_2 - a_0), \forall(a_0 < x < a_2) \end{cases}, \quad (2)$$

$$B_\alpha = \left[b_1(\alpha), b_2(\alpha)\right] = \left[b_1 + \alpha(b_0 - b_1), b_2 - \alpha(b_2 - b_0)\right], \quad (3)$$

$$\mu_{\underset{\sim}{B}}(x) = \begin{cases} 0, \forall(x \le b_1) \cup (x \ge b_2) \\ F_{left}(x, b_1, b_0), \forall(b_1 < x \le b_0) \\ F_{right}(x, b_0, b_2), \forall(b_0 < x < b_2) \end{cases} = \begin{cases} 0, \forall(x \le b_1) \cup (x \ge b_2) \\ (x - b_1)/(b_0 - b_1), \forall(b_1 < x \le b_0) \\ (b_2 - x)/(b_2 - b_0), \forall(b_0 < x < b_2) \end{cases}. \quad (4)$$

The operation of the FNs-minimum $\left(\underset{\sim}{C} = \underset{\sim}{A}(\wedge)\underset{\sim}{B}\right)$ based on α-cuts [13] can be written as

$$\begin{aligned} C_\alpha = A_\alpha(\wedge)B_\alpha = [a_1(\alpha),\ a_2(\alpha)](\wedge)[b_1(\alpha),\ b_2(\alpha)] = \\ = [a_1(\alpha) \wedge b_1(\alpha),\ a_2(\alpha) \wedge b_2(\alpha)] = [c_1(\alpha),\ c_2(\alpha)]. \end{aligned} \quad (5)$$

We further describe in detail the proposed approach to the synthesis of the analytic FNs-minimum models for TrFNs.

Let us analyze primarily the separate intersections of the left branches of TrFNs

$$F_{left}(x, a_1, a_0) \cap F_{left}(x, b_1, b_0) : \underset{\sim}{A}, \underset{\sim}{B} \in R \quad (6)$$

and right branches of TrFNs

$$F_{right}(x, a_0, a_2) \cap F_{right}(x, b_0, b_2) : \underset{\sim}{A}, \underset{\sim}{B} \in R. \quad (7)$$

If left branch $F_{left}(x, a_1, a_0)$ of TrFN $\underset{\sim}{A}$ has an intersection point with left branch $F_{left}(x, b_1, b_0)$ of TrFN $\underset{\sim}{B}$, then we can write $a_1(\alpha) = b_1(\alpha) = c_1(\alpha)$. Taking into account that $a_1(\alpha) = a_1 + \alpha(a_0 - a_1)$ and $b_1(\alpha) = b_1 + \alpha(b_0 - b_1)$ we can form the equation $a_1 + \alpha(a_0 - a_1) = b_1 + \alpha(b_0 - b_1)$ and, after simple transformations, it is possible to find a parameter α which corresponds to the intersection point (6)

$$\alpha = (b_1 - a_1)/(a_0 - a_1 - b_0 + b_1). \quad (8)$$

This value (8) α, if $\alpha \in [0,1]$, corresponds to the vertical coordinate α^* (Fig. 2) of the intersection point (6) with correct conditions $\mu_{\underset{\sim}{A}}(x) \in [0,1]$ and $\mu_{\underset{\sim}{B}}(x) \in [0,1]$:

$\alpha^* = \mu_{\underset{\sim}{A}}(x^*) = \mu_{\underset{\sim}{B}}(x^*) = \mu_{\underset{\sim}{C}}(x^*)$, where x^* is a horizontal coordinate of the intersection point (6). In this case, $a_1(\alpha^*) = b_1(\alpha^*) = c_1(\alpha^*)$, and we can present the intersection point (6) by two coordinates $(a_1(\alpha^*), \alpha^*)$ or $(x^*, \mu_{\underset{\sim}{A}}(x^*))$, where $x^* = a_1(\alpha^*)$, $\mu_{\underset{\sim}{A}}(x^*) = \alpha^*$, taking into account the interconnections between inverse and direct models of TrFNs. Using direct model (2) we can find

$$\mu_{\underset{\sim}{A}}(x^*) = (x^* - a_1) / (a_0 - a_1) \tag{9}$$

and substituting $x^* = a_1(\alpha^*)$, $\mu_{\underset{\sim}{A}}(x^*) = \alpha^*$ we can obtain the coordinate $a_1(\alpha^*)$:

$$a_1(\alpha^*) = a_1 + \alpha^*(a_0 - a_1) = a_1 + \frac{(b_1 - a_1)(a_0 - a_1)}{a_0 - a_1 - b_0 + b_1} \in [\min(a_1, b_1), \min(a_0, b_0)]. \tag{10}$$

Thus, the coordinates $(a_1(\alpha^*), \alpha^*)$ of the intersection (6) can be calculated using the parameters (a_1, a_0, b_1, b_0) of the TrFNs $(\underset{\sim}{A}, \underset{\sim}{B})$ and the universal models (10) and (8).

Let us analyze the right branches of the fuzzy numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$ in a similar fashion. If the right branch $F_{right}(x, a_0, a_2)$ of TrFN $\underset{\sim}{A}$ has an intersection (7) with the right branch $F_{right}(x, b_0, b_2)$ of TrFN $\underset{\sim}{B}$, then we can write $a_2(\alpha) = b_2(\alpha) = c_2(\alpha)$.

Taking into account that $a_2(\alpha) = a_2 - \alpha(a_2 - a_0)$ and $b_2(\alpha) = b_2 - \alpha(b_2 - b_0)$, it is possible to form the equation $a_2 - \alpha(a_2 - a_0) = b_2 - \alpha(b_2 - b_0)$ and, after simple transformations, we can find a parameter α which corresponds to the intersection (7)

$$\alpha = (b_2 - a_2) / (b_2 - b_0 - a_2 + a_0). \tag{11}$$

This value of (11) α, if $\alpha \in [0,1]$, corresponds to the vertical coordinate α^{**} (Fig. 2) of the intersection point (7) with conditions that $\mu_{\underset{\sim}{A}}(x) \in [0,1]$ and $\mu_{\underset{\sim}{B}}(x) \in [0,1]$: $\alpha^{**} = \mu_{\underset{\sim}{A}}(x^{**}) = \mu_{\underset{\sim}{B}}(x^{**}) = \mu_{\underset{\sim}{C}}(x^{**})$, where x^* is a horizontal coordinate of the intersection point (7). In this case $a_2(\alpha^{**}) = b_2(\alpha^{**}) = c_2(\alpha^{**})$, and we can present the intersection (7) by two coordinates $(a_2(\alpha^{**}), \alpha^{**})$ or $(x^{**}, \mu_{\underset{\sim}{A}}(x^{**}))$, where $x^{**} = a_1(\alpha^{**})$, $\mu_{\underset{\sim}{A}}(x^{**}) = \alpha^{**}$. Using the direct models (2) we can find

$$\mu_{\underset{\sim}{A}}(x^{**}) = (a_2 - x^{**}) / (a_2 - a_0) \tag{12}$$

and substituting $x^* = a_1(\alpha^*)$, $\mu_{\underset{\sim}{A}}(x^*) = \alpha^*$ we can obtain the coordinate $a_2(\alpha^{**})$:

$$a_2\left(\alpha^{**}\right)=a_2-\alpha^{**}\left(a_2-a_0\right)=a_2-\frac{\left(b_2-a_2\right)\left(a_2-a_0\right)}{b_2-b_0-a_2+a_0}\in\left[\min\left(a_0,b_0\right),\min\left(a_2,b_2\right)\right]. \quad (13)$$

Thus, the coordinates $\left(a_2\left(\alpha^{**}\right),\alpha^{**}\right)$ of the intersection (7) can be calculated using the parameters $\left(a_0,a_2,b_0,b_2\right)$ of the TrFNs $\left(\underset{\sim}{A},\underset{\sim}{B}\right)$ and the universal models (13) and (11).

Using the vertical coordinates α^* (8) and α^{**} (11) of the intersections (6) and (7), we can synthesize the resulting inverse (14) and direct (15) models of the minimum of TrFNs $\underset{\sim}{A}=\left(a_1,a_0,a_2\right)$, $\underset{\sim}{B}=\left(b_1,b_0,b_2\right)$ for conditions $a_1<b_1,a_0>b_0,a_2<b_2$:

$$\begin{aligned}
C_\alpha &= A_\alpha(\wedge)B_\alpha=\left[a_1(\alpha)\wedge b_1(\alpha),\ a_2(\alpha)\wedge b_2(\alpha)\right]=\left[c_1(\alpha),\ c_2(\alpha)\right]= \\
&=\left[\begin{cases} a_1(\alpha),\forall\alpha\middle|\alpha\in\left[0,\alpha^*\right] \\ b_1(\alpha),\forall\alpha\middle|\alpha\in\left[\alpha^*,1\right]\end{cases},\ \begin{cases} a_2(\alpha),\forall\alpha\middle|\alpha\in\left[0,\alpha^{**}\right] \\ b_2(\alpha),\forall\alpha\middle|\alpha\in\left[\alpha^{**},1\right]\end{cases}\right]= \\
&=\left[\begin{cases} a_1+\alpha\left(a_0-a_1\right),\forall\alpha\middle|\alpha\in\left[0,\alpha^*\right] \\ b_1+\alpha\left(b_0-b_1\right),\forall\alpha\middle|\alpha\in\left[\alpha^*,1\right]\end{cases},\ \begin{cases} a_2-\alpha\left(a_2-a_0\right),\forall\alpha\middle|\alpha\in\left[0,\alpha^{**}\right] \\ b_2-\alpha\left(b_2-b_0\right),\forall\alpha\middle|\alpha\in\left[\alpha^{**},1\right]\end{cases}\right]
\end{aligned}, \quad (14)$$

where $c_1(0)=a_1$; $c_2(0)=a_2$; $c_1(1)=c_2(1)=b_0$.

$$\mu_{\underset{\sim}{C}}(x)=\begin{cases} 0,\forall\left(x\le a_1\right)\cup\left(x\ge a_2\right) \\ F_{left}\left(x,a_1,a_0\right),\forall\left(a_1<x\le a_1\left(\alpha^*\right)\right) \\ F_{left}\left(x,b_1,b_0\right),\forall\left(a_1\left(\alpha^*\right)<x\le b_0\right) \\ F_{right}\left(x,b_0,b_2\right),\forall\left(b_0<x<a_2\left(\alpha^{**}\right)\right) \\ F_{right}\left(x,a_0,a_2\right),\forall\left(a_2\left(\alpha^{**}\right)<x<a_2\right)\end{cases}=\begin{cases} 0,\forall\left(x\le a_1\right)\cup\left(x\ge a_2\right) \\ \left(x-a_1\right)/\left(a_0-a_1\right),\forall\left(a_1<x\le a_1\left(\alpha^*\right)\right) \\ \left(x-b_1\right)/\left(b_0-b_1\right),\forall\left(a_1\left(\alpha^*\right)<x\le b_0\right) \\ \left(b_2-x\right)/\left(b_2-b_0\right),\forall\left(b_0<x<a_2\left(\alpha^{**}\right)\right) \\ \left(a_2-x\right)/\left(a_2-a_0\right),\forall\left(a_2\left(\alpha^{**}\right)<x<a_2\right)\end{cases}. \quad (15)$$

4 The Library of Resulting Direct Models for FNs-Minimum

The inverse C_α (14) and the direct $\mu_{\underset{\sim}{C}}(x)$ (15) models for the FNs-minimum are validated only for TrFNs $\underset{\sim}{A}=\left(a_1,a_0,a_2\right)$ and $\underset{\sim}{B}=\left(b_1,b_0,b_2\right)$ under the following conditions: $a_1<b_1,a_0>b_0,a_2<b_2$. At the same time a lot of real input values for fuzzy processing can be presented as TrFNs with different relations between parameters: $a_1\square\, b_1,a_0\square\, b_0,a_2\square\, b_2$, $\square\in\left\{(<),(=),(>)\right\}$. Therefore, for each special case it is necessary to develop a separate analytic model of resulting fuzzy set for implementation of “FNs-minimum” if the TrFNs $\left(\underset{\sim}{A},\underset{\sim}{B}\right)$ have different relations □ between parameters $\left(a_1,b_1;a_0,b_0;a_2,b_2\right)$.

In this section the authors aim to develop a library of inverse and direct analytic models of the resulting fuzzy sets $\underset{\sim}{C}$ for realization of the “minimum” as arithmetic operation with TrFNs $\underset{\sim}{A}$ and $\underset{\sim}{B}$ and various combinations of the relations □ . Following [19-21] we can determine a mask

$$\text{Mask }\left(\underset{\sim}{A},\underset{\sim}{B}\right)=\left\{d,g,p\right\} \tag{16}$$

for any pair of the TrFNs $\underset{\sim}{A}$ and $\underset{\sim}{B}$, where indicators d,g and p are defined as

$$d=\begin{cases}0, & \text{if } a_1>b_1\\ 1, & \text{if } a_1<b_1\end{cases};\quad g=\begin{cases}0, & \text{if } a_0>b_0\\ 1, & \text{if } a_0<b_0\end{cases};\quad p=\begin{cases}0, & \text{if } a_2>b_2\\ 1, & \text{if } a_2<b_2\end{cases}.$$

The Mask (16) is a the basis for forming a 8-component’s library (Table 1) of the resulting mathematical models $\left\{M_1...M_8\right\}$ for FNs-minimum with all possible combinations of TrFNs $\left(\underset{\sim}{A},\underset{\sim}{B}\right)$ and different relations □ . The library of the developed direct $\mu_{\underset{\sim}{C}}\left(x\right)$ models $\left\{M_1,M_2,...,M_8\right\}$ is represented in the Table 2.

Table 1. Masks and models for pairs of TrFNs $\underset{\sim}{A},\underset{\sim}{B}\in R$

$\{d,g,p\}$	$\{1,1,1\}$	$\{1,1,0\}$	$\{1,0,1\}$	$\{1,0,0\}$	$\{0,1,1\}$	$\{0,1,0\}$	$\{0,0,1\}$	$\{0,0,0\}$
M_i	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8

Table 2. Library of the direct models $\mu_{\underset{\sim}{C}}(x)$ for resulting fuzzy number $\underset{\sim}{C}=\underset{\sim}{A}(\wedge)\underset{\sim}{B}$

	Direct model $\mu_{\underset{\sim}{C}}\left(x\right)$, based on:	
	the functions $\left(F_{left},F_{right}\right)$	the parameters $\left(a_1,b_1,a_0,b_0,a_2,b_2\right)$
M_1	$\begin{cases}0,\forall\left(x\le a_1\right)\cup\left(x\ge a_2\right)\\ F_{left}\left(x,a_1,a_0\right),\forall\left(a_1<x\le a_0\right)\\ F_{right}\left(x,a_0,a_2\right),\forall\left(a_0<x<a_2\right)\end{cases}$	$\begin{cases}0,\forall\left(x\le a_1\right)\cup\left(x\ge a_2\right)\\ \left(x-a_1\right)/\left(a_0-a_1\right),\forall\left(a_1<x\le a_0\right)\\ \left(a_2-x\right)/\left(a_2-a_0\right),\forall\left(a_0<x<a_2\right)\end{cases}$
M_2	$\begin{cases}0,\forall\left(x\le a_1\right)\cup\left(x\ge b_2\right)\\ F_{left}\left(x,a_1,a_0\right),\forall\left(a_1<x\le a_0\right)\\ F_{right}\left(x,a_0,a_2\right),\forall\left(a_0<x<a_2\left(\alpha^{**}\right)\right)\\ F_{right}\left(x,b_0,b_2\right),\forall\left(a_2\left(\alpha^{**}\right)<x<b_2\right)\end{cases}$	$\begin{cases}0,\forall\left(x\le a_1\right)\cup\left(x\ge b_2\right)\\ \left(x-a_1\right)/\left(a_0-a_1\right),\forall\left(a_1<x\le a_0\right)\\ \left(a_2-x\right)/\left(a_2-a_0\right),\forall\left(a_0<x<a_2\left(\alpha^{**}\right)\right)\\ \left(b_2-x\right)/\left(b_2-b_0\right),\forall\left(a_2\left(\alpha^{**}\right)<x<b_2\right)\end{cases}$

M_3	$\begin{cases} 0, \forall\left(x \le a_1\right) \cup \left(x \ge a_2\right) \\ F_{left}\left(x, a_1, a_0\right), \forall\left(a_1 < x \le a_1\left(\alpha^*\right)\right) \\ F_{left}\left(x, b_1, b_0\right), \forall\left(a_1\left(\alpha^*\right) < x \le b_0\right) \\ F_{right}\left(x, b_0, b_2\right), \forall\left(b_0 < x < a_2\left(\alpha^{**}\right)\right) \\ F_{right}\left(x, a_0, a_2\right), \forall\left(a_2\left(\alpha^{**}\right) < x < a_2\right) \end{cases}$	$\begin{cases} 0, \forall\left(x \le a_1\right) \cup \left(x \ge a_2\right) \\ \left(x - a_1\right) / \left(a_0 - a_1\right), \forall\left(a_1 < x \le a_1\left(\alpha^*\right)\right) \\ \left(x - b_1\right) / \left(b_0 - b_1\right), \forall\left(a_1\left(\alpha^*\right) < x \le b_0\right) \\ \left(b_2 - x\right) / \left(b_2 - b_0\right), \forall\left(b_0 < x < a_2\left(\alpha^{**}\right)\right) \\ \left(a_2 - x\right) / \left(a_2 - a_0\right), \forall\left(a_2\left(\alpha^{**}\right) < x < a_2\right) \end{cases}$
M_4	$\begin{cases} 0, \forall\left(x \le a_1\right) \cup \left(x \ge b_2\right) \\ F_{left}\left(x, a_1, a_0\right), \forall\left(a_1 < x \le a_1\left(\alpha^*\right)\right) \\ F_{left}\left(x, b_1, b_0\right), \forall\left(a_1\left(\alpha^*\right) < x \le b_0\right) \\ F_{right}\left(x, b_0, b_2\right), \forall\left(b_0 < x < b_2\right) \end{cases}$	$\begin{cases} 0, \forall\left(x \le a_1\right) \cup \left(x \ge b_2\right) \\ \left(x - a_1\right) / \left(a_0 - a_1\right), \forall\left(a_1 < x \le a_1\left(\alpha^*\right)\right) \\ \left(x - b_1\right) / \left(b_0 - b_1\right), \forall\left(a_1\left(\alpha^*\right) < x \le b_0\right) \\ \left(b_2 - x\right) / \left(b_2 - b_0\right), \forall\left(b_0 < x < b_2\right) \end{cases}$
M_5	$\begin{cases} 0, \forall\left(x \le b_1\right) \cup \left(x \ge a_2\right) \\ F_{left}\left(x, b_1, b_0\right), \forall\left(b_1 < x \le a_1\left(\alpha^*\right)\right) \\ F_{left}\left(x, a_1, a_0\right), \forall\left(a_1\left(\alpha^*\right) < x \le a_0\right) \\ F_{right}\left(x, a_0, a_2\right), \forall\left(a_0 < x < a_2\right) \end{cases}$	$\begin{cases} 0, \forall\left(x \le b_1\right) \cup \left(x \ge a_2\right) \\ \left(x - b_1\right) / \left(b_0 - b_1\right), \forall\left(b_1 < x \le a_1\left(\alpha^*\right)\right) \\ \left(x - a_1\right) / \left(a_0 - a_1\right), \forall\left(a_1\left(\alpha^*\right) < x \le a_0\right) \\ \left(a_2 - x\right) / \left(a_2 - a_0\right), \forall\left(a_0 < x < a_2\right) \end{cases}$
M_6	$\begin{cases} 0, \forall\left(x \le b_1\right) \cup \left(x \ge b_2\right) \\ F_{left}\left(x, b_1, b_0\right), \forall\left(b_1 < x \le a_1\left(\alpha^*\right)\right) \\ F_{left}\left(x, a_1, a_0\right), \forall\left(a_1\left(\alpha^*\right) < x \le a_0\right) \\ F_{right}\left(x, a_0, a_2\right), \forall\left(a_0 < x < a_2\left(\alpha^{**}\right)\right) \\ F_{right}\left(x, b_0, b_2\right), \forall\left(a_2\left(\alpha^{**}\right) < x < b_2\right) \end{cases}$	$\begin{cases} 0, \forall\left(x \le b_1\right) \cup \left(x \ge b_2\right) \\ \left(x - b_1\right) / \left(b_0 - b_1\right), \forall\left(b_1 < x \le a_1\left(\alpha^*\right)\right) \\ \left(x - a_1\right) / \left(a_0 - a_1\right), \forall\left(a_1\left(\alpha^*\right) < x \le a_0\right) \\ \left(a_2 - x\right) / \left(a_2 - a_0\right), \forall\left(a_0 < x < a_2\left(\alpha^{**}\right)\right) \\ \left(b_2 - x\right) / \left(b_2 - b_0\right), \forall\left(a_2\left(\alpha^{**}\right) < x < b_2\right) \end{cases}$
M_7	$\begin{cases} 0, \forall\left(x \le a_1\right) \cup \left(x \ge a_2\right) \\ F_{left}\left(x, b_1, b_0\right), \forall\left(b_1 < x \le b_0\right) \\ F_{right}\left(x, b_0, b_2\right), \forall\left(b_0 < x < a_2\left(\alpha^{**}\right)\right) \\ F_{right}\left(x, a_0, a_2\right), \forall\left(a_2\left(\alpha^{**}\right) < x < a_2\right) \end{cases}$	$\begin{cases} 0, \forall\left(x \le b_1\right) \cup \left(x \ge a_2\right) \\ \left(x - b_1\right) / \left(b_0 - b_1\right), \forall\left(b_1 < x \le b_0\right) \\ \left(b_2 - x\right) / \left(b_2 - b_0\right), \forall\left(b_0 < x < a_2\left(\alpha^{**}\right)\right) \\ \left(a_2 - x\right) / \left(a_2 - a_0\right), \forall\left(a_2\left(\alpha^{**}\right) < x < a_2\right) \end{cases}$
M_8	$\begin{cases} 0, \forall\left(x \le b_1\right) \cup \left(x \ge b_2\right) \\ F_{left}\left(x, b_1, b_0\right), \forall\left(b_1 < x \le b_0\right) \\ F_{right}\left(x, b_0, b_2\right), \forall\left(b_0 < x < b_2\right) \end{cases}$	$\begin{cases} 0, \forall\left(x \le b_1\right) \cup \left(x \ge b_2\right) \\ \left(x - b_1\right) / \left(b_0 - b_1\right), \forall\left(b_1 < x \le b_0\right) \\ \left(b_2 - x\right) / \left(b_2 - b_0\right), \forall\left(b_0 < x < b_2\right) \end{cases}$

5 Modeling Results

Let's consider an example with realisation of the arithmetic operation "minimum" for the pair $\left(\underset{\sim}{A},\underset{\sim}{B}\right)$ of TrFNs: $\underset{\sim}{A}=(3,10,17)$, $\underset{\sim}{B}=(5,7,24)$. In this case, we have: $a_1=3;\ b_1=5;\ a_0=10;\ b_0=7;\ a_2=17;\ b_2=24$. Using (16) we can automatically determine (a) the corresponding Mask $\left(\underset{\sim}{A},\underset{\sim}{B}\right)=\{d,g,p\}=\{1,0,1\}$ for the conditions $a_1<b_1;\ a_0>b_0;\ a_2<b_2$ and (b) the corresponding model M_3 from the library of models $\{M_1,M_2,...,M_8\}$ (Table 1).

Let's calculate the coordinates $\left(a_1\left(\alpha^*\right),\alpha^*\right)$ and $\left(a_2\left(\alpha^{**}\right),\alpha^{**}\right)$ for intersection points (6) and (7) of the fuzzy numbers $\left(\underset{\sim}{A},\underset{\sim}{B}\right)$ according to (10), (8), (13) and (11):

$$a_1\left(\alpha^*\right)=3+\frac{(5-3)(10-3)}{10-3-7+5}=5.8;\qquad \alpha^*=\frac{5-3}{10-3-7+5}=0.4\,;$$

$$a_2\left(\alpha^{**}\right)=17-\frac{(24-17)(17-10)}{24-7-17+10}=12.1;\qquad \alpha^{**}=\frac{24-17}{24-7-17+10}=0.7\,.$$

Then (for recognized M_3) we can choose the corresponding direct model $\mu_{\underset{\sim}{C}}(x)$ from the library (Table 2). We further present the resulting inverse $C_\alpha=A_\alpha(\wedge)B_\alpha$ and direct $\mu_{\underset{\sim}{C}}(x)$ models (Fig.2) for $\underset{\sim}{C}=\underset{\sim}{A}(\wedge)\underset{\sim}{B}$:

$$\mu_{\underset{\sim}{C}}(x)=\begin{cases}0,\forall\left(x\le a_1\right)\cup\left(x\ge a_2\right)\\ F_{left}\left(x,a_1,a_0\right),\forall\left(a_1<x\le a_1\left(\alpha^*\right)\right)\\ F_{left}\left(x,b_1,b_0\right),\forall\left(a_1\left(\alpha^*\right)<x\le b_0\right)\\ F_{right}\left(x,b_0,b_2\right),\forall\left(b_0<x<a_2\left(\alpha^{**}\right)\right)\\ F_{right}\left(x,a_0,a_2\right),\forall\left(a_2\left(\alpha^{**}\right)<x<a_2\right)\end{cases}=\begin{cases}0,\forall\left(x\le 3\right)\cup\left(x\ge 17\right)\\ (x-3)/7,\forall\left(3<x\le 5.8\right)\\ (x-5)/2,\forall\left(5.8<x\le 7\right)\\ (24-x)/17,\forall\left(7<x<12.1\right)\\ (17-x)/7,\forall\left(12.1<x<17\right)\end{cases},$$

$$C_\alpha=A_\alpha(\wedge)B_\alpha=\left[\begin{Bmatrix}3+7\alpha,\forall\alpha\,|\,\alpha\in[0,0.4]\\ 5+2\alpha,\forall\alpha\,|\,\alpha\in[0.4,1]\end{Bmatrix},\ \begin{Bmatrix}17-7\alpha,\forall\alpha\,|\,\alpha\in[0,0.7]\\ 24-17\alpha,\forall\alpha\,|\,\alpha\in[0.7,1]\end{Bmatrix}\right].$$

6 The Application of the FNs-Minimum Library

The implementation of the developed library of direct analytic models (Table 2) for calculation of the resulting MFs $\mu_{\underset{\sim}{C}}(x)$, according to given TrFNs with various relationships between parameters a_1,a_0,a_2,b_1,b_0,b_2 of MFs, allows researchers to use one-step-automation-mode for arithmetic operation "FNs-minimum" $\underset{\sim}{C}=\underset{\sim}{A}(\wedge)\underset{\sim}{B}$.
We further consider some examples of the developed analytic models library application (Table 2) for solving real-life decision-making problems under uncertainty.

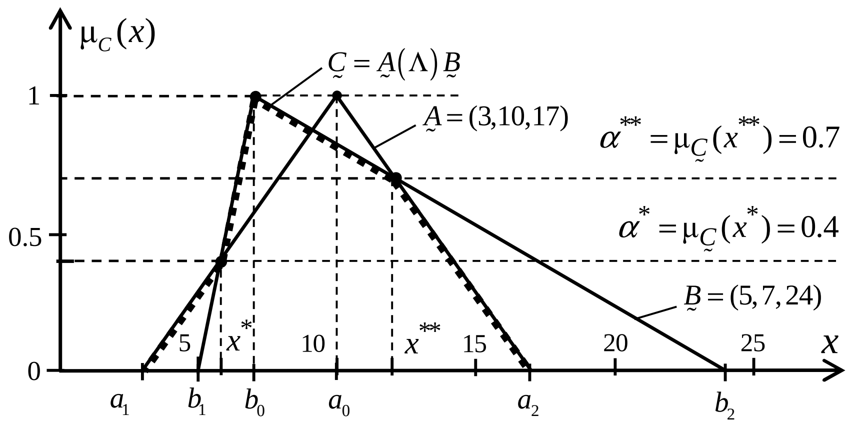

Fig. 2. FNs-Minimum $\underset{\sim}{C} = \underset{\sim}{A}(\wedge)\underset{\sim}{B}$ of the TrFNs $\underset{\sim}{A} \in R$ and $\underset{\sim}{B} \in R$

6.1 Capacitive Vehicle Routing Problem with Fuzzy Demands at Nodes

The trucks or bunkering tankers provide corresponding cargo transportation and unloading operations for various nodes served and located in different destinations, for example, (a) in the cities – for the trucks; (b) in the marine ports and open sea points – for bunkering tankers. Taking into account the limited capacity of the transporting unit (i.e., truck or tanker), we need to solve capacitive vehicle routing problem (CVRP). The efficiency of the preliminary vehicle routes planning can be evaluated by its ability to serve all nodes' orders with maximum possible quantity of unloaded cargo and minimum length of the total vehicle routes.

Transport logistic practice shows that very often the information about cargo demands of served nodes are uncertain. These demands can be modeled by TrFNs [9,11,22]. For example, such uncertain demands as (a) "approximately a_0 " or (b) "value between a_1 and a_2 "can be modelled by TrFNs $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ represented in Fig.1. The CVRP with fuzzy demands $\underset{\sim}{A}_j = \left(a_{1j}, a_{0j}, a_{2j}\right)$ at nodes $j \in \{1, 2, ..., r\}$ is considered in [9], where a_{0j} is the value of MF of TrFN $\underset{\sim}{A}_j$ with $\mu\left(a_{0j}\right) = 1$; a_{1j} and a_{2j} are the lowest and highest possible values of demand, respectively, $\mu\left(a_{1j}\right) = 0$, $\mu\left(a_{2j}\right) = 0$; $D_{\max}$ - a cargo capacity of the vehicle. All fuzzy demands $\underset{\sim}{A}_1, \underset{\sim}{A}_2, ..., \underset{\sim}{A}_r$ may have various parameters of their TrMFs models [9,22].

Solving such kinds of decision-making problems deals with vehicle route planning (*Route* 1 in Fig. 4) [9,22], when vehicle should start from deport D_0 and serve nodes with fuzzy demands one-by-one taking into account the current fuzzy value of remaining cargo at the vehicle $\underset{\sim}{D}_j = \left(d_{1j}, d_{0j}, d_{2j}\right)$ and fuzzy demand of the next node-candidate $\underset{\sim}{A}_{j+1}$.

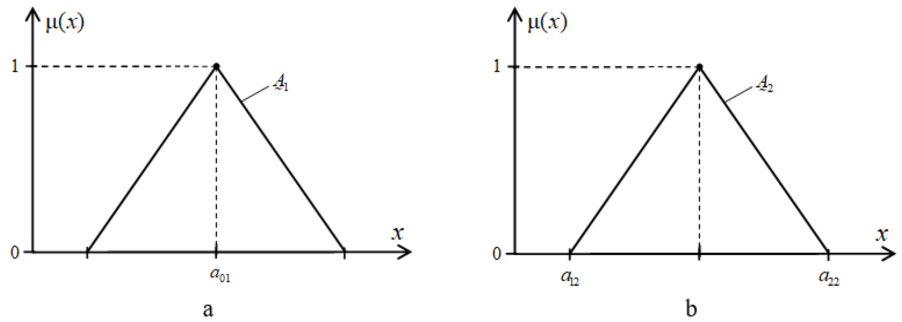

Fig. 3. Models of fuzzy demands as TrFNs $\underset{\sim}{A}_1$ (a) and $\underset{\sim}{A}_2$ (b)

Fig. 2 illustrates the situation when the next node-candidate from the unserved node set $\{S_6, S_7, S_8, S_9\}$ can be chosen to be included in the *Route* 2 planning process, otherwise, the vehicle should return to the deport D_0 if the value of its remaining cargo $\underset{\sim}{D}_5 = (d_{15}, d_{05}, d_{25})$ is insufficient [9].

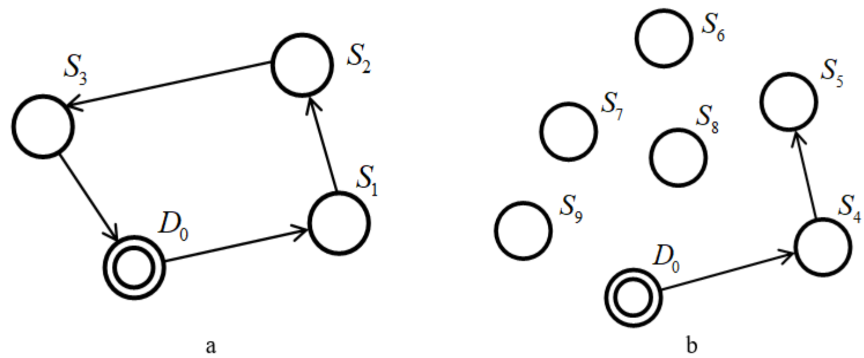

Fig. 4. Solving CVRP: planned *Route* 1 (a) and planning *Route* 2 (b)

The decision about including the node-candidate S_6 with fuzzy demand $\underset{\sim}{A}_6$ can be made by analyzing the resulting MFs $\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5$ for the arithmetic operation FNs-minimum with TrFNs $\underset{\sim}{A}_6$ and $\underset{\sim}{D}_5$. Calculating the distance $Dist_{65}\left(\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5, \underset{\sim}{A}_6\right)$ between the resulting TrFN $\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5$ and TrFN $\underset{\sim}{A}_6$ (fuzzy demand at node S_6) with application of one of the well-known methods for measuring distance between two fuzzy numbers (Hausdorff-, Euclid-, Hemming-distance, etc.) [23,24,25], we can include the node S_6 to the *Route* 2 in the condition if

$$Dist_{65}\left(\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5, \underset{\sim}{A}_6\right) \le \Delta_{des}, \tag{17}$$

where Δ_{des} is a desired value of the deviation between abovementioned TrFNs $\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5$ and $\underset{\sim}{A}_6$.

The diverse cases represented in Fig. 5 – Fig. 7 depend on the relationship between the parameters a_{16}, a_{06}, a_{26} of TrFN $\underset{\sim}{A}_6$ and the parameters d_{15}, d_{05}, d_{25} of TrFN $\underset{\sim}{D}_5$. In particular, in the occasions when condition $Dist_{65}\left(\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5, \underset{\sim}{A}_6\right) = 0$ the node S_6 will be included in the *Route* 2 planning (Fig.5a, Fig. 6a). The inclusion of the respective node will be relevant when $Dist_{65}\left(\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5, \underset{\sim}{A}_6\right) < \Delta_{des}$ (Fig.6b) and $Dist_{65}\left(\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5, \underset{\sim}{A}_6\right) = \Delta_{des}$ (Fig. 7a). If the condition (17) is not fulfilled, for example $Dist_{65}\left(\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5, \underset{\sim}{A}_6\right) > \Delta_{des}$ (Fig. 7b) or $Dist_{65}\left(\underset{\sim}{A}_6 (\wedge) \underset{\sim}{D}_5, \underset{\sim}{A}_6\right) >> \Delta_{des}$ (Fig. 5b), then the node S_6 should not be included in the *Route* 2 planning.

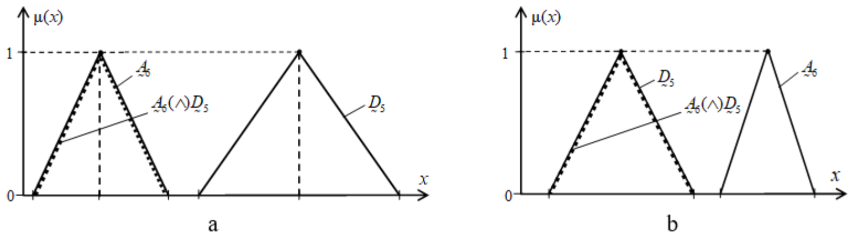

Fig. 5. The cases for *Route* 2 when (a) $Dist_{56} = 0$ and (b) $Dist_{56} >> \Delta_{des}$

Therefore, the decision maker will include any node-candidate S_{j+1} to the corresponding *Route* in the planning process if the condition

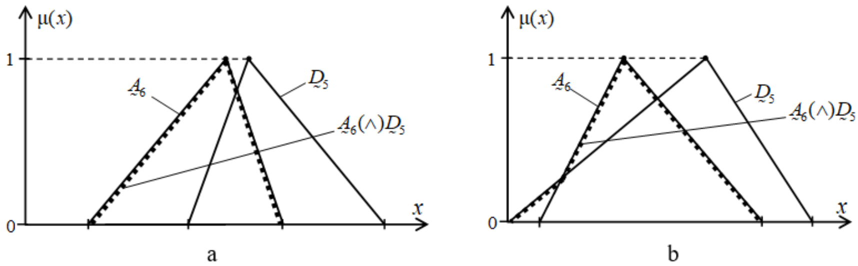

Fig. 6. The node S_6 is included in *Route* 2: (a) $Dist_{56} = 0$; (b) $Dist_{56} < \Delta_{des}$

$$Dist_{j+1,j}\left(\underset{\sim}{A}_{j+1} (\wedge) \underset{\sim}{D}_j, \underset{\sim}{A}_{j+1}\right) \le \Delta_{des} \tag{18}$$

is fulfilled. The desired value Δ_{des} can be preliminary determined using simulation approach based on the generation of random sequences [26,27] for modelling fuzzy and crisp demands at nodes [28].

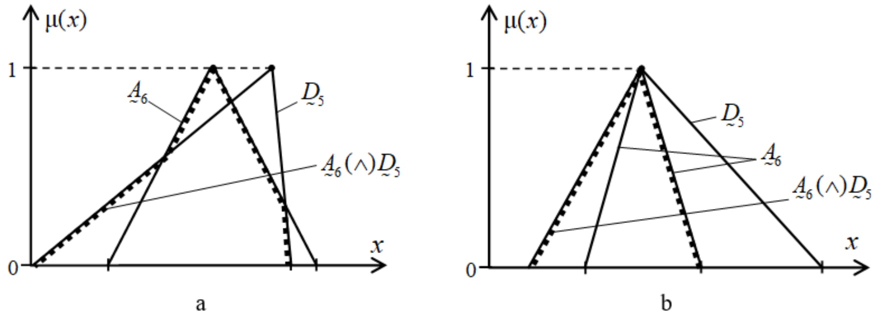

Fig. 7. The illustration of *Route* 2: (a) $Dist_{56} = \Delta_{des}$; (b) $Dist_{56} > \Delta_{des}$

6.2 Using FNs-minimum for the Evaluation of the Rating of Professional Skills or Knowledge Level

Another effective way of the implementation of the fuzzy arithmetic operation "FNs-minimum" may be an evaluation process of (a) the knowledge level of the students [29] or (b) the employees' professional skills assessment. It is of outmost importance to pay special attention to the evaluation procedures with providing minimal (or fixed) level of knowledge for each domain of the knowledge specified.

The proposed approach is based on the following five steps:

(a) formation of the fuzzy number which is a fuzzy model of the corresponding grade, for example, $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}$;

(b) formation of the fuzzy number $\underset{\sim}{S}_{ans}$ equivalent to the student's knowledge level, for example, in three different domains (parts);

(c) calculation of the resulting MF of the arithmetic operation FNs-minimum for TrFN $\underset{\sim}{S}_{ans}$ and TrFN, which corresponds to the desired score;

(d) comparison of the resulting FNs-minimum with TrFN $\underset{\sim}{S}_{ans}$;

(e) conclusion of the desired score outcome.

7 Conclusions

The minimum of fuzzy sets is a very important fuzzy arithmetic operation, which requires a lot of time for its realization. The implementation of the developed direct analytic models' library (Table 2) allows using one step automation mode for operation "FNs-minimum" $\underset{\sim}{C} = \underset{\sim}{A}(\wedge)\underset{\sim}{B}$. Modeling results confirm the efficiency of proposed universal analytic models for different applications. In some cases, such direct analytic models $\mu_{\underset{\sim}{C}}(x) = \mu_{\underset{\sim}{A}(\wedge)\underset{\sim}{B}}(x)$ provide an efficient solution to the fuzzy processing in evaluation, control and decision-making processes, in particular, for the financial analysis [12], automatic evaluation of the student's knowledge [29], group anonymity [30], and model design process [31,32], soft computing based on reconfigurable technology [33], analysis of the big data during testing of computer systems and their components [34,35], optimization in transport logistics [9,11,22,36], redesigning social inquiry [37,38], partner selection [39], fuzzy-algorithmic reliability analysis of complex systems in economics, management and engineering [26,40-43] and others.

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