Unique colorings of bi-hypergraphs*

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Abstract

We discuss the properties of uniform hypergraphs which have precisely one partition (i.e., a unique coloring apart from permutation of the colors) under the condition that in each edge, there exist three vertices which belong to precisely two classes of the partition. In particular, we investigate the relation between unique colorability, number of colors, and the cardinalities of color classes.

1 Introduction

A *mixed hypergraph* [18] is a triple $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$, where V is the vertex set and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of V, the $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively. A *proper k-coloring* of a mixed hypergraph is a mapping from the vertex set X into a set of k colors $1, 2, \ldots, k$ so that each $\mathcal{C}$-edge has two vertices with a **c**ommon color and each $\mathcal{D}$-edge has two vertices with **d**ifferent colors. A mixed hypergraph is *k-colorable* if it has a proper coloring with at most k colors.

A *strict* k-coloring is a proper k-coloring using all the k colors. The maximum number of colors in a strict coloring of $\mathcal{H}$ is the *upper chromatic number* $\bar{\chi}(\mathcal{H})$;

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the minimum number is the *lower chromatic number* $\chi(\mathcal{H})$. Thus, colorings of hypergraphs in the classical sense (see e.g. [1]) represent the particular case of mixed hypergraphs restricted to $\mathcal{C} = \emptyset$. For each k, let $r_k = r_k(\mathcal{H})$ be the number of partitions of the vertex set into k nonempty parts (color classes) such that the coloring constraint is satisfied on each edge. The vector $R(\mathcal{H}) = (r_1, \ldots, r_n) = (0, \ldots, 0, r_\chi, \ldots, r_{\bar{\chi}}, 0, \ldots, 0)$ is the *chromatic spectrum* of $\mathcal{H}$, introduced in [16]. The set of values k such that $\mathcal{H}$ has a strict k-coloring is the *feasible set* of $\mathcal{H}$, written $S(\mathcal{H})$; this is the set of indices i such that $r_i > 0$.

Definition 1 A mixed hypergraph $\mathcal{H}$ is called *uniquely colorable* [15] (uc hypergraph, or uc for short) if it has precisely one strict coloring apart from permutations of colors.

Equivalently, $\mathcal{H}$ is uc if it allows exactly one feasible partition of the vertex set V into color classes. Let us agree that the expression "unique coloring" means in the sequel "unique partition" into the corresponding number of color classes. Evidently, if $\mathcal{H}$ is a uc hypergraph, then $\chi(\mathcal{H}) = \bar{\chi}(\mathcal{H}) = \chi$ and $r_\chi(\mathcal{H}) = 1$, therefore $R(\mathcal{H}) = (0, \ldots, 0, 1, 0, \ldots, 0)$ (and conversely, a hypergraph with this spectrum is uc).

Definition 2 A mixed hypergraph $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ is a *bi-hypergraph* [17, 14] if $\mathcal{C} = \mathcal{D}$.

In contrast to classical colorings of hypergraphs, mixed hypergraphs may have no colorings at all. A mixed hypergraph having no colorings is *uncolorable* [16, 17]. Otherwise it is called *colorable*. The first paper about uncolorable mixed hypergraphs is [14]. The *colorability problem* takes a mixed hypergraph as input, and asks whether it admits at least one coloring.

The problem which is closest to the colorability problem is that of *unique colorability*. In the classical coloring theory of graphs, the only objects having this property are the complete graphs (cliques). In the literature the term "uniquely colorable" has been used to denote graphs which have a unique coloring *with the mimimum number* of colors. But such graphs, other than cliques, have additional colorings using more colors and are therefore not uniquely colorable when viewed as mixed hypergraphs.

It turns out that uniquely colorable mixed hypergraphs represent the relevant generalizations of cliques from the point of view of colorings. The first paper on uniquely colorable mixed hypergraphs is [15]. As it was shown there, the structure of uniquely colorable mixed hypergraphs is unexpectedly rich. Namely, every colorable mixed hypergraph can be embedded into some uniquely colorable mixed hypergraph as an induced subhypergraph. In addition, in [15] the authors investigated the role of uniquely colorable subhypergraphs being separators, studied recursive operations (orderings and subset contractions) and unique colorings, and proved that it is *NP*-hard to decide whether a mixed hypergraph is uniquely colorable.

The following weaker property was discussed in [15]: a mixed hypergraph which has a unique coloring with $\bar{\chi}$ colors and a unique coloring with χ colors is called *weakly uniquely colorable*. This trivially includes all uniquely colorable graphs in the usual sense.

Motivated by [15], the following classes of uniquely colorable mixed hypergraphs have been characterized: those with $\chi = n-1$ and $\chi = n-2$ in [12]; mixed hypertrees in [11]; and circular mixed hypergraphs in [19], see also [13]. Moreover, based on the idea of unique colorability, pseudo-chordal mixed hypergraphs as a generalization of chordal graphs have been introduced and described in [20].

A mixed hypergraph is *r-uniform* if all the $\mathcal{C}$- and $\mathcal{D}$-edges are of size r. In this paper we consider the class of *r-uniform mixed bi-hypergraphs*. It is convenient to make no difference between the expressions like "bi-edge," "r-tuple," and "subset of r vertices which is a $\mathcal{C}$- and $\mathcal{D}$-edge at the same time." The special case $r = 3$ of such hypergraphs derived from Steiner Triple Systems (each block considered as a $\mathcal{C}$- and $\mathcal{D}$-edge) has been investigated recently in [3, 7, 8, 9, 10] and in some further papers. These STS-bi-hypergraphs are generally not uniquely colorable. The study of unique colorability in important particular cases like them may lead to a better understanding of the behavior of colorings in more general settings.

There is, however, one more motivation for the study of r-uniform uc bi-hypergraphs. The fact discovered recently is that mixed hypergraphs may have gaps in their chromatic spectra, i.e. zeros may occur between positive values for the number of strict colorings [5, 6]. In fact, much stronger results have been proved, for instance:

Theorem 1 [6] *A finite set S of positive integers is the feasible set of some mixed hypergraph whose spectrum has $r_i \in \{0, 1\}$ for each i if and only if $1 \notin S$ or $\max(S) \leq 2$.*

Such mixed hypergraphs found so far are not uniform bi-hypergraphs, though they are "everywhere uc" in the sense that for each k, if there exists a strict k-coloring then there is precisely one.

Theorem 2 [6] *For each integer $r \geq 3$, there exists an r-uniform bi-hypergraph $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ whose chromatic spectrum contains a gap.*

These mixed hypergraphs are not "everywhere uc." Hence, it remains an open question whether every feasible set of integers may be a feasible set of some r-uniform bi-hypergraph whose spectrum has $r_i \in \{0, 1\}$ for all $\chi \leq i \leq \bar{\chi}$. The study of the properties of uniform uc bi-hypergraphs will contribute to the knowledge necessary for finding the answer.

2 Characterizations of unique colorability

We call an r-uniform bi-hypergraph *saturated* if it is uniquely colorable and adding any r-tuple to it makes it uncolorable. Throughout the paper, r will be at least 3, and "hypergraph" means "r-uniform bi-hypergraph."

Notation For a given partition (coloring) $\mathcal{P}_0$ of a hypergraph $\mathcal{H}$, and for a given r, $\mathcal{H}_r(\mathcal{P}_0) = \mathcal{H}(\mathcal{P}_0)$ is the hypergraph consisting of all those r-tuples which have some pair of vertices with a common color and also some pair with distinct colors — that is, the r-tuples which are consistent with the given coloring.

Let us fix r. A partition (coloring) $\mathcal{P}_0$ is called *unique* if $\mathcal{H}(\mathcal{P}_0)$ is uniquely colorable. Our aim is to give a characterization of unique colorings in this sense.

2.1 The Main Lemma

As one can easily see, for any given r, the number k of colors in a unique coloring must be at least $r-1$. Indeed, otherwise we can split a color class into two, without violating the conditions on coloring, and hence producing another coloring, contrary to the assumption of uniqueness.

The goal of this section is to prove an assertion that we call the Main Lemma. It contains necessary and sufficient conditions to be an "alternative" of a given coloring. This important notion is defined as follows:

Definition 3 For a given coloring $\mathcal{P}_0$, $\mathcal{P}$ is an *alternative* (of $\mathcal{P}_0$) if it is a proper coloring of $\mathcal{H}(\mathcal{P}_0)$. A coloring is always an alternative of itself. Here we emphasize that "$\mathcal{P}$ is an alternative of $\mathcal{P}_0$" is not a symmetric relation.

We next introduce two conditions for colorings:

Union Condition: Every pair of color classes has a union of size at least r.

Size Condition: The underlying vertex set has more than $(r-1)^2$ vertices.

These two conditions are called the **Basic Conditions**. Clearly, a unique coloring satisfies the Union Condition. It also satisfies the Size Condition since otherwise we could make another coloring with color classes all smaller than r. Consequently, **every unique coloring satisfies the Basic Conditions.**

Given a coloring $\mathcal{P}_0$, a set is *crossing* if it intersects every color class of $\mathcal{P}_0$ in at most one point.

Remark Though, for an arbitrary set, "polychromatic" (i.e., having all the colors different) and "crossing" mean the same property, we have chosen two different expressions, since r-tuples and color classes of an alternative play different roles.

Main Lemma *Given a coloring $\mathcal{P}_0$, satisfying the basic conditions, the coloring $\mathcal{P} \neq \mathcal{P}_0$ is an alternative of $\mathcal{P}_0$ if and only if it has fewer than r color classes, and for all of its "big" classes X (i.e., for $|X| \geq r$)*

– *either X is the subset of some class in $\mathcal{P}_0$ ("subclass")* (1)

– *or X is crossing.* (2)

Proof. For the 'if' part, take a $\mathcal{P}$ satisfying all properties in the assumption, and take some r-tuple R from $\mathcal{H}(\mathcal{P}_0)$. If it is not properly colored by $\mathcal{P}$, then there are two possibilities: R is polychromatic or it is contained in some class X of $\mathcal{P}$. The first possibility can be excluded since $|\mathcal{P}| < r$. For the second possibility, $|X| \geq r$

thus X is a subclass or crossing, and, appropriately, R is monochromatic in $\mathcal{P}_0$, or polychromatic in $\mathcal{P}$. Both cases are impossible, proving the assertion.

For the 'only if' part, first we prove some claims. We begin with the second condition of the Main Lemma.

Claim 1 *If $\mathcal{P}$ is an alternative, then each of its big classes has property (1) or (2).*

Proof. Otherwise, let us pick a class X of $\mathcal{P}$, satisfying neither (1) nor (2). Let c, c' be two common vertices of X and of some class $X_0 \in \mathcal{P}_0$. There exists a Y_0 in $\mathcal{P}_0$, different from X_0, intersecting X in some vertex c'', since $r \geq 3$. Extending the set $\{c, c', c''\}$ to an r-element subset of X, we get an edge of $\mathcal{H}(\mathcal{P}_0)$ which is not properly colored by $\mathcal{P}$, and hence $\mathcal{P}$ is not an alternative. This contradiction proves Claim 1. □

Claim 2 *If $|\mathcal{P}| \geq r$ and $\mathcal{P}$ is an alternative, then $\mathcal{P}_0$ is its refinement, i.e. every color class of $\mathcal{P}_0$ is a subset of some color class of $\mathcal{P}$.*

Proof. Otherwise we find a color class X_0 in $\mathcal{P}_0$ and two classes X and Y in $\mathcal{P}$ which intersect X_0. Now we prove that for all such X_0, X, and Y, necessarily every class in $\mathcal{P}$ different from X and Y is contained in X_0. Suppose not, and let some F in $\mathcal{P}$ have some vertex $f \notin X_0$. Let c and d be a common vertex of X_0 with X and Y, respectively. Then the triple $\{c, d, f\}$ can be extended to an r-tuple which is properly colored by $\mathcal{P}_0$ but not by $\mathcal{P}$. So, for the original X and Y, this statement is true and because of $r \geq 3$, there exists some third class X' in $\mathcal{P}$ which is contained in X_0. The set X_0 cannot cover the whole V, thus at least one among X and Y does have some vertices outside X_0. Say, this class is X. Then, taking X' instead of X, and repeatedly applying the statement above, we obtain that $X \subseteq X_0$, contrary to the choice of X. This proves Claim 2. □

Claim 3 *All alternatives different from $\mathcal{P}_0$ have at most $r-1$ color classes.*

Proof. Otherwise, by Claim 2, $\mathcal{P}_0$ is a refinement of $\mathcal{P}$. They are different, so $\mathcal{P}$ has some class X which is the union of several $\mathcal{P}_0$-classes. If $|X| < r$, then we violate the Union Condition, and if $|X| \geq r$ then, by Claim 1, it has to be a subclass or crossing. The former is impossible, while the latter implies that all the $\mathcal{P}_0$-classes contained in X have exactly one element, contradicting the Union Condition. □

By the claims above, the 'only if' part, and thus the whole Main Lemma, is proved. □

2.2 The Extension Lemma and unique $(r-1)$-colorings

In the next two sections, in two lemmas (Lemmas 3 and 4, the "implication lemmas"), we will show how the uniqueness of a coloring can be derived from that of another one. Furthermore, in this section, we characterize the unique $r-1$-colorings.

First we introduce the following concept.

Strong Union Condition: Every pair of color classes has a union of size at least $2r-1$.

Lemma 1 *If a unique coloring has exactly $r-1$ color classes, then it satisfies the strong union condition.*

Proof. Otherwise, let us consider two color classes X and Y with union of size at most $2r-2$. Taking another coloring $\mathcal{P}'$ by splitting the union into two classes of sizes at most $r-1$, the new coloring will be an alternative by the Main Lemma. (Since $r \geq 3$, a set of $2r-2$ elements admits more than one splitting.) □

The next lemma will be used repeatedly.

Lemma 2 *If a unique coloring has exactly $r-1$ color classes, then one of the following two cases occurs:*

- *all of its color classes are of size at least r, or*
- *one color class is of size $r-1$ and the others are of size $\geq r$.*

Proof. Consider the unique coloring. First, suppose there exists some color class X of size at most $r-2$ in it. By the Size Condition, we have some color class Y of cardinality at least r. Putting a vertex from Y into X, we get another coloring which is an alternative by the Main Lemma, a contradiction.

Thus we may now assume that we have more than one class of size $r-1$. Then the strong union condition is not satisfied, contradicting Lemma 1. □

Definition 4 To extend a color class C, means to replace it by a set $C' \supset C$ such that every $v \in C' - C$ is a new vertex. Of course, in this way the underlying set will become larger, too.

Lemma 3 *If we extend a color class of a unique coloring (using any number of colors), then the new coloring will be unique, too.*

Proof. Let the original coloring be $\mathcal{P}$ and the new one be $\mathcal{P}'$. We may assume that $\mathcal{P}'$ is obtained from $\mathcal{P}$ by adding one vertex f to a color class F of $\mathcal{P}$. Suppose for a contradiction that $\mathcal{P}'$ is not unique and there exists an alternative $\widetilde{\mathcal{P}} \neq \mathcal{P}'$.

The Main Lemma can be applied for $\mathcal{P}'$ and $\widetilde{\mathcal{P}}$. So, $\widetilde{\mathcal{P}}$ has fewer than r color classes, and every class of at least r elements either is contained in some class of $\mathcal{P}'$ or is crossing. The "trace" $\mathcal{TR}$ of $\widetilde{\mathcal{P}}$ on $\mathcal{P}$ is defined as the following coloring:

$$\mathcal{TR} = \{T = \widetilde{X} \cap V : \widetilde{X} \in \widetilde{\mathcal{P}}, T \neq \emptyset\}$$

where V denotes the set of vertices of $\mathcal{P}$.

Obviously, $|\mathcal{TR}| < r$. Let T be a class in $\mathcal{TR}$, with $|T| \geq r$. Let T be the trace of some $X \in \widetilde{\mathcal{P}}$, then $|X| \geq r$, consequently X is a "subclass" or crossing. In the first case, T will be contained in some color class of $\mathcal{P}$ as well, while in the second case,

T will be crossing in $\mathcal{P}$. So, $\mathcal{TR}$ is an alternative of $\mathcal{P}$, but the latter is supposed to be unique, thus the only possibility is $\mathcal{TR} = \mathcal{P}$.

Thus $|\mathcal{P}| < r$, so that $|\mathcal{P}| = r - 1$.

Furthermore, $\mathcal{TR} = \mathcal{P}$ implies that $\widetilde{\mathcal{P}}$ can be obtained by adding f to a color class Z of $\mathcal{P}$ different from F. This Z is the trace of a color class $\widetilde{Z}$ in $\widetilde{\mathcal{P}}$. By Lemma 2, $|Z| \geq r - 1$ so that $|\widetilde{Z}| \geq r$. Thus by the Main Lemma, $\widetilde{Z}$ is either a subclass or crossing in $\mathcal{P}'$. It is clearly not a subclass, and if it is crossing then $|Z| = 1$ which is a contradiction since we have seen that $|Z| \geq r - 1 \geq 2$. □

In the next section we shall give another implication lemma, for the addition of color classes.

Now we give the characterization of unique colorings with exactly $r - 1$ colors, which is fairly simple.

Theorem 3 *A coloring with exactly $r - 1$ colors is unique if and only if one of the following two cases occurs:*
– *all of its color classes are of size at least r, or*
– *one color class is of size $r - 1$ and the others are of $\geq r$.*

Proof *"Only if":* This is just Lemma 2.
"If": By Lemma 3, it is enough to prove the statement for the coloring $\mathcal{P}_0$ with one color class of size $r - 1$ and with the other classes of size exactly r. Take an alternative $\mathcal{P}$. Since there are no crossing classes, every big class of $\mathcal{P}$ coincides with some class in $\mathcal{P}_0$. If $\mathcal{P}$ has more than one "small" class, then it does not cover the underlying set; and otherwise it is equal to $\mathcal{P}_0$. □

2.3 The Addition Lemma

In this section we prove a lemma on the effect of adding a color class to a coloring. First we prove a claim.

Claim 4 *Let the coloring $\mathcal{Q}$ with r colors have a 1-element color class and another class of $r-1$ vertices, and let all the remaining classes have exactly r vertices. Then $\mathcal{Q}$ is unique.*

Proof. Let V be the underlying set of $\mathcal{Q}$. Let us consider an alternative $\mathcal{C}$ different from $\mathcal{Q}$, if any exists. The Main Lemma implies that $|\mathcal{C}| \leq r - 1$ and every class of $\mathcal{C}$ has at most r vertices. At the same time, $|V| = (r - 1)r$. Therefore, each class in $\mathcal{C}$ must have exactly r vertices. All the subclasses have to be identical to the color classes of $\mathcal{C}$ containing them, and so there are two cases: all the classes of $\mathcal{C}$ are subclasses or all of them are crossing. The first case is obviously impossible ($\mathcal{C} = \mathcal{Q}$), and the second one is also excluded since the presence of a 1-element color class in $\mathcal{Q}$ does not allow more than one crossing class in $\mathcal{C}$. □

Remark $\mathcal{Q}$ is an example for a unique coloring which does not satisfy the Strong Union Condition.

Lemma 4 *If we add a new color class to a unique coloring in such a way that the Union Condition remains valid, then the new coloring will be unique, too.*

Proof. Let the original coloring be $\mathcal{P}$ and the new one be $\mathcal{P}'$. Suppose that $\mathcal{P}'$ is not unique and there exists an alternative $\widetilde{\mathcal{P}} \neq \mathcal{P}'$. The basic conditions are satisfied by $\mathcal{P}'$ (since we assumed the Union Condition for it, while the Size Condition holds already for $\mathcal{P}$) so that the Main Lemma can be applied. Similarly to the proof of Lemma 3, the trace $\mathcal{TR}$ can be defined, and $\mathcal{TR}$ will be an alternative of $\mathcal{P}$. Since the latter is unique, the only possibility is $\mathcal{TR} = \mathcal{P}$, and thus $|\mathcal{P}| = |\mathcal{TR}| < r$. Now consider a coloring $\mathcal{P}'$ from a unique coloring $\mathcal{P}$ having $r-1$ colors, applying Lemma 2 we know the possible forms of $\mathcal{P}$. It follows, in particular, that $\mathcal{P}'$ can be constructed by extending the color classes of the coloring $\mathcal{Q}$ in Claim 4. So, Lemma 3 can be applied and the uniqueness of $\mathcal{P}'$ is verified. □

2.4 More than $r-1$ colors

In this section we characterize unique colorings for at least r colors, and give a slightly simpler result for the particular case of exactly r colors.

We need some preliminaries for obtaining the main result, Theorem 4 below.

Throughout this section $\mathcal{P}_0$ is assumed to satisfy the Basic Conditions.

For a fixed coloring $\mathcal{P}_0$, and an alternative $\mathcal{P}$ of it, the three parts of $\mathcal{P}$ can be seen in Figure 3: the big subclasses, the big crossing classes and the union of small classes. Their number will be denoted by g, c and s, respectively. The number of big classes in $\mathcal{P}_0$ is denoted by t.

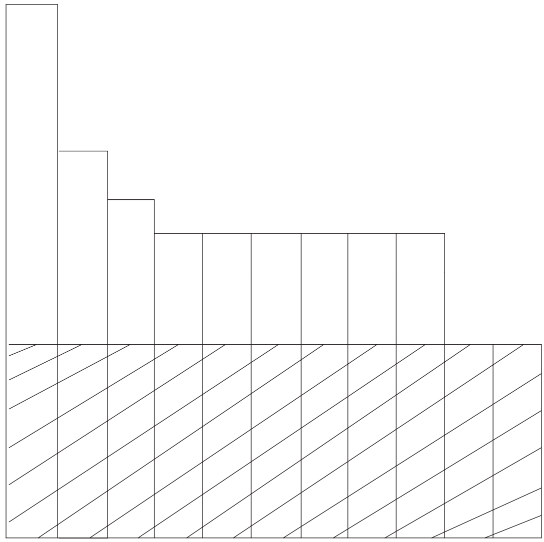
Claim 5 *We may assume that in $\mathcal{P}$, every big subclass is identical to the class in $\mathcal{P}_0$ containing it.*

Proof. If a given subclass is a proper subset of a color class P_0 in $\mathcal{P}_0$ and we change it to P_0, then some crossing classes will decrease by 1 and some small classes will become one smaller, which means (by the Main Lemma) that the coloring will remain an alternative. (Some classes of $\mathcal{P}$ may become empty.) If necessary, we execute this transformation several times.

The only problem could be if the new coloring is identical to $\mathcal{P}_0$. But the transformation does not increase the number of colors, thus the original coloring $\mathcal{P}$ would have at least r colors, which contradicts the Main Lemma. □

Claim 6 *We may assume that the big subclasses are the largest parts $X_1, ...X_g$.*

Proof. Otherwise, suppose we have a $P = X_i \in \mathcal{P}$ and for some $j < i$, $B = X_j$ has no big subclass. Let us change P to B and for all $P' \in \mathcal{P}$ intersecting B, replace $P' \cap B$ with a subset $S' \subseteq X_i$, such that $|S'| \leq |P' \cap B|$ and the modified classes cover X_i. So we get an alternative of $\mathcal{P}_0$, by the arguments above. After using this transformation possibly several times, the assumption stated by Claim 6 will be true for the alternative obtained and this alternative will differ from $\mathcal{P}_0$, similarly as above. □



$g = 0$

Figure 1: No big subclasses, 5 big crossing classes.

Let N denote the union of the big crossing classes and the small classes of $\mathcal{P}$. In other words, N is the union of the set system

$$\mathcal{P}_0 \setminus \{X \in \mathcal{P}_0 \cap \mathcal{P}, X \text{ is big}\}.$$

In Figures 1, 2 and 3, we have drawn the classes of $\mathcal{P}_0$ in decreasing order of their cardinalities as columns: $X_1, X_2, \ldots, X_k$. The height of each X_i is proportional to its size.

For $g \leq k - r$ let $h = h(g) = |X_{g+r}|$ and for $g > k - r$ let $h(g) = 0$. We may assume that the big crossing classes of $\mathcal{P}$ will be sets of the form

$$Y_i := \{v \in N : \text{the height of } v \text{ is } i\}$$

with some $1 \leq i \leq h$. Thus, the maximum possible number c of big crossing classes is $h(g)$. Furthermore, g (the number of big subclasses in $\mathcal{P}$) clearly determines $c = c_g$.

For $g > k - r$, c is necessarily 0.

In Figure 3 also we have drawn a possible alternative coloring $\mathcal{P}$ of $\mathcal{P}_0$. In Figures 1 and 2 we have shown the colorings determined by $g = 0$ and $g = t$. These colorings are not alternatives of $\mathcal{P}_0$. All the three figures correspond to the example below.

Let us define the set of "remaining vertices":

$$U = U_g := V - \bigcup_{j=1}^{g} X_j - \bigcup_{i=1}^{c} Y_i.$$

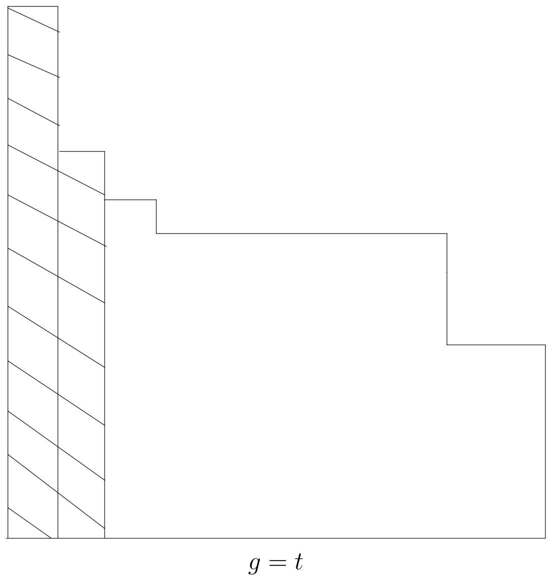


Figure 2: 2 big subclasses, no big crossing classes.

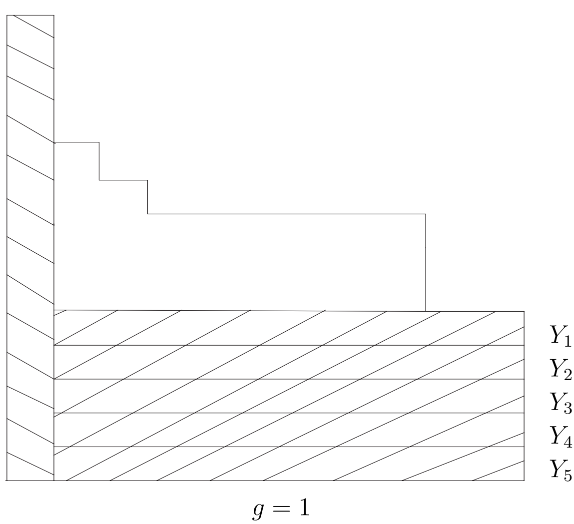


Figure 3: 1 big subclass and 5 big crossing classes

(For $c = 0$, $\bigcup_{i=1}^{c} Y_i = \emptyset$, of course.)

If we fix g, the problem is to cover the set U with small classes of $\mathcal{P}$. By the Main Lemma, we have that $|\mathcal{P}| \leq r-1$, thus we can use at most $r-1-g-c$ classes for them. Therefore, if $\mathcal{P}_0$ is not unique, then for at least one g, the inequality

$$|U| \leq (r-1)(r-1-g-c) \tag{3}$$

must hold. And also conversely, if (3) holds for some g, then an appropriate alternative $\mathcal{P} = \mathcal{P}_g$ exists and $\mathcal{P}_0$ is not unique. In this way, we have proved the following result :

Theorem 4 *Suppose that the partition $\mathcal{P}_0$ with $k \geq r$ classes satisfies the Basic Conditions. Then $\mathcal{P}_0$ is not unique if and only if there exists some $0 \leq g \leq t$ such that the inequality (3) is valid.*

Example Observe that g cannot be restricted to the values 0 and t in Theorem 4.

Using the same notation as above, let $r \geq 10$ be an even number, $X_1 = 3r/2$, $X_2 = r$, $X_3 = X_4 = ... = X_{r/2-2} = r-1$, $X_{r/2-1} = ...X_{r-1} = r-2$, finally $X_r = X_{r+1} = r/2$. Here the values 0 and t do not give any alternative different from $\mathcal{P}_0$, while the value $g = 1$ does. (Figures 1, 2 and 3 show these cases for $r = 10$.)

For the particular case of $k = r$, the situation is much simpler:

Theorem 5 *Let $\mathcal{P}_0$ with $k = r$ satisfy the Basic Conditions. Then $\mathcal{P}_0$ is not unique if and only if for $g = 0$ or $g = t$, the inequality (3) is valid.*

Problem Is there a nice analogue of Theorem 3 for more than r colors?

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