AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **64(3)** (2016), Pages 420–431

Taylor expansions for the m-Catalan numbers

Sheng-Liang Yang* Li-Juan Wang

Department of Applied Mathematics
Lanzhou University of Technology
Lanzhou, 730050, Gansu
P.R. China
slyang@lut.cn 903623012@qq.com

Abstract

By a Taylor expansion of a generating function, we mean that the remainder of the expansion is a functional of the generating function itself. In this paper, we consider the Taylor expansion for the generating function $\mathcal{B}_m(t)$ of the m-Catalan numbers. In order to give combinatorial interpretations of the coefficients of these expansions, we study a new collection of partial Grand Dyck paths, that is, (i, j)-balance m-Dyck paths, and we obtain some new Chung-Feller type results.

1 Introduction

The m-Catalan numbers $C_n^{(m)}$ are defined, for $m \geq 1$, by

$$C_n^{(m)} = \frac{1}{mn+1}\binom{mn+1}{n}, \ n \geq 0;$$

they are also called the Fuss-Catalan numbers [11, 15, 16, 20, 25]. It can easily be seen that the ordinary Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ are the special case $m = 2$, i.e., $C_n = C_n^{(2)}$. The generating function $\mathcal{B}_m(t) = \sum_{n=0}^{\infty} C_n^{(m)} t^n$ for the m-Catalan numbers is the so-called generalized binomial series (see [11]), and it satisfies the function equation

$$\mathcal{B}_m(t) = 1 + t\mathcal{B}_m(t)^m. \tag{1}$$

Equivalently, the m-Catalan numbers satisfy the recurrence

$$C_n^{(m)} = \sum_{i_1+i_2+\cdots+i_m=n-1} C_{i_1}^{(m)} C_{i_2}^{(m)} \ldots C_{i_m}^{(m)},$$

* Research was supported by the National Natural Science Foundation of China (Grant No. 11261032).

with initial condition $C_0^{(m)} = C_1^{(m)} = 1$.

Here we consider m-Catalan numbers as enumerators of m-Dyck paths (see [6, 15, 16, 18, 23]). For a positive integer m, an m-Dyck path of length mn is a path from the origin to $(mn, 0)$ using the steps $u = (1, 1)$ (i.e., north-east, up steps) and $d = (1, 1 - m)$ (i.e., south-east, down steps) and staying weakly above the x-axis.

It is well-known that the number of m-Dyck paths of length mn is given by the m-Catalan number $C_n^{(m)}$. In a lattice path, a descent is a maximal sequence of consecutive down steps, whereas an ascent is a maximal sequence of consecutive up steps. The points on the x-axis, except for the initial point of a lattice path, are called return points. We say that an m-Dyck path is prime if it has exactly one return point, i.e., the only vertices at the x-axis are the start and end points.

Every m-Dyck path can be decomposed into prime blocks according to its returns. When $m = 3$, the corresponding 3-Catalan numbers are

$$1,\ 1,\ 3,\ 12,\ 55,\ 273,\ 1428, \ldots,$$

and they count the number of 3-Dyck paths. In the literature, the 3-Dyck paths are often called ternary paths, so the Fuss-Catalan numbers $C_n^{(3)}$ are also called ternary numbers (see [8, 19]).

A Grand m-Dyck path of length mn is a lattice path from the origin to $(mn, 0)$, consisting of up-steps $u = (1, 1)$ and down-steps $d = (1, 1 - m)$, but without the requirement of staying above the x-axis. We define a negative m-Dyck path to be a Grand m-Dyck path of nonzero length which has no vertices with positive y-coordinates. A partial Grand m-Dyck path is just a Grand m-Dyck path but without the requirement of ending on the x-axis. Every up step of a partial Grand m-Dyck path α which lies below the x-axis is called a flaw of α. The number of flaws of α is denoted by $p(\alpha)$. Every partial Grand m-Dyck path can be encoded by a word in a language $\mathbb{S}$ on the alphabet $\{u, d\}$.

The classical Chung-Feller theorem [7, 9, 10] states that the number of Grand Dyck paths of length $2n$ with j flaws is independent of j and is equal to the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. For example, with $n = 2$, of the 6 paths consisting of 2 up and 2 down steps, 2 paths have no flaw: *uudd*, *udud*; 2 paths have 1 flaw: *uddu*, *duud*; and 2 paths have 2 flaws: *dduu*, *dudu*.

Eu, Liu and Yeh [9] proved that the generating function of the Catalan numbers satisfies

$$C(t) = \sum_{i=0}^{n-1} C_i t^i + t^n F_n(C(t)),\ n \geq 1, \tag{2}$$

where $F_n(x) = \sum_{k=1}^{n} \frac{k}{2n-k}\binom{2n-k}{n-k} x^{k+1}$. This is called the n-th Taylor expansion of $C(t)$, and the expansion provides a simple proof for the classical Chung-Feller theorem. In Eu, Fu and Yeh [9], a strengthening of the classical Chung-Feller theorem is obtained.

In this paper, we consider the Taylor expansion for the generating function $\mathcal{B}_m(t)$ of the m-Catalan numbers. In order to give combinatorial interpretations of the coefficients of these expansions, we study a new collection of partial Grand Dyck paths, that is, (i,j)-balance m-Dyck paths, and obtain some new Chung-Feller type results.

2 Taylor expansions for the m-Catalan numbers

By a Taylor expansion of a generating function, we mean that the remainder of the expansion is a functional of the generating function itself. For example, the n-th Taylor expansion of $C(t)$ in equation (2) can be expressed in the form [9]

$$C(t) = \sum_{i=0}^{n-1} C_i t^i + t^n \sum_{k=0}^{n-1} a(n-1,k)C(t)^{k+2},$$

where the coefficient $a(n,k)$ is the number of Dyck paths from $(0,0)$ to $(2n+2,0)$ that begin with an ascent of length $k+1$, and it is also the number of Dyck paths of semi-length $n+1$ with $k+1$ returns. The matrix $A = (a(n,k))$ is precisely the Catalan triangle [17]. In order to provide a combinatorial interpretation for coefficients in the remainder of the Taylor expansion of $\mathcal{B}_m(t)$, we need to introduce a new matrix which is the ECO matrix of m-Dyck paths.

The ECO (Enumerating Combinatorial Objects) method is a constructive method to produce all the objects of a given class, according to the growth of a certain parameter (in terms of the size) of the objects. The core of the ECO method is a recursive description of a class of combinatorial objects. This should be done in such a way that, if $\mathcal{O}_n$ denotes the set of objects of size n, each object $O' \in \mathcal{O}_{n+1}$ is achieved from one and only one object $O \in \mathcal{O}_n$. We say that O' is a successor of O.

We assign a label (k), $k \in \mathbb{N}^+$ (the positive integers), to each object. An object's label gives the number of successors of that object. The succession rule dictates the labels of these successors. The rule also includes an axiom (a), $a \in \mathbb{N}^+$, which gives the label of the smallest object. In the basic case a succession rule is written as

$$\Omega : \begin{cases} (a), \\ (k) \rightsquigarrow (e_1(k))(e_1(k))\dots(e_k(k)). \end{cases} \tag{3}$$

where $e_i(k) : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ gives the labels of the k successors of an object with label k. It is natural to think of a succession rule as a generating tree: the root of the tree is the axiom, (a), and if a node has label (k), it has k children labelled $e_1(k)$, $e_2(k)$, …, $e_k(k)$. The production matrix $P = (p_{i,j})_{i,j\geq 0}$ of a succession rule is defined as follows. List all labels of a rule. Let this list be l_0, l_1, …. Then $p_{i,j}$ equals the number of successors with label l_j produced by an object with label l_i. A succession rule defines an infinite matrix F whose (n,k) entry represents the number of nodes labelled l_k at level n. The matrix F is called the ECO matrix associated with Ω. Obviously F contains all the enumerative information provided by the succession rule Ω. The main references concerning these topics are [2,3], whereas for the applications of the ECO method we refer the reader to [1,4,14].

Now we apply the ECO method to m-Dyck paths. Let $\mathcal{GD}_n$ be the set of m-Dyck paths of length mn. Given $\alpha \in \mathcal{GD}_n$, we construct a set of paths of $\mathcal{GD}_{n+1}$ as follows: if the length of the first ascent of α is k, then we insert a peak of form $u_1 \ldots u_{m-1}d$, where $u_1 = \cdots = u_{m-1} = u$, into any point of the first ascent of α. The succession rule Ω describing the above construction is:

$$\Omega : \begin{cases} (m), \\ (k) \rightsquigarrow (m)(m+1)\ldots(k)(k+1)\ldots(k+m-1). \end{cases} \tag{4}$$

It corresponds to the production matrix $\mathcal{P}_m = (p_m(i,j))_{i,j\geq 0}$, where

$$p_m(i,j) = \begin{cases} 1, \text{ if } 0 \leq j \leq m+i-1, \\ 0, \text{ otherwise.} \end{cases}$$

The entries of the ECO matrix $\mathcal{A}_m = (a_m(i,j))_{i,j\geq 0}$ obey the recursion:

$$a_m(n+1,k) = a_m(n,k-m+1) + a_m(n,k-m+2) + \cdots + a_m(n,n(m-1)), \tag{5}$$

for all $n, k \geq 0$, with initial conditions $a_m(0,0) = 1$, and $a_m(0,k) = 0$ for all $k \neq 0$. Here we summarize the result as the following theorem.

Theorem 2.1. *The general entry $a_m(i,j)$ of the ECO matrix $\mathcal{A}_m$ is the number of m-Dyck paths from $(0,0)$ to $(mi+m,0)$ whose first ascent is of length $j+m-1$.*

Let $d_k(t)$ denote the generating function for the kth column of ECO matrix $\mathcal{A}_m$, i.e., $d_k(t) = \sum_{n=k}^{\infty} a_m(n,k)t^n$. Then, by the succession rule (4), we have $d_0(t) = \mathcal{B}_m(t)$, $d_1(t) = d_2(t) = \cdots = d_{m-1}(t) = \mathcal{B}_m(t) - 1 = t\mathcal{B}_m(t)^m$, and for $k \geq m$,

$$d_k(t) = t\mathcal{B}_m(t)d_{k-m+1}(t) + t\mathcal{B}_m(t)^2 d_{k-m+2}(t) + \cdots + t\mathcal{B}_m(t)^{m-1}d_{k-1}(t).$$

For example, the ECO matrix $\mathcal{A}_2$ is the Catalan triangle [6,13], and it is the sequence A033184 in [22]. The first few rows of the ECO matrix $\mathcal{A}_2$ are:

$$\mathcal{A}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 14 & 14 & 9 & 4 & 1 & 0 & 0 & 0 & 0 \\ 42 & 42 & 28 & 14 & 5 & 1 & 0 & 0 & 0 \\ 132 & 132 & 90 & 48 & 20 & 6 & 1 & 0 & 0 \\ 429 & 429 & 297 & 165 & 75 & 27 & 7 & 1 & 0 \\ 1430 & 1430 & 1001 & 572 & 275 & 110 & 35 & 8 & 1 \end{pmatrix}.$$

The first few rows of the array $\mathcal{A}_3$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 12 & 9 & 6 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 55 & 55 & 55 & 43 & 31 & 19 & 10 & 4 & 1 & 0 & 0 & 0 & 0 \\ 273 & 273 & 273 & 218 & 163 & 108 & 65 & 34 & 15 & 5 & 1 & 0 & 0 \\ 1428 & 1428 & 1428 & 1155 & 882 & 609 & 391 & 228 & 120 & 55 & 21 & 6 & 1 \end{pmatrix}.$$

Starting from the equation $\mathcal{B}_m(t) = 1 + t\mathcal{B}_m(t)^m$, we can obtain successively:

$$\begin{aligned} \mathcal{B}_m(t) &= 1 + t\mathcal{B}_m(t)^m = 1 + t + t^2 \sum_{k=1}^{m} \mathcal{B}_m(t)^{m+k-1} \\ &= 1 + t + mt^2 + t^3 (\sum_{k=1}^{m} m\mathcal{B}_m(t)^{m+k-1} + \sum_{k=m+1}^{2m-1} (2m-k)\mathcal{B}_m(t)^{m+k-1}). \end{aligned}$$

In general, we can obtain the Taylor expansion of $\mathcal{B}_m(t)$. The coefficients in the reminder of the expansion are the entries of ECO matrix $\mathcal{A}_m$.

Theorem 2.2. *The generating function of the m-Catalan numbers satisfies*

$$\mathcal{B}_m(t) = \sum_{i=0}^{n-1} C_i^{(m)} t^i + t^n \sum_{k=0}^{(m-1)(n-1)} a_m(n-1,k)\mathcal{B}_m(t)^{k+m}. \tag{6}$$

where $a_m(n,k)$ is the number of m-Dyck paths from $(0,0)$ to $(mn+m,0)$ that begin with exactly $k+m-1$ rise steps.

Proof. We suppose that $\mathcal{B}_m(t) = \sum_{i=0}^{n-1} C_i^{(m)} t^i + t^n \sum_{k=0}^{(m-1)(n-1)} b_m(n-1,k)\mathcal{B}_m(t)^{k+m}$. We need only show that the coefficients $b_m(n-1,k)$ satisfy the recurrence relation and the initial conditions as those of $a_m(n-1,k)$. By (1), we have $\mathcal{B}_m(t) = 1 + t\mathcal{B}_m(t)^m$. Multiplying the two sides of the equation by $\mathcal{B}_m(t)$ and substituting the the equation again, we get $\mathcal{B}_m(t)^2 = 1 + t(\mathcal{B}_m(t)^m + \mathcal{B}_m(t)^{m+1})$. Iterating this procedure, we have

$$\mathcal{B}_m(t)^k = 1 + t(\mathcal{B}_m(t)^m + \mathcal{B}_m(t)^{m+1} + \cdots + \mathcal{B}_m(t)^{m+k-1}).$$

Substituting this relation into

$$\mathcal{B}_m(t) = \sum_{i=0}^{n-1} C_i^{(m)} t^i + t^n \sum_{k=0}^{(m-1)(n-1)} b_m(n-1,k)\mathcal{B}_m(t)^{k+m},$$

we obtain

$$\begin{aligned}
\mathcal{B}_m(t) &= \sum_{i=0}^{n-1} C_i^{(m)} t^i + t^n \sum_{k=0}^{(m-1)(n-1)} b_m(n-1,k)\mathcal{B}_m(t)^{k+m} \\
&= \sum_{i=0}^{n-1} C_i^{(m)} t^i + t^n \sum_{k=0}^{(m-1)(n-1)} b_m(n-1,k)(1 + t \sum_{j=0}^{k+m-1} \mathcal{B}_m(t)^{j+m}) \\
&= \sum_{i=0}^{n-1} C_i^{(m)} t^i + \sum_{k=1}^{(m-1)(n-1)} (t^n b_m(n-1,k) + t^{n+1} b_m(n-1,k) \sum_{j=0}^{k+m-1} \mathcal{B}_m(t)^{j+m}) \\
&= \sum_{i=0}^{n} C_i^{(m)} t^i + t^{n+1} \sum_{j=0}^{(m-1)n} (\sum_{k=j-m+1}^{(m-1)(n-2)} b_m(n-1,k))\mathcal{B}_m(t)^{j+m}.
\end{aligned}$$

Therefore, we have $b_m(n+1,k) = \sum_{i=k-m+1}^{(m-1)(n-1)} b_m(n,i)$, with initial conditions $b_m(0,0) = 1$, and $b_m(0,k) = 0$ for all $k \neq 0$. Comparing with equation (5), we get $b_m(n,k) = a_m(n,k)$. This complete the proof. □

Corollary 2.3. *If $n \geq r \geq 1$, then we have*

$$C_n^{(m)} = [t^{n-r}] \sum_{k=0}^{(m-1)(r-1)} a_m(r-1,k)\mathcal{B}_m(t)^{k+m}.$$

Corollary 2.4. *If $n \geq r \geq 1$, then we have*

$$C_n = [t^{n-r}] \sum_{k=0}^{r-1} a(r-1,k)C(t)^{k+2},$$

$$C_n^{(3)} = [t^{n-r}] \sum_{k=0}^{2r-2} a_3(r-1,k)\mathcal{B}_3(t)^{k+3}.$$

3 Some identities related to the m-Catalan numbers

An infinite lower triangular matrix D is called a Riordan array [12,21,24] if its column k has generating function $g(t)(tf(t))^k$, $k = 0, 1, 2, \ldots$, where $g(t)$ and $f(t)$ are formal power series with $g(0) = 1$ and $f(0) \neq 0$. For instance, as defined in Section 2, the ECO matrix of Catalan numbers is the Riordan array (see [5,17]) $\mathcal{A}_2 = (C(t), tC(t))$. However, when $m \geq 3$, the ECO matrix $\mathcal{A}_m$ of m-Catalan numbers is not the Riordan array.

Theorem 3.1. *Let the lower triangular matrix* $\mathcal{T}_m = (T_m(n,k))_{n,k\geq 0}$ *be defined by the Riordan array* $\mathcal{T}_m = (\mathcal{B}_m(t), t\mathcal{B}_m(t))$. *Then* $T_m(n,k)$ *counts the number of partial m-Dyck paths from the origin to* $(mn - mk + k, k)$, *and* $T_m(n,k) = \frac{k+1}{m(n-k)+k+1}\binom{m(n-k)+k+1}{n-k}$.

Proof. Each partial m-Dyck path from the origin to $(mn - mk + k, k)$ consists of $(m-1)(n-k)+k$ "up" steps and $n-k$ "down" steps, and can be decomposed as indicated in the following figure.

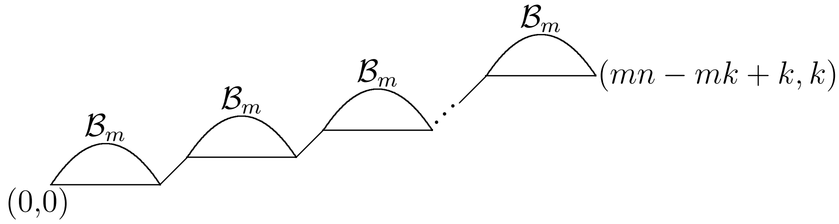

Figure 1: An m-Dyck path from $(0,0)$ to $(mn - mk + k, k)$.

Hence the generating function for these paths is given by $t^k\mathcal{B}_m(t)^{k+1}$. By the definition of a Riordan array, we have

$$\begin{aligned} T_m(n,k) &= [t^n]\mathcal{B}_m(t)(t\mathcal{B}_m(t))^k = [t^{n-k}]\mathcal{B}_m(t)^{k+1} \\ &= \frac{k+1}{m(n-k)+k+1}\binom{m(n-k)+k+1}{n-k} \\ &= \frac{k+1}{mn-mk+k+1}\binom{mn-mk+k+1}{n-k}, \end{aligned}$$

and $T_m(n,k)$ is the number of partial m-Dyck paths starting at $(0,0)$ and ending at $(mn - mk + k, k)$. □

The row sum sequence $b_n = \sum_{k=0}^{n} T_m(n,k)$ counts the number of the partial m-Dyck paths from $(0,0)$ to the line $x + (m-1)y = mn$. For example, the first few rows of the lower triangular matrix $\mathcal{T}_3 = (\mathcal{B}_3(t), t\mathcal{B}_3(t))$ are (sequence A110616 in [22]):

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 7 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 55 & 30 & 12 & 4 & 1 & 0 & 0 & 0 & 0 \\ 273 & 143 & 55 & 18 & 5 & 1 & 0 & 0 & 0 \\ 1428 & 728 & 273 & 88 & 25 & 6 & 1 & 0 & 0 \\ 7752 & 3876 & 1428 & 455 & 130 & 33 & 7 & 1 & 0 \\ 43263 & 21318 & 7752 & 2448 & 700 & 182 & 42 & 8 & 1 \end{pmatrix}.$$

Theorem 3.2. *For $n \geq r \geq 1$, the following identity holds:*

$$\sum_{k=0}^{(m-1)(r-1)} a_m(r-1,k)T_m(n+k+m-r-1,k+m-1) = C_n^{(m)} \tag{7}$$

Proof From Corollary 2.2, we obtain

$$\begin{aligned} C_n^{(m)} &= [t^{n-r}] \sum_{k=0}^{(m-1)(r-1)} a_m(r-1,k)\mathcal{B}_m(t)^{k+m} \\ &= \sum_{k=0}^{(m-1)(r-1)} a_m(r-1,k)[t^{n-r}]\mathcal{B}_m(t)^{k+m} \\ &= \sum_{k=0}^{(m-1)(r-1)} a_m(r-1,k)T_m(n+k+m-r-1,k+m-1). \end{aligned}$$

□

Corollary 3.3. *For $0 \leq i,j < n$, we have*

$$\sum_{k=0}^{(m-1)i} a_m(i,k)T_m(n-j+k+m-2,k+m-1) = C_{n+i-j}^{(m)}. \tag{8}$$

Corollary 3.4. *For $i,j \geq 0$, we have*

$$\sum_{k=0}^{(m-1)i} a_m(i,k)T_m(j+k+m-1,k+m-1) = C_{i+j+1}^{(m)}. \tag{9}$$

Example 3.1. The most important special case arises when $m = 2$. We then have $\mathcal{A}_2 = \mathcal{T}_2$. Thus, for $i,j \geq 0$,

$$\sum_{k=0}^{i} a(i,k)T_2(j+k+1,k+1) = C_{i+j+1}.$$

When $0 \leq i,j \leq 3$, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 5 & 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 & 14 \\ 1 & 3 & 9 & 28 \\ 1 & 4 & 14 & 48 \\ 1 & 5 & 20 & 75 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \\ 14 & 42 & 132 & 429 \end{pmatrix}$$

Example 3.2. For $m = 3$, we have

$$\sum_{k=0}^{2i} a_3(i,k)T_3(j+k+2,k+2) = C_{i+j+1}^{(3)}.$$

When $0 \le i, j \le 3$, we have the matrix identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 2 & 1 & 0 & 0 \\ 12 & 12 & 12 & 9 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 12 & 55 \\ 1 & 4 & 18 & 88 \\ 1 & 5 & 25 & 130 \\ 1 & 6 & 33 & 182 \\ 1 & 7 & 42 & 245 \\ 1 & 8 & 52 & 320 \\ 1 & 9 & 63 & 408 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 12 & 55 \\ 3 & 12 & 55 & 273 \\ 12 & 55 & 273 & 1428 \\ 55 & 273 & 1428 & 7752 \end{pmatrix}.$$

4 A Chung-Feller property of the m-Catalan paths

Definition 4.1. Let $i, j \ge 1$. A partial Grand m-Dyck path α is called a (i, j)-balance m-Dyck path if it can be decomposed in the form $\alpha = \beta\gamma$ satisfying:

(1) β is a negative m-Dyck path of length mi;

(2) γ is a partial m-Dyck path with j down steps;

(3) If the length of the last ascent of β is k, then the end point of γ is at height k, where $m - 1 \le k \le mi - i$.

An example of a $(2, 2)$-balance 3-Dyck path is illustrated in Figure 2.

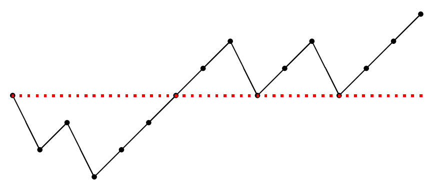

Figure 2: A $(2, 2)$-balance 3-Dyck path.

Theorem 4.1. *The number of (i, j)-balance m-Dyck paths is given by $C_{i+j}^{(m)}$.*

Proof. By Theorem 1.1, $a_m(i-1, k)$ is the number of negative m-Dyck paths from $(0, 0)$ to $(mi, 0)$ whose last ascent is of length $k+m-1$. By Theorem 3.1, $T_m(j+k+m-1, k+m-1)$ is the number of partial m-Dyck paths with j down steps and whose last point is at height $k+m-1$. Hence $\sum_{k=0}^{(m-1)i} a_m(i-1, k)T_m(j+k+m-2, k+m-1)$ is the total number of (i, j)-balance Grand m-Dyck paths. By Corollary 3.3, we have

$$\sum_{k=0}^{(m-1)(i-1)} a_m(i-1, k)T_m(j + k + m - 2, k + m - 1) = C_{i+j}^{(m)}.$$

Therefore the number of (i, j)-balance m-Dyck paths is given by $C_{i+j}^{(m)}$. $\square$

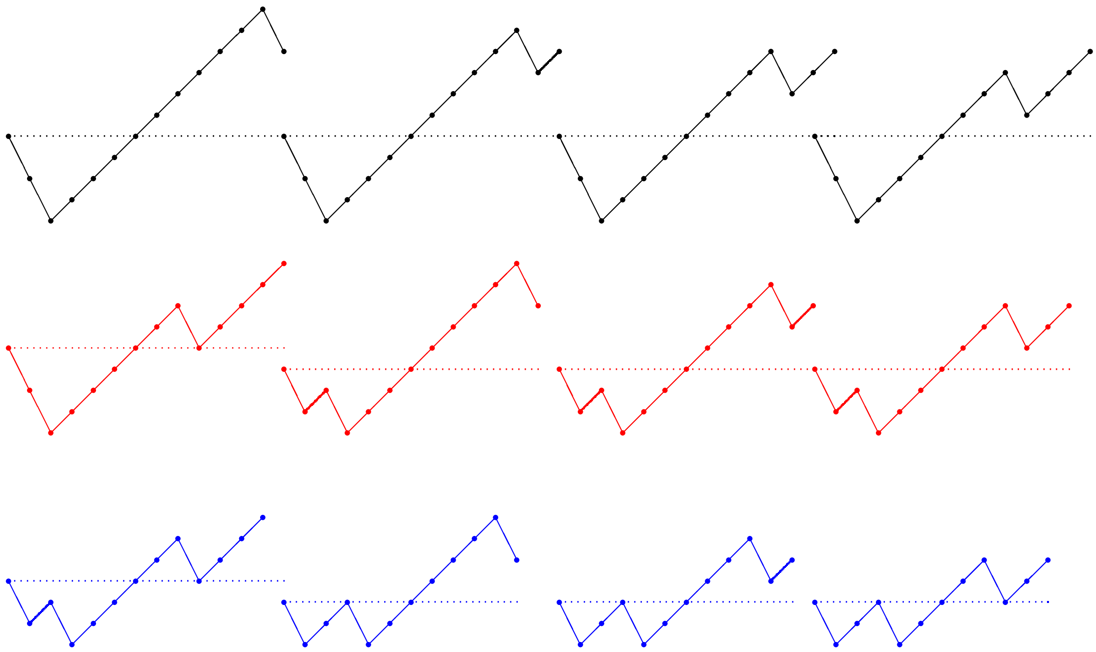

Figure 3: The $(2,1)$-balance 3-Dyck paths.

Theorem 4.2. *For $n \geq i \geq 1$, the number of $(i, n-i)$-balance m-Dyck paths is counted by $C_n^{(m)}$, and is independent of i.*

Example 4.1. For $m = 3$, $i = 2$, and $j = 1$, the number of $(2,1)$-balance 3-Dyck paths is equal to $C_3^{(3)} = 12$. These paths are illustrated in Figure 3.

Acknowledgements

The authors would like to thank the editor and anonymous referees for their careful reading and helpful comments, which have improved the final version of this paper.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy and D. Gouyou-Beauchamps, Generating functions for generating trees, *Discrete Math.* 246 (1-3) (2002), 29–55.

[2] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, ECO: a methodology for the enumeration of combinatorial objects, *J. Differ. Equations Appl.* 5 (4-5) (1999), 435–490.

[3] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, A methodology for plane trees enumeration, *Discrete Math.* 180 (1-3) (1998), 45–64.

[4] E. Barcucci, E. Pergola, R. Pinzani and S. Rinaldi, ECO method and hill-free generalized Motzkin paths, *Séminaire Lotharingien de Combinatoire* 46 (2001), Article B46b.

[5] P. Barry, A note on a one-parameter family of Catalan-like numbers, *J. Integer Seq.* 12(2009), Article 09.5.4.

[6] G.-S. Cheon, H. Kim and L.W. Shapiro, Combinatorics of Riordan arrays with identical A and Z sequences, *Discrete Math.* 312 (2012), 2040–2049.

[7] K.L. Chung and W. Feller, On fluctuations in coin-tossing, *Proc. Nat. Acad. Sci. USA* 35 (1949), 605–608.

[8] E.Y.P. Deng and T. Mansour, Three Hoppy path problems and ternary paths, *Discrete Appl. Math.* 156 (2008), 770–779.

[9] S.-P. Eu, S.-C. Liu and Y.-N. Yeh, Taylor expansions for Catalan and Motzkin numbers, *Adv. Appl. Math.* 29 (2002), no. 3, 345-357.

[10] S.-P. Eu, T.-S. Fu and Y.-N. Yeh, Refined Chung-Feller theorems for lattice paths, *J. Combin. Theory Ser. A* 112 (2005), no. 1, 143-162.

[11] R. Graham, D. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, New York, 1989.

[12] T.X. He and R. Sprugnoli, Sequence characterization of Riordan arrays, *Discrete Math.* 309 (2009), 3962–3974.

[13] T.X. He, Parametric Catalan numbers and Catalan triangles, *Linear Alg. Appl.* 438 (2013), 1467–1484.

[14] T.X. He, Matrix characterizations of Riordan arrays, *Linear Algebra Appl.* 465 (2015), 15–42.

[15] S. Heubach, N.Y. Li and T. Mansour, Staircase tilings and k-Catalan structures, *Discrete Math.* 308 (24) (2008), 5954–5964.

[16] P. Hilton and J. Pedersen, Catalan numbers, their generalization, and their uses, *Math. Intell.* 13 (1991), 64–75.

[17] A. Luzón, D. Merlini, M. Morón and R. Sprugnoli, Identities induced by Riordan arrays, *Linear Alg. Appl.* 436 (2011), 631–647.

[18] T. Mansour and Y. Sun, Bell Polynomials and k-generalized Dyck Paths, *Discrete Appl. Math.* 156 (2008), 2279–2292.

[19] T. Mansour, Protected points in k-ary trees, *Appl. Math. Lett.* 24 (2011), 478–480.

[20] W. Młotkowski, Fuss-Catalan Numbers in Noncommutative Probability, *Documenta Mathematica* 15 (2010), 939–955.

[21] L.W. Shapiro, S. Getu, W.-J. Woan and L. Woodson, The Riordan Group, *Discrete Appl. Math.* 34 (1991), 229–239.

[22] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at `http://oeis.org`, 2014.

[23] C. Song, The generalized Schröder theory, *Electron. J. Combin.* 12 (2005), #R53.

[24] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* 132 (1994), 267–290.

[25] R.P. Stanley, *Enumerative Combinatorics (vol. 2)*, Cambridge University Press, Cambridge / New York, 1999.

(Received 15 Feb 2015; revised 23 Oct 2015, 7 Dec 2015)