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The Fibonacci hypercube

Fred J. Rispoli

Department of Mathematics and Computer Science
Dowling College, Oakdale, NY 11769
U.S.A.

Steven Cosares

Department of Business Computer Information Systems
Hofstra University, Hempstead, NY 11549
U.S.A.

Abstract

The Fibonacci Hypercube is defined as the polytope determined by the convex hull of the "Fibonacci" strings, i.e., binary strings of length n having no consecutive ones. We obtain an efficient characterization of vertex adjacency and use this to study the graph of the Fibonacci Hypercube. In particular we discuss a decomposition of the graph into self-similar subgraphs that are also graphs of Fibonacci hypercubes of lower dimension, we obtain vertex degrees, a recurrence formula for the number of edges, show that the graph is Hamiltonian and study some additional connectivity properties. We conclude with some related open problems.

1 Introduction

For each positive integer n, a *Fibonacci string of order* n is defined to be a binary string of length n having no two consecutive ones, and V_n denotes the set of all Fibonacci strings of order n. Constructing a graph with vertices V_n was introduced by Hsu [2], who defined the *Fibonacci Cube* as the subgraph of the n-cube Q_n with vertices V_n, where two vertices are adjacent if and only if their Hamming distance is 1. Hsu was motivated by the possibility of using the Fibonacci Cube as a interconnection topology for multicomputers. Here, we consider the Fibonacci strings as n-dimensional $\{0, 1\}$-vectors in $\mathbb{R}^n$, and define the n-dimensional *Fibonacci Hypercube* as the convex hull of the elements in V_n. The graph of this polytope, denoted by FQ_n, consists of the vertices V_n together with edges of the Fibonacci Hypercube. Illustrations of FQ_3 and FQ_4 are given in Figure 1. Observe that a Fibonacci Hypercube consists of a Fibonacci Cube with some additional edges. Moreover, the Fibonacci Hypercube is a special case of the Fibonacci Polytopes investigated by Rispoli [5].

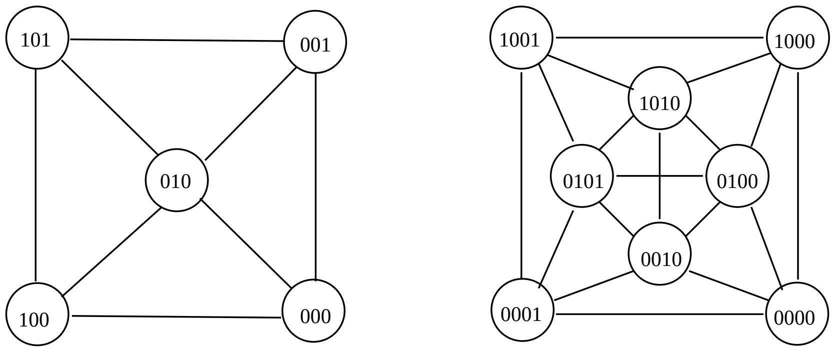

Figure 1: The graphs FQ_3 and FQ_4

The main result of this paper is a vertex adjacency citerion for FQ_n of two Fibonacci strings in terms of their bits. It tells us that two vertices $x = \{x_1, x_2, ..., x_n\}$ and $y = \{y_1, y_2, ..., y_n\}$ are adjacent if and only if all the coordinates $\{i : x_i \neq y_i\}$ make a subsequence, consisting of consecutive elements of the sequence $\{1, 2, ..., n\}$. The characterization is used to obtain a decomposition of FQ_n into a subgraph isomorphic to FQ_{n-1}, a subgraph isomorphic to FQ_{n-2}, plus some additional edges. This leads to a recurrence relation that can be used to compute the number of edges in FQ_n. In [2], the Fibonacci Cube was shown to preserve some (but not all) of the favorable connectivity qualities of the n-cube with respect to a communications network. For example, the Fibonacci Cube is neither Hamiltonian nor n-connected. This is because its edges represent only a subset of the edges that are in the convex hull of V_n. The graph FQ_n on the other hand, describes the entire Fibonacci Hypercube which is an n-dimensional polytope, so it boasts n-connectedness as one of its qualities. Furthermore, we constructively show that FQ_n is Hamiltonian.

The remainder of the paper is organized as follows. First we characterize vertex adjacency for FQ_n and obtain a formula for the degree of each vertex in the graph. Next we derive a recurrence relation for the number of edges in FQ_n, obtain the diameter of FQ_n, show that it is n-connected and contains a Hamilton circuit for all $n \geq 2$. We conclude this paper with a brief discussion of the edge expansion rate of FQ_n and identify some open problems related to FQ_n.

2 The vertices and edges of FQ_n

The Fibonacci numbers, denoted by F_n, are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. It is well known that $|V_n| = F_{n+2}$. Let $[n] = \{1, 2, \ldots, n\}$. Given a pair of vertices x, $y \in V_n$, let $D(x, y) = \{i \in [n] : x_i \neq y_i\}$. We define a *maximal* *run* to be a subset of $D(x,y)$ whose elements consist of consecutive integers and is inclusion-wise maximal (i.e., it is not contained in any larger set of consecutive integers.) For example, suppose that $x = 01010101$ and $y = 00100001$. Then $D(x,y) = \{2,3,4,6\}$, and $D(x,y)$ is the union of the two maximal runs $\{2,3,4\}$ and $\{6\}$. We point out that a maximal run corresponds to a substring of alternating 0 and 1 bits within both x and y. Furthermore, a valid Fibonacci string can always be obtained by interchanging the 0s and 1s in either x or y along the full extent of any maximal run.

Given any convex polytope P, two vertices x and y of P are *adjacent* if and only if for every $0 < \lambda < 1$, the point $\lambda x + (1-\lambda)y$ cannot be expressed as a convex combination of any other points in P. For a reference on convex polytopes, see [1] or [7].

Proposition 1 *Two vertices x and y are adjacent in the Fibonacci Hypercube if and only if $D(x,y)$ consists of a single maximal run.*

Proof. Let $x \neq y$ be two Fibonacci strings and suppose that $D(x,y)$ consists of two or more maximal runs. Let R_1 and R_2 be any two of these runs. Construct the Fibonacci string u from x by interchanging the bits along R_1, and similarly, construct v from y by interchanging bits along R_2. Now, observe that $\frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}u + \frac{1}{2}v$, and hence, x and y are not adjacent.

Now, suppose that $D(x,y)$ consists of exactly one maximal run, say $D(x,y) = \{p, p+1, p+2, ..., p+q\}$ where $p \in [n]$ and $q \geq 0$. If $q = 0$, then the Hamming distance between x and y is 1, which implies that x and y must be adjacent in the Fibonacci Hypercube. So assume that $q \geq 1$. Suppose also that there exists some subset of Fibonacci strings z^1, ..., z^m, positive reals $\alpha_1, ..., \alpha_m$ satisfying $\sum_{j=1}^{m} \alpha_j = 1$, and some λ such that $0 < \lambda < 1$ and $\sum_{j=1}^{m} \alpha_j z^j = \lambda x + (1-\lambda)y$. Notice that $\alpha_j > 0$, for every $j = 1, 2, ..., m$, implies that if $x_i = y_i = 0$, for some $i \in [n]$, then $z_i^j = 0$, for every j. In addition, $\sum_{j=1}^{m} \alpha_j = 1$ implies that if $x_i = y_i = 1$, for some $i \in [n]$, then $z_i^j = 1$, for every j. Consequently, for all $i < p$ and all $i > p+q$, we have that $x_i = y_i = z_i^1 = z_i^2 = ... = z_i^m$. Since $D(x,y) = \{p, p+1, p+2, ..., p+q\}$, without loss of generality, we may assume that $x_p = 1$, $y_p = 0$, and that both x and y have alternating 0's and 1's over the indices in $D(x,y)$. Therefore, $\lambda x_p + (1-\lambda)y_p = \lambda$ and $\lambda x_{p+1} + (1-\lambda)y_{p+1} = (1-\lambda)$.

For convenience, we relabel the z^j such that $z^1, z^2, ..., z^v$ have a one in the pth coordinate and $z^{v+1}, ..., z^m$ have a zero in the pth coordinate. Now, $\lambda = \sum_{j=1}^{v} \alpha_j z_p^j = \sum_{j=1}^{v} \alpha_j$. Since the z^j do not have consecutive ones, for $j = 1, 2, ...v$, we must have that $z_{p+1}^j = 0$. Hence, $(1-\lambda) = \sum_{j=1}^{m} \alpha_j z_{p+1}^j = \sum_{j=v+1}^{m} \alpha_j z_{p+1}^j$. Since $\sum_{j=1}^{m} \alpha_j = 1$ and $\sum_{j=1}^{v} \alpha_j =$

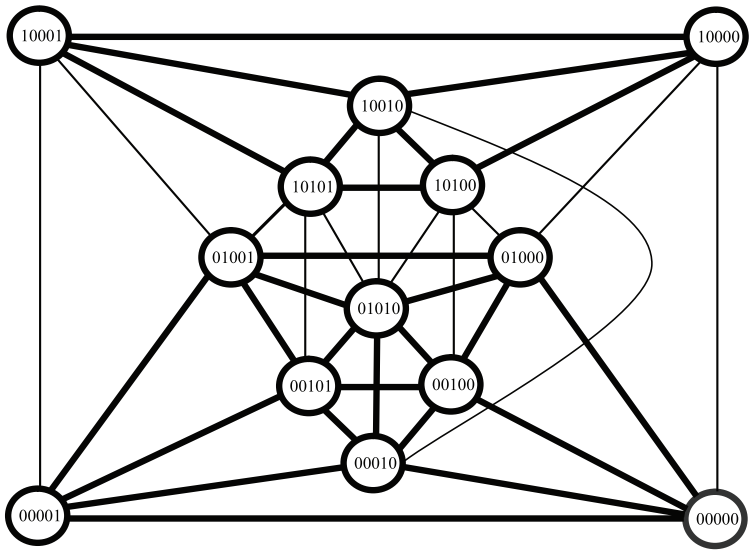

Figure 2: The Fibonacci Hypercube graph FQ_5. Thick edges illustrate the composition from FQ_3 and FQ_4.

λ, we know that $\sum_{j=v+1}^{m} \alpha_j = 1 - \lambda$. Therefore, $z^j_{p+1} = 1$, for $j = v + 1, ..., m$.

If $q \geq 2$, we may repeat the above argument observing the following: $\lambda x_{p+2} + (1 - \lambda)y_{p+2} = \lambda$, $\lambda x_{p+3} + (1 - \lambda)y_{p+3} = (1 - \lambda)$, $z^j_{p+2} = 1$, for $j = 1, 2, ...v$, and $z^j_{p+2} = 0$, for $j = v + 1, .., m$. This implies that $z^1 = z^2 = ... = z^v = x$ and $z^{v+1} = z^{v+2} = ... = z^m = y$, and consequently x and y are adjacent. $\square$

Figure 2 provides an illustration of FQ_5 and also indicates a decomposition. In particular, we can partition the vertices of FQ_5 into a subset of Fibonacci strings of the form (0,*) (i.e., strings that begin with 0) that induce a subgraph isomorphic to FQ_4, plus another subgraph with vertices with form (1,0,*) isomorphic to FQ_3. For suppose that x and y are Fibonacci strings in V_4 and $(0, x)$ and $(0, y)$ are the Fibonacci strings in V_5 starting with 0 followed by the bits in x and y respectively. Then $D(x, y) = D(\ (0, x), (0, y)\ )$. Similarly, if x and y are Fibonacci strings in V_3 and $(1, 0, x)$ and $(1, 0, y)$ are the Fibonacci strings in V_5 starting with 10 followed by the bits in x and y respectively. Then $D(x, y) = D(\ (1, 0, x), (1, 0, y)\ )$. Hence, the adjacency structure within the subgraph in FQ_5 induced by strings starting with 0 will be isomorphic to FQ_4, and the subgraph induced by strings starting with 10 in FQ_5 will be isomorphic to FQ_3.

In general, let H_0 and H_1 be the subgraphs of FQ_n induced by vertices starting with 0 and 10 respectively. Then we have the following isomorphisms: $H_0 \simeq FQ_{n-1}$

and $H_1 \simeq FQ_{n-2}$. Thus the total number of edges in FQ_n include the edges from these two graphs, plus edges existing between the two subgraphs, i.e., those connecting a vertex of the form (1,0,*) to a vertex of the form (0,*). This can be characterized by the following Proposition.

Proposition 2. (a) *For $n \geq 3$, the number of edges in FQ_n satisfies the non-homogeneous recurrence relation $E_n = E_{n-1} + E_{n-2} + F_{n+2} - 1$, where $E_1 = 1$, $E_2 = 3$.*

(b) *For $n \geq 6$, the number of edges in FQ_n satisfies $E_n > F_{n+4}$.*

Proof. If there exists an edge between vertices, $x \in H_0$ and $y \in H_1$, which differ in the first bit, then $\{1\}$ must be a subset of the single maximal run in $D(x, y)$. Thus, if this run is $\{1, 2, ...k\}$, for some $k \geq 1$, x takes the form (0101.., s) and y takes the form (1010..., s), or vice versa, where s is a Fibonacci substring that starts with a 0, or it is the empty string if $k = n$. When $k = n$, the number of such adjacent vertex pairs is $1 = F_1$. When $k = n - 1$, $s = (0)$ and the number of associated adjacent vertex pairs is again $1 = F_2$. As k decreases, the number of valid s follows the Fibonacci sequence. Finally, for $k = 1$, the number of valid substrings s is equal to the number of Fibonacci strings of length $n - 2$, which is F_n. So the total number of edges between H_0 and H_1 is given by $\sum_{j=1}^{n} F_j$, which, by a well known identity, is equal to $F_{n+2} - 1$.

(b) The proof is by induction. For $n = 6$ we have that $E_6 = 76$ and $F_{10} = 55$. By part (a), $E_{n+1} = E_n + E_{n-1} + F_{n+3} - 1$. By the induction assumption, $E_n > F_{n+4}$ and $E_{n-1} > F_{n+3}$. Hence, $E_{n+1} > F_{n+4} + F_{n+3} + F_{n+3} - 1 > F_{n+5}$. □

It should be pointed out that since $|V_n| = F_{n+2}$ an alternative recurrence relation for the number of edges in FQ_n is given by $E_n = E_{n-1} + E_{n-2} + |V_n| - 1$. Proposition 2 allows us to compare the difference in the number of edges in the Fibonacci Hypercube over the Fibonacci Cube. If we let $\widehat{E}_n$ denote the number of edges in the Fibonacci Cube, then a recurrence relation for $\widehat{E}_n$, given in [2], is $\widehat{E}_n = \widehat{E}_{n-1} + \widehat{E}_{n-2} + F_n$, *where* $\widehat{E}_1 = 1$, $\widehat{E}_2 = 2$.

It is obvious from Figures 1 and 2, that FQ_n is not a regular graph. For a given vertex x, the degree may be as small as n, but could be much larger if there are vertices y for which $D(x, y)$ contains a single maximal run. We note from our earlier observations, that x and y are adjacent if their binary strings match everywhere except where x contains an alternating substring, e.g., (0101...) and y contains the complement substring (1010...). Thus the number of potential neighbors of x is maximal when all of its bits alternate. Otherwise, its degree is limited by the number of breaks in its alternating pattern, i.e., where a consecutive pair of bits have the same value (0). We define the *segments* of x to be the maximal sequences of alternating bits between these breaks. For example, $x = 100101000101$ contains the segments $10, 01010$, 0, and 0101.

Proposition 3 (a) *Let x be a Fibonacci string that contains $p \geq 2$ segments. Let*

k_i be the number of occurrences of the substring 010 in the i^{th} segment. Then the degree of vertex x in FQ_n is $n + \sum_{i=1}^{p} \binom{k_i+1}{2}$.

(b) *For every vertex $x \in V_n$ the degree of x satisfies $n \le deg(x) \le n + \binom{\lceil \frac{n}{2} \rceil}{2}$.*

Proof. (a) Let S represent some segment within the string for vertex x. Let $\overline{S}$ represent the alternating sequence of complement bits. One could obtain a vertex that is adjacent to x if S were replaced by $\overline{S}$. Suppose S were partitioned into a pair of non-empty substrings such that $S = S_1S_2$. If $|S| = m$, then there are $m-1$ such partitions. For each, another adjacent vertex is obtained by replacing S with exactly one of $\overline{S_1}S_2$ or $S_1\overline{S_2}$. Thus every segment of length m contributes m to the degree of x, for a subtotal of n.

Suppose S has k occurrences of the substring 010. Additional vertices adjacent to x can be obtained by replacing one such instance of 010 in S with 000. This gives k additional neighbors. Next we can take an occurrence of two consecutive 010 substrings, which has the form 01010, and replace it with 00100. This gives $k-1$ additional neighbors of x. We may continue to group the 010 substrings, three at at time, then four at a time, and so on. Hence each segment contributes an additional $k + (k-1) + (k-2) + \ ... + 1 = \binom{k+1}{2}$ to the degree of x.

(b) The lower bound on the degree of x is obvious. The upper bound follows from the fact that $\sum_{i=1}^{p} \binom{k_i+1}{2}$ is maximum when $p = 1$ and $k_1 = \lceil \frac{n}{2} \rceil - 1$. □

3 Connectivity Properties

Given a graph G the *distance* between any pair of vertices is the number of edges in a shortest path joining the vertices. The *diameter* of G is the maximum distance among all pairs of vertices. A *Hamilton circuit* is a circuit that visits every vertex in G exactly once, and a *Hamilton path* is a path in G that visits every vertex exactly once. The *edge connectivity* $\lambda(G)$of a connected graph G is the smallest number of edges whose removal disconnects G. When $\lambda(G) \ge k$, the graph G is called *k-connected.* For example we can see from Figure 1 that FQ_4 is 4-connected. For more details on basic graph terminology, see [6].

Proposition 4 (a) *The distance between a pair of vertices x and y in FQ_n, is equal to the number of maximal runs in $D(x,y)$.*
(b) *The diameter of FQ_n is $\lceil \frac{n}{2} \rceil$.*
(c) *For every $n \ge 2, FQ_n$ is n-connected.*

Proof. (a) Suppose that there are $p \ge 2$ maximal runs in $D(x,y)$ and denote these by $R_1, R_2, ..., R_p$. Recall from our previous discussion that a vertex x^1, adjacent to x, can be obtained by interchanging the 0 and 1 bits along the full extent of any maximal run, e.g., R_1. Hence, we can construct a path in FQ_n, from x to x^1 to x^2 to ... to x^{p-1} to $x^p = y$, where x^j is the vertex associated with exchanging the bits from x^{j-1} along run R_j. Thus the distance from x to y is p.

(b) Observe that the number of maximal runs in $D(x, y)$ is at most $\lceil \frac{n}{2} \rceil$, which occurs when $D(x, y)$ consists of every other integer, {1,3,5,...}, i.e., when $x = 0000\ldots$ and $y = 10101\ldots$ In this case, the distance is $\lceil \frac{n}{2} \rceil$.

(c) The result follows from Balinski's Theorem (see [7]), which states that the graph of any n-dimensional polyhedron is n-connected. □

Next we show that for every $n \geq 2$, FQ_n contains a Hamilton path joining the vertices 00...0 to 10...0. Since these vertices are adjacent, this fact implies the existence of a Hamilton circuit in FQ_n. The proof uses the following two base cases to anchor the induction. For $n = 2$ and $n = 3$ we have the Hamilton paths:

$$00 \rightarrow 01 \rightarrow 10 \qquad \text{and} \qquad 000 \rightarrow 001 \rightarrow 010 \rightarrow 101 \rightarrow 100$$

Suppose that FQ_2, FQ_3, ..., FQ_n all contain Hamilton paths from 00...0 to 10...0, for some $n \geq 3$. Consider FQ_{n+1}. By the inductive hypothesis, there exist Hamilton paths P_1 in FQ_n and P_2 in FQ_{n-1} with both paths joining 0...0 to 10...0. Let $\overline{P_1}$ be the path in FQ_{n+1} obtained from P_1 by concatenating a 0 on the left of all bit strings in V_n. Then $\overline{P_1}$ joins 00...0 to 010...0. Let $\overline{P_2}$ be the path in FQ_{n+1} obtained from P_2 by concatenating 10 on the left of all bit strings in V_{n-1}. Then $\overline{P_2}$ joins 10...0 to 1010...0. Since 010...0 is adjacent to 1010...0 in FQ_{n+1}, the path obtained by following $\overline{P_1}$ from 00...0 to 010...0 and then $\overline{P_2}$ in reverse from 1010...0 to 10...0 is the desired Hamilton path in FQ_{n+1}. We have proved the following.

Proposition 5 *For every $n \geq 2, FQ_n$ contains a Hamilton circuit.*

The following Hamilton circuit for FQ_4 illustrates the method of proof.

$$0000 \rightarrow 0001 \rightarrow 0010 \rightarrow 0101 \rightarrow 0100 \rightarrow 1010 \rightarrow 1001 \rightarrow 1000 \rightarrow 0000$$

Next we consider the edge to vertex ratio rate of growth. By Proposition 2(a) and (b), for $n \geq 6$,

$$\frac{E_n}{V_n} = \frac{E_{n-1} + E_{n-2} + F_{n+2} - 1}{F_{n+2}} > \frac{F_{n+3} + 2F_{n+2} - 1}{F_{n+2}} > 3.$$

A related growth parameter that has been investigasted recently is the *edge expansion* of $G = (V, E)$, denoted $\chi(G)$, and defined as

$$\chi(G) = min \left\{ \frac{|\delta(U)|}{|U|} : U \subset V, U \neq \emptyset, |U| \leq \frac{|V|}{2} \right\}$$

where $\delta(U)$ is the set of all edges with one end node in U and the other one in $V - U$. The edge expansion rate for graphs of polytopes with 0-1 coordinates has been recently studied and is an important parameter for a variety of reasons ([4]). It is known that the hypercube, Q_n has edge expansion 1 [3]. It is easy to see that FQ_2, which is simply a triangle, has $\chi(FQ_2) = 2$, and from Figure 1, $\chi(FQ_3) = 2$.

Now consider FQ_4 which has 4 degree 4 nodes, and 4 degree 5 nodes. Since every vertex has degree at least 4, every pair of vertices U has $|\delta(U)| \geq 7$. Hence $\frac{|\delta(U)|}{|U|} \geq \frac{7}{2}$ for every subset with $|U| = 2$. If $|U| = 3$, then the sum of the degrees of vertices in U is at least 12. Furthermore, the subgraph induced by U can have at most 3 edges, implying that $|\delta(U)| \geq 12 - (3)(2) = 6$. Thus for any subset with $|U| = 3$, we have $\frac{|\delta(U)|}{|U|} \leq \frac{6}{3} = 2$. If $|U| = 4$, there is only one possible subset with 4 nodes of degree 4 which is $U = \{1001, 1000, 0001, 0000\}$. In this case we have $\frac{|\delta(U)|}{|U|} = \frac{8}{4} = 2$. Any other subset with 4 nodes must have at least 2 nodes of degree 5, so a sum of degrees of at least 18. In addition, any subgraph induced by a subset of 4 nodes other than $\{1001, 1000, 0001, 0000\}$ contains at most 5 edges. Therefore $|\delta(U)| \geq 18-(5)(2) = 8$ and $\frac{|\delta(U)|}{|U|} \geq \frac{8}{4} = 2$, for every subset with $|U| = 4$. This shows that $\chi(FQ_4) = 2$.

Proposition 6 (a) *For* $n \geq 6$, *the edge to vertex ratio satisfies* $\frac{E_n}{V_n} > 3$.
(b) *For every* $n \geq 5$, *the edge expansion of* FQ_n *satisfies* $\chi(FQ_n) < 1+\phi$, *where* $\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$.

Proof The proof of (a) is given above. For (b) consider FQ_n and the cut created from the decomposition $FQ_n = FQ_{n-1} \cup FQ_{n-2} \cup U$ where U is the set of $F_{n+2} - 1$ edges described in the proof of Proposition 2(a). Since FQ_{n-2} has F_n vertices and FQ_{n-1} has F_{n+1} vertices, we have

$$\chi(FQ_n) \leq \frac{F_{n+2} - 1}{F_n} = \frac{F_{n+1} + F_n - 1}{F_n} = 1 + \frac{F_{n+1} - 1}{F_n} < 1 + \frac{F_{n+1}}{F_n} < 1 + \phi.$$

4 Conclusions

In this paper we have introduced a new graph called the Fibonacci Hypercube. The graph is easy to describe and arises naturally in a geometric context. Figures 1 and 2 illustrate the graphs for dimensions 3, 4 and 5 and show that these graphs may be drawn in a symmetric manner. The graph also exhibits many important connectivity properties which may make it useful as a communications network in the same sense as the Fibonacci Cube [2]. In the previous section we showed that the expansion rate is bounded above by $1 + \phi$. As for lower bounds for $\chi(FQ_n)$, we know that $\chi(FQ_n) = 2$, for $n = 2, 3$ and 4. By using the decomposition of FQ_5 described earlier, one can obtain an enumerative proof showing that $\chi(FQ_5) = \frac{13}{6}$. The key step is when considering subsets with 6 nodes, examine the cases with k nodes in H_0 and $6 - k$ nodes in H_1, where $k = 0, 1, 2, 3, 4, 5$. The authors conjecture that for

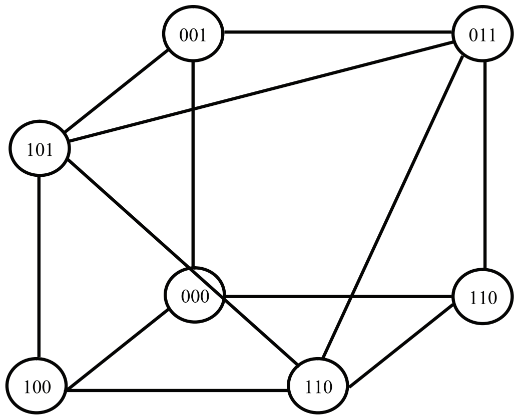

Figure 3: The Fibonacci 3-polytope of order 3.

every $n \geq 2$, FQ_n satisfies $\chi(FQ_n) \geq 2$. Moreover, as n approaches infinity, $\chi(FQ_n)$ approaches $1 + \phi$ from below.

In [5] the Fibonacci d-polytope of order k, denoted by $FP_d(k)$, is defined as the convex hull of the set of $\{0, 1\}$-vectors having d entries and no consecutive k ones. For example, the Fibonacci 3-polytope of order 3 is given in Figure 3. The Fibonacci Hypercube is the special case where $k = 2$. We may observe that from Figure 3 the adjacency criterion given in Proposition 1 fails for $k = 3$. In particular the vertices corresponding to $x = 110$ and $y = 011$ are adjacent, but $D(x, y) = \{1, 3\}$. In this situation adjacency must be checked using the definition of adjacency on a polytope given above. This leads to the following question.

Open Problem Find an efficient vertex adjacency criterion for $FP_d(k)$, when $k = 3$, and in general for all $k \geq 3$.

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