Strategy-Proof House Allocation with Existing Tenants over Social Networks

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ABSTRACT

Mechanism design over social networks, whose goal is to incentivize agents to diffuse the information of a mechanism to their followers, as well as to report their true preferences, is one of the new trends in market design. In this paper, we reconsider the traditional house allocation problem with existing tenants from the perspective of mechanism design over social networks. Since our model is a generalization of the networked housing market investigated by Kawasaki et al. [9], no mechanism simultaneously satisfies strategy-proofness, individual rationality and Pareto efficiency for general social network structures. We therefore examine the cases where the social network has a tree structure. We first show that even for the restricted structure, a weaker welfare requirement called non-wastefulness is not achievable by any strategy-proof and individually rational mechanism. We then show that a non-trivial modification of *You Request My House - I Get Your Turn mechanism (YRMH-IGYT)* is individually rational, strategy-proof, and weakly non-wasteful. Furthermore, it chooses an allocation in the strict core for neighbors and satisfies weak group strategy-proofness.

KEYWORDS

House Allocation; Mechanism Design; Information Diffusion

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1 INTRODUCTION

Mechanism design, as a sub-field of game theory, is one of the theoretical foundations of multi-agent decision making where multiple agents have conflicting interests. It has been widely studied in the intersection among economics, computer science and mathematics. More specifically, designing *strategy-proof* (SP) mechanisms, in which reporting true preferences to the mechanism is a dominant strategy for each agent, is a main objective of the traditional mechanism design theory. The theory of mechanism design can be applied to various domains, e.g., spectrum allocation, hospital-residency matching, live organ exchange, and political decision makings.

Allocating scarce resources to agents without monetary compensation is a typical mechanism design problem. In particular, a specific problem called *housing market* has been extensively investigated, where each agent initially owns an indivisible object, a house, and the objective is to find a socially efficient, e.g., Pareto efficient, reallocation of the objects. Gale's Top-Trading-Cycles (TTC) algorithm always returns a Pareto efficient (PE) and individually rational (IR) allocation and satisfies SP [15]. One of the most well-studied extensions of the housing market is the model where some agents have no initial endowment house and some initially-vacant houses are also available in the market, which is usually called *house allocation with existing tenants*. For house allocation with existing tenants, a mechanism called *You Request My House - I Get Your Turn (YRMH-IGYT)* satisfies all of the three properties above [1].

Besides the above development of the theory of mechanism design, a new direction of mechanism design, called *mechanism design over social networks*, has been initiated by Li et al. [11]. In mechanism design over social networks, the set of agents, as well as the mechanism designer, are assumed to be distributed over a social network, and the mechanism designer wants to widely propagate the information of the mechanism through the social network to attract more participants. In this model, SP additionally requires that, for each agent, inviting as many followers as possible to the mechanism is an optimal strategy, even if she has the option not to invite any of them. There is an existing work that investigates the housing market problem from the perspective of mechanism design over social networks [9]. They proposed a modified TTC to achieve SP reallocation of houses over social networks [9]. We call this mechanism *Kawasaki mechanism.* However, no work has studied house allocation with existing tenants over social networks.

This paper is the first to consider house allocation with existing tenants over social networks. Since our model is a generalization of Kawasaki et al. [9], it is impossible to find a mechanism that simultaneously satisfies SP, IR, and PE, or SP and a property called *weak core for neighbors (WC4N)*, for general network structures. We therefore focus on a special (but still common) structure, i.e., a tree.

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We first show that even for a tree-structured network, a weaker welfare property called *non-wastefulness (NW)* is incompatible with the combination of SP and IR. We then show that a relaxation of NW called *weak non-wastefulness (weak NW)*, as well as SP and IR, are achieved by a non-trivial modification of YRMH-IGYT [1], called *YRMH-IGYT for tree structured social networks (YRMH-IGYT for TSSNs)*. We further show that YRMH-IGYT for TSSNs also satisfies a condition called *strict core for neighbors (SC4N)* (which is stronger than WC4N), and weak group strategy-proofness (weak GSP).

The rest of this paper is organized as follows. Section 2 gives a literature review. Section 3 describes our model. Section 4) shows that the straightforward application of YRMH-IGYT works only in a very restrictive setting. Motivated by this, Section 5 proposes YRMH-IGYT for TSSNs, which satisfies SP, IR, weak NW, and SC4N. Section 6 examines several alternative mechanisms and shows that they fail to satisfy our desiderata. Section 7 shows that YRMH-IGYT for TSSNs satisfies weak GSP. Finally, Section 8 concludes the paper.

2 LITERATURE REVIEW

Shapley and Scarf [15] proposed TTC and showed that it always chooses a unique strict core for traditional housing markets. Ma [13] showed the uniqueness of TTC as the mechanism that simultaneously satisfies PE, IR, and SP. In recent years, various extensions of the housing market have been proposed, including taking into account indifference in preferences [2, 3, 20], preferences with externalities [10, 14], multiple houses per agent [5, 16, 23], and asymmetry over houses [21]. Sönmez [17] showed that choosing the strict core is SP if and only if the strict core is essentially single-valued.

Abdulkadiroğlu and Sönmez [1] initiated the research on house allocation with existing tenants as an extension of the housing market. They proposed YRMH-IGYT and showed its equivalence to a variation of TTC. Sönmez and Ünver [18] showed an equivalence of the following two mechanisms: TTC applied to a housing market when each newcomer is randomly assigned a vacant house and YRMH-IGYT where all the newcomers are placed at the top of the priority order. Sönmez and Ünver [19] characterized YRMH-IGYT by five properties, namely SP, IR, PE, weak neutrality, and consistency, where weak neutrality requires that renaming vacant houses does not affect the final outcome. Karakaya et al. [7] then investigated which mechanisms arise by dropping weak neutrality. Ekici [4] considered the cases where some subset of existing tenants is discriminately treated so that they are to be placed at the bottom of the priority order used in YRMH-IGYT.

Li et al. [11] proposed a new framework of mechanism design over social networks. They focused on single-item auctions and proposed an SP mechanism. Both Zhao et al. [24] and Kawasaki et al. [8] studied a multi-unit unit-demand auction via social networks, where each unit is identical and each buyer requires a unit. Liu et al. [12] studied a reverse auction where the buyer has a budget constraint from the perspective of mechanism design over social networks. Takanashi et al. [22] focused on the efficiency of such auctions. Kawasaki et al. [9] has investigated resource allocation without money from the perspective of mechanism design over social networks. However, they restricted themselves to a classic housing market and did not consider any vacant houses owned by the moderator.

3 MODEL

We extend the *networked housing market* model of Kawasaki et al. [9] by introducing initially vacant houses and newcomers. In a networked housing market, there exists a group A of n agents, which is divided into two disjoint groups A_E and A_N; $A_E := \{1, 2, \dots, |A_E|\}$ is a set of existing tenants, while $A_N := \{|A_E| + 1, |A_E| + 2, \dots, n\}$ is a set of newcomers. There also exists a set of n' indivisible objects H, usually referred to as *houses*. H is also divided into two disjoint groups H_O and H_V, where $H_O := \{h_1, h_2, \dots, h_{|A_E|}\}$ denotes a finite set of houses occupied by the existing tenants, while $H_V := \{h_{|A_E|+1}, h_{|A_E|+2}, \dots, h_{n'}\}$ is a finite set of initially vacant houses. We assume enough houses exist to accommodate all agents, i.e., $n' \geq n$ holds. Besides the above n agents, there exists a special agent s called the *source* or *moderator* in the market, who initially owns all the vacant houses. Note that s is not a strategic agent; she is willing to give away vacant houses (e.g., s is a social planner). Each existing tenant $i \in A_E$ owns one house h_i in H_O. An allocation m is a mapping from $A \cup \{\emptyset\}$ to $H \cup \{\emptyset\}$. Let $m(i)$ denote the house assigned to agent i under m, and $m(h)$ denote the agent who is assigned to h. If no house is allocated to i (or h is not assigned to any agent), we assume $m(i) = \emptyset$ (or $m(h) = \emptyset$) holds. Also, let $m(s)$ denote the set of houses that are not assigned to any agent in A by m. Let M be the set of all possible allocations. Furthermore, let $\widehat{m}$ denote the initial allocation, where it holds that $\widehat{m}(i) = h_i$ for all $i \in A_E$ and $\widehat{m}(i) = \emptyset$ for all $i \in A_N$, as well as $\widehat{m}(s) = H_V$.

For each agent $i \in A \cup \{s\}$, let $r_i \subseteq A \setminus \{i\}$ denote her neighbors. Agent i is directly connected to each agent in r_i, meaning that agent i knows the agents in r_i and can invite them (and only them) to the mechanism. Note that the neighborhood relation can be asymmetric. Each agent i has a strict preference $\succ_i$ over H. We write $h \succ_i h'$ if agent i prefers house h over h', and $h \succeq_i h'$ if either $h = h'$ or $h \succ_i h'$ holds. Each agent prefers any house over nothing, i.e., $\forall i \in A$, $\forall h \in H$, $h \succ_i \emptyset$. In sum, for each agent $i \in A$, her type (also known as her *private information*) θ_i is given as $(\succ_i, r_i)$.

Now we are ready to describe the mechanism design model considered in this paper. We restrict our attention to direct revelation mechanisms, to which each agent declares her type $\theta'_i = (\succ'_i, r'_i)$. Note that we assume partial verification is possible, i.e., agent i can only declare $\theta'_i = (\succ'_i, r'_i)$ s.t., $r'_i \subseteq r_i$. This partial verification scheme obviously satisfies the well-known *Nested Range Condition* [6], which guarantees that the revelation principle holds. We believe this assumption is realistic and widely used in existing works on mechanism design in social networks. Thus, we can restrict our attention to direct revelation mechanisms without loss of generality.[1] Let $R(\theta_i)$ denote the set of all possible types that agent i with true type θ_i can declare. That is, for any $i \in A$ and any $\theta_i = (\succ_i, r_i)$, $R(\theta_i) := \{(\succ'_i, r'_i) \mid r'_i \subseteq r_i\}$.

Let $\theta = (\theta_1, \dots, \theta_n)$ denote the profile consisting of the agents' true types, while $\theta' = (\theta'_1, \dots, \theta'_n)$ denotes a declared profile of types. We also use the following standard notations: θ'_{-i} is a profile of the agents' declared types except for agent i, $(\theta'_i, \theta'_{-i})$ is a profile of the agents' declared types where agent i declares θ'_i and others declare θ'_{-i}, and $R(\theta_{-i})$ is a set of profiles that agents (except i) can

[1]When describing the behavior of a mechanism, we use expressions like "agent i chooses a house", etc. for simplicity. Strictly speaking, in a direct revelation mechanism, the mechanism acts on behalf of each agent according to her declared type.

jointly declare when their true type profile is θ_{-i}. We also use a similar notation for a group of agents $S \subset A$, i.e., θ'_{-S} is a profile of the agents' declared types except for the agents in S, $(\theta'_S, \theta'_{-S})$ is a profile of the agents' declared types where each agent i in S declares θ'_i (where $\theta'_S = (\theta'_i)_{i \in S}$), and other agents declare θ'_{-S}. Also, $R(\theta_S)$ is a set of profiles that agents in S can jointly declare when their true type profile is θ_S, and $R(\theta_{-S})$ is a set of profiles that agents not in S can jointly declare when their true type profile is θ_{-S}.

For notation simplicity, let $r' = (r'_i)_{i \in A}$ denote the neighborhood relations for the declared profile. We also refer to r' as a (declared) *social network*. Social network r' can be represented as a directed graph, $G = (V, E)$, where $V := A \cup \{s\}$, and for any pair $i, j \in A \cup \{s\}$, an edge $(i, j) \in E$ if and only if $j \in r'_i$. Also, let r denote the social network when all agents report their neighbors truthfully. Agent i is *connected* if there is a path $s \to \cdots \to i$ in G defined by r'. Let $d(i)$ denote the distance from s to i in r', i.e., the length of the shortest path from s to i. The moderator does not a-priori know the edges/vertices of the social network (beyond its direct neighbors). Thus, the moderator must construct the network based on the declared agents' types. Disconnected agents and their initial endowment houses do not participate in the mechanism.

A (direct revelation) mechanism f is defined as a function that takes θ' as input and returns $f(\theta') \in M$. Let $f_i(\theta')$ denote the house assigned to agent i. In this paper, we further restrict our attention to *feasible* mechanisms, in which only connected agents can exchange or obtain a vacant house. Formally, allocation $m \in M$ is feasible under reported r' if for any agent i who is disconnected under r', $m(i) = h_i$ holds when $i \in A_E$, and $m(i) = \emptyset$ holds when $i \in A_N$. A mechanism is feasible if, for any reported θ', $f(\theta')$ is feasible.

We introduce four desirable properties for allocation mechanisms: *strategy-proofness* (SP), *individual rationality* (IR), *Pareto efficiency* (PE), and the *strict core* (SC). Intuitively, SP requires that for each agent $i \in A$, reporting her true type θ_i is a dominant strategy. IR requires that for each agent $i \in A_E$, reporting her true type θ_i guarantees a house that she weakly prefers over her initial endowment h_i. Then, allocation $m \in M$ is PE, if there exists no allocation m' that is weakly better for all agents $i \in A$ and strictly better for at least one agent $j \in A$. Finally, any allocation in SC ensures that no group of agents has an incentive to leave the market and exchange their initial endowments among themselves.

Definition 1 (Strategy-Proofness (SP)). *For a networked housing market, mechanism f is said to satisfy* strategy-proofness *(SP) if, for any $i \in A$, any θ_i, any $\theta'_i \in R(\theta_i)$, and any θ_{-i}, it holds that $f_i(\theta_i, \theta_{-i}) \succeq_i f_i(\theta'_i, \theta_{-i})$.*

Definition 2 (Individual Rationality (IR)). *For a networked housing market, feasible allocation $m \in M$ is said to be* individually rational *for given θ if $m(i) \succeq_i h(i)$ for any $i \in A_E$. Let $IR(\theta)$ be a set of such allocations under θ. Mechanism f is said to satisfy* individual rationality *(IR) if $f(\theta) \in IR(\theta)$ for any θ.*

Definition 3 (Pareto Efficiency (PE)). *For a networked housing market, feasible allocation m Pareto dominates another feasible allocation m' under θ if $m(i) \succeq_i m'(i)$ for all $i \in A$ and $m(j) \succ_j m'(j)$ for some $j \in A$. Let $PE(\theta)$ be a set of feasible allocations that are not Pareto dominated by any other feasible allocation. Mechanism f is said to satisfy* Pareto efficiency *(PE) if $f(\theta) \in PE(\theta)$ for any θ.*

Definition 4 (Strict Core (SC)). *For a networked housing market with θ and feasible allocation m, we say $S \subseteq A$ is a blocking coalition if there exists another allocation $m' \neq m$ that satisfies the following conditions: $\bigcup_{i \in S} m'(i) = \bigcup_{i \in S} \widehat{m}(i)$, (a) $\forall i \in S$, $m'(i) \succeq_i m(i)$ and (b) $\exists j \in S$, $m'(j) \succ_j m(j)$ holds.*
We say m is in a strict core (SC) under θ, if there exists no blocking coalition. Let $SC(\theta)$ be the set of such allocations for given θ. For a networked housing market, mechanism f is said to satisfy SC if for any θ, $f(\theta) \in SC(\theta)$ holds.

Agents in a blocking coalition have an incentive to deviate from the mechanism and exchange their initial endowments among themselves. Note that if S is a blocking coalition, for each $i \in S$, $i \in A_E$ holds; a newcomer cannot be included in a blocking coalition. This is because a newcomer has no initial endowment. Deviating from the mechanism with a newcomer will lead to the result that one agent in the group will be assigned no house.

We can analogously define the *weak core* (WC) by replacing conditions (a) and (b) with (c) $m'(i) \succ_i m(i)$ for any $i \in S$. In this case, we say m' *strongly* dominates m for coalition S. Let $\mathrm{WC}(\theta)$ be the set of weak cores for the given θ, and mechanism f is said to satisfy WC if $f(\theta) \in \mathrm{WC}(\theta)$ for any θ. The following is a weaker version of the core that takes the network structure into account.

Definition 5 (Strict Core for Neighbors (SC4N)). *For a networked housing market, an outcome is said to be in* the strict core for neighbors *(SC4N) under profile θ, if there exists no blocking coalition $S \subseteq A$, such that either (i) $|S| = 1$, or (ii) $|S| = \{i, j\}$, and there is a path $s \to \cdots \to i \to j$ under θ. Let SC4N(θ) be the set of such allocations for a given θ. A mechanism is said to satisfy* SC4N *if $f(\theta) \in$ SC4N(θ) for any θ.*

We can analogously define the weak core for neighbors (WC4N). For given θ, let $\mathrm{WC4N}(\theta)$ be the set of allocations in the weak core for neighbors. Note that WC4N implies IR. If a mechanism satisfies WC4N and an existing tenant is assigned a house that is worse than her initial endowment, she would form a blocking coalition of size 1 in which she keeps her initial endowment house. Note that WC4N implies IR. WC4N can be considered as the weakest variant of the core for social networks (except IR) since it only cares about coalitions by two agents in a parent-child relation (which we call *a blocking pair*), as well as the deviations by each single agent. However, it turns out that even this weakest variant is incompatible with SP. More specifically, since our model is a strict generalization of Kawasaki et al. [9], it inherits their impossibility results, i.e., the following theorem holds.

Corollary 1. *(from Theorem 4.3 and 4.7 of Kawasaki et al. [9]) For a general network structure, where $|A| \geq 3$, no mechanism simultaneously satisfies either of the following sets of properties: (i) SP, IR, and PE, or (ii) SP and WC4N.*

4 APPLYING EXISTING MECHANISM

For *network housing markets*, we first investigate the applicability of an existing mechanism, the *You Request My House - I Get Your Turn (YRMH-IGYT)* mechanism [1], in the network model.

YRMH-IGYT requires a linear priority order among agents. Let $\rhd$ denote such a priority defined among agents who are invited to the housing market.

Definition 6 (YRMH-IGYT for SNs).

(1) *Each agent declares her preference and a list of her neighbors. Agents are sorted by a given priority order* $\rhd$.
(2) *A remaining top agent, say* i*, is selected. Based on her preference, we select her most preferred house that still remains in the market, say* h.
(3) *If* h *is currently vacant, agent* i *obtains it and leaves the market. If she has an initial endowment house, it becomes vacant. Go to 2.*
(4) *If* h *is currently occupied, we let agent* i *point to the owner of* h*. Then, we check whether a loop is formed. A loop is an ordered list of agents* $(i_1, i_2, \ldots, i_\ell)$ *such that* i_1 *points to* i_2*,* i_2 *points to* i_3*, . . ., and* i_ℓ *points to* i_1 *(if an agent prefers her own initial endowment house, we assume she forms a self-loop).*
   (a) *If no loop exists, let the owner of this house, say* j*, become the first agent (and* i *becomes the second). Go to 2.*
   (b) *If a loop is formed, assign each agent in the loop the house she is requesting. These agents leave the market. Go to 2.*

YRMH-IGYT is known to be equivalent to the following variation of TTC [1]. Each agent points to her favorite house, each house with an existing tenant points to its tenant, and each vacant house points to the remaining top agent according to $\rhd$. There exists at least one cycle. Each agent included in a cycle obtains the house she points and leaves the market, and so on.

YRMH-IGYT is SP for any priority order in the traditional housing market. However, in a networked housing market, the priority order must be carefully designed such that each agent has an incentive to invite her neighbors. In this paper, we assume the priority order is *distance-based.*

Definition 7 (Distance-Based Priority). *Given social network* G*, a priority order of agents,* $\rhd$*, is said to be* distance-based *if, for any* $i, j \in A$*,* $[d(i) < d(j)] \Rightarrow i \rhd j$ *holds.*

Now, we are ready to present our results on the properties of YRMH-IGYT for SNs with a distance-based priority. When YRMH-IGYT for SNs satisfies SP, it is easy to show that it also satisfies PE and IR. Thus, we focus on showing when it satisfies SP.

Theorem 2. *Assume that YRMH-IGYT for SNs is defined with a distance-based priority order* $\rhd$*. Also assume that the preference domain is general, i.e., each agent can have/report any preference. Then, the mechanism is SP if and only if all the existing tenants are directly connected to the moderator, i.e.,* $A_E \subseteq r_s$.

Proof. The only if direction can be shown as a corollary of Theorem 6.1 of Kawasaki et al. [9]. This is because, by assuming $A_N = \emptyset$, and for each agent i and any vacant house $h \in H_V$, $\widehat{m}(i) \succ_i h$ holds, i.e., there exists no newcomer and each existing tenant prefers her own initial endowment house over any vacant house, our model becomes equivalent to their model. As a result, we can easily find an example input where the outcome by YRMH-IGYT for SNs becomes equivalent to Kawasaki mechanism.

To complete the proof, we show that YRMH-IGYT for SNs satisfies SP under the condition above. Furthermore, since it is known that no agent can benefit from misreporting her preference under YRMH-IGYT, it suffices to show that no agent can benefit by not inviting some of her followers, i.e., by reporting $\theta'_i = (\succ_i, r'_i)$ s.t., $r'_i \subset r_i$. Assuming this misreport can affect the mechanism, two consequences are possible: (i) agent $j \in r_i \setminus r'_i$ is moved to a lower priority (i.e., if the distance becomes larger), or (ii) agent $j \in r_i \setminus r'_i$ is removed from the market. In either case, j must be a newcomer, since we assume all existing tenants are directly connected to the moderator. Furthermore, since $\rhd$ is distance-based and i's report influences j's priority, it has to hold $i \rhd j$ (without misreporting). Since each newcomer initially has no house, j cannot take any other agent's turn and is thus first activated after i left the market. Therefore, no assignment until i gets a house is affected by the manipulation. Thus, this manipulation is useless for agent i. □

Note that the above definition of YRMH-IGYT for SNs is just an implementation of the original YRMH-IGYT for our model, where the priority over agents is defined based on the distance from s. However, the following example shows that this implementation is almost necessary to guarantee SP. That is, when the priority is not distance-based, we can find an instance where YRMH-IGYT fails to satisfy SP, even if all existing tenants are directly connected to s.

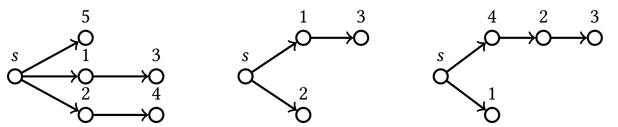
Example 1. *Two agents exist in the market, existing tenant* 1*, and newcomer* 2*, where* $r_s = \{1\}$*,* $r_1 = \{2\}$*,* $r_2 = \emptyset$*,* $h_2 \succ_1 h_1$*, and* $h_2 \succ_2 h_1$*. The house, initially owned by* 1*, is* h_1*;* h_2 *is vacant. Note that there exists only one existing tenant* 1 *who is directly connected to the moderator. In this case, if* $\rhd := (2, 1)$*, which is not distance-based, the allocation decided by YRMH-IGYT should be* $m(1) = h_1$*, and* $m(2) = h_2$*. However, if agent* 1 *does not invite agent* 2*, the allocation decided by YRMH-IGYT will be* $m(1) = h_2$*, and* $h_2 \succ_1 h_1$ *holds. Thus, YRMH-IGYT for SNs fails to satisfy SP.*

5 NEW WEAKLY NON-WASTEFUL MECHANISM FOR TREES

As shown in the previous section, YRMH-IGYT for SNs satisfies SP only under a very strict assumption, i.e., all existing tenants are directly connected to the moderator. Hence, in this section, we focus on one common social network structure, i.e., a tree, in which each agent i can be invited by at most one agent j (we say j is i's parent) and existing tenants are not necessarily to be direct neighbors of the moderator. We examine whether SC4N and SP can be compatible with a general preference together with any welfare requirement weaker than PE. One natural candidate for a welfare requirement is *non-wastefulness* (NW), but we show it is incompatible with IR and SP for a tree. Therefore, we consider a further relaxation called weak non-wastefulness (weak NW). We propose a non-trivial modification of YRMH-IGYT called *YRMH-IGYT for tree-structured social networks (YRMH-IGYT for TSSNs)* that takes into account the tree structure and obtain a positive result; YRMH-IGYT for TSSNs satisfies SP, SC4N, and weak NW.

Definition 8 (Non-Wastefulness (NW)). *Allocation* m *is* non-wasteful *(NW) if, for each agent* $i \in A$ *and any house* $h \in m(s)$*,* $m(i) \succ_i h$ *holds.*

NW requires that no agent wants to exchange her assigned house to any vacant house in allocation m. Clearly, if m is Pareto efficient, it is also non-wasteful, but not vice versa. The following theorem shows that NW is incompatible with the combination of IR and SP.



(a) Social network where SP, IR, and NW are incompatible **(b) Examples used in comparison with other mechanisms** **(c) Counter example for strong GSP**

Figure 1: Network structures used in examples/proofs.

Theorem 3. *No mechanism simultaneously satisfies SP, IR, and NW, even if the graph is restricted to a tree.*

Proof. By way of contradiction, we assume there exists a mechanism that satisfies these three properties.

Assume there exist four existing tenants 1, 2, 3, 4 and a newcomer 5, with $r_s = \{1, 2, 5\}, r_1 = \{3\}, r_2 = \{4\}$, and $r_3 = r_4 = r_5 = \emptyset$ (see Fig. 1a). Each existing tenant i (where $1 \leq i \leq 4$) owns h_i. Also, there exist two initially vacant houses, h_5 and h_6. Consider the following preferences:

$$\begin{array}{lcl} \succ_1 & : & h_4 \succ h_1 \succ \cdots \\ \succ_2 & : & h_3 \succ h_2 \succ \cdots \\ \succ_3 & : & h_5 \succ h_3 \succ \cdots \\ \succ_4 & : & h_5 \succ h_4 \succ \cdots \\ \succ_5 & : & h_6 \succ \cdots \succ h_5 \end{array}$$

That is, every agent except 5 prefers exactly one house over her initial endowment. Consider a case where all agents are truthful. Since the mechanism is NW, h_6 cannot be vacant; otherwise agent 5 prefers h_6 over her assignment. Also, since the mechanism is IR, h_6 can be assigned only to agent 5. Similarly, h_5 must be assigned to either agent 3 or 4.

If h_5 is given to agent 3, her initial endowment house h_3 becomes vacant and has to be assigned to agent 2 to satisfy NW. In this case, both agent 1 and 4 receive their initial endowments. Now, assume 1 does not invite her child 3. Then, h_5 has to be assigned to 4 and h_4 must be assigned to 1 to satisfy NW. Since 1 prefers h_4 over h_1, this manipulation is beneficial; this mechanism fails to satisfy SP.

If h_5 is given to 4 instead, her initial endowment house h_4 becomes vacant and has to be assigned to 1 to satisfy NW. Then, by the analogous argument to above, it follows that 2 has an incentive not to invite her child 4. Thus, h_5 can be assigned to neither 3 nor 4, a contradiction. □

Hence, in order to evaluate whether a mechanism performs well in terms of efficiency, we introduce a weaker notion of allocative efficiency, called *weak non-wastefulness* (weak NW).

Definition 9 (Weak Non-Wastefulness). *Allocation m is* weakly non-wasteful *(weakly NW) if* $\forall i \in A$ *and* $\forall h \in m(s) \cap \widehat{m}(s)$, $m(i) \succ_i h$ *holds.*

Weak NW ensures that no house that started out and remained vacant is preferred by any agent i over her allocated house $m(i)$.

In the rest of this section, we introduce YRMH-IGYT for TSSNs and show that it satisfies SP, IR, weak NW, and SC4N if the social network is a tree. We say agent j is an ancestor of agent i if there exists a path $s \to \cdots \to j \to \cdots \to i$ in the social network. Let a_i denote the set of i's ancestors.

Mechanism 1 YRMH-IGYT for tree-structured social networks

function YRMH-IGYT-TSSNs($\triangleright, \theta', A_E, A_N, H_V, H_O$)
  $H'_V \leftarrow H_V \cup \{\varnothing\}$
  while $|\triangleright| > 0$ **do**
    $i \leftarrow \triangleright_1, H_C \leftarrow \{\widehat{m}(j) \mid j \in r'_i\}$
    $H_A \leftarrow \{\widehat{m}(j) \mid j \in a_i$ and j is included in $\triangleright\}$
    if $i \in A_E$ **then** $H' \leftarrow H'_V \cup H_C \cup H_A \cup \{\widehat{m}(i)\}$
    else $H' \leftarrow H'_V \cup H_C$
    end if
    Find $h \in H'$ s.t. $\forall h' \in H' \setminus \{h\}, h \succ_i h'$
    if $h \in H'_V$ **then**
      $m(i) \leftarrow h, \triangleright \leftarrow \triangleright \setminus \{i\}$
      if $i \in A_E$ **then** $H'_V \leftarrow (H'_V \setminus \{h\}) \cup \{\widehat{m}(i)\}$
      else if $h \neq \varnothing$ **then** $H'_V \leftarrow H'_V \setminus \{h\}$
      end if
    else if $h = \widehat{m}(i)$ **then** $m(i) \leftarrow h, \triangleright \leftarrow \triangleright \setminus \{i\}$
    else if $h \in H_A$ **then**
      $\mathcal{L} \leftarrow (j_1, \ldots, j_\ell)$ s.t. $\widehat{m}(j_1) = h, j_\ell = i$,
       and each j_k points to j_{k+1}
      $m(i) \leftarrow h, \triangleright \leftarrow \triangleright \setminus \{i\}$
      $\forall k \in \{1, 2, \ldots, \ell - 1\}$,
       $m(j_k) \leftarrow \widehat{m}(j_{k+1}), \triangleright \leftarrow \triangleright \setminus \{k\}$
    else
      Let i points to $\widehat{m}(h)$
      Find $\triangleright_\ell$ s.t. $\triangleright_\ell = \widehat{m}(h)$ holds
      $\forall k \in \{1, 2, \ldots, \ell - 1\}, \triangleright_{k+1} \leftarrow \triangleright_k, \triangleright_1 \leftarrow \ell$
    end if
  end while
  $m(s) \leftarrow H'_V \setminus \{\varnothing\}$
  return m
end function

Definition 10 (YRMH-IGYT for TSSNs).

(1) *Each agent declares her preference and a list of her children. Based on the declaration, a tree is constructed. Agents are sorted according to a given distance-based priority order* $\triangleright$.
(2) *The remaining top agent is selected. Based on her preference, the most preferred house is selected among the following four cases: (i) any currently vacant house, (ii) a house owned by an ancestor (assuming the ancestor is still in the market), (iii) her initial endowment house, or (iv) any house owned by her children.*
(3) *For case (i), she obtains the house and leaves the market. If she has her initial endowment house, it becomes vacant. Go to 2.*
(4) *For case (ii), assuming agent i points to her ancestor, there exists a loop* $(j_1, j_2, \ldots, j_\ell)$, *where* $j_\ell = i$ *and each* j_k *points to* j_{k+1}, *and* j_ℓ *points to* j_1. *Each agent obtains the house she is requesting and leaves the market. Go to 2.*
(5) *For case (iii), she obtains her initial endowment house and leaves the market. Go to 2.*
(6) *For case (iv), she remains in the market while pointing to her child who owns the house that she requested. Her child becomes the top among the remaining agents (and she becomes the second). Go to 2.*

Note that in case (ii), agent i is selected and her ancestor is still in the market. This is possible only when i is an existing tenant, and there exists a sequence of i's ancestors $(j_1, j_2, \ldots, j_\ell)$, where $j_\ell = i$ and each j_k is requesting the house owned by j_{k+1}.

A more precise description of YRMH-IGYT for TSSNs is given as Mechanism 1. Here, $\rhd_i$ denotes the i^{th} agent in $\rhd$. Also, $|\rhd|$ denotes the number of agents in $\rhd$, and $\rhd \setminus \{i\}$ denotes the order where agent i is removed from $\rhd$. H'_V denotes the set of currently vacant houses (including $\emptyset$). The mechanism returns feasible allocation m.

Let us illustrate how YRMH-IGYT for TSSNs works.

Example 2. *Assume there exist tenants* 1, 2, 3, 4 *and a newcomer* 5*. The social network is given as Fig 1a. Each existing tenant i (where* $1 \le i \le 4$*) owns* h_i*. Also, there exist two initially vacant houses,* h_5 *and* h_6*. Consider the following preferences.*

$$\begin{array}{lcl} \succ_1 & : & h_3 \succ h_1 \succ \cdots \\ \succ_2 & : & h_4 \succ h_2 \succ \cdots \\ \succ_3 & : & h_1 \succ h_3 \succ \cdots \\ \succ_4 & : & h_5 \succ h_4 \succ \cdots \\ \succ_5 & : & h_6 \succ h_5 \succ \ldots \end{array}$$

Assume $\rhd := (5, 1, 2, 3, 4)$*, which is distance-based. The set of vacant houses is* $\{h_5, h_6\}$*. First, agent* 5 *is selected.* 5 *is a newcomer, and she has no child. Therefore, she can choose only from the currently vacant houses* $\{h_5, h_6\}$*. Since her favorite house is* h_6*, she obtains it and leaves the market. Now the set of vacant houses becomes* $\{h_5\}$*. Next, agent* 1 *is selected. Since she is an existing tenant, she can choose* h_1 *(her initial endowment house),* h_3 *(her child's house), or* h_5*. Her favorite house among them is* h_3*. Since it is her child's house, she remains in the market and agent* 3 *becomes the top. Then, agent* 3 *is selected. She can choose* h_1 *(a house owned by her parent* 1 *who still remains in the market),* h_3*, or* h_5*. Her favorite house is* h_1*. Thus, agent* 1 *and* 3 *exchange their initial endowment houses and leave the market. Next, agent* 2 *is selected. She can choose* h_2*,* h_5*, or* h_4 *(a house owned by her child). Her favorite house among them is* h_4*. Since it is her child's house, she remains in the market and agent* 4 *becomes the top. Then, agent* 4 *is selected. She can choose* h_2 *(a house owned by her parent* 2 *who still remains in the market),* h_4*, or* h_5*. Her favorite house is* h_5*. Since it is vacant, she obtains it and leaves the market. The set of currently vacant houses becomes* $\{h_4\}$*. Next, agent* 2 *is selected again. She can choose* h_4 *or* h_2*. Her favorite house is* h_4*, which is currently vacant. Thus, she obtains it and leaves the market.*

In contrast to the original YRMH-IGYT, YRMH-IGYT for TSSNs restricts set of houses an agent can choose. A tricky part is that this set changes dynamically; if a tenant obtains a vacant house, her initial endowment house is added to this set. As a result, proving SP of YRMH-IGYT for TSSNs becomes non-trivial.

Theorem 4. *YRMH-IGYT for TSSNs is SP for any tree.*

Proof. The proof is straightforward from the following three lemmas. □

Lemma 1. *Assume when agent i is selected, i request the house owned by her child j (i.e., i points to j) and the set of currently vacant houses is* H'_V*. When she is selected again, either (a) the set of currently vacant houses has not changed (i.e., it is still* H'_V*) or (b) h is the sole additional house that has become vacant and exactly one house* $h' \in H'_V$ *is no longer vacant.*

Proof. Assume the set of vacant houses changes before i is selected again. Clearly, no newcomer can be activated before i is activated again (since she does not own a house that her parent can desire). This means that the first time the set of vacant houses changes, agent k (who is a descendent of i) exchanges her endowment house h_k for vacant house $h' \in H'_V$ and leaves the market. Next, k's parent ℓ is activated again. Since ℓ prefers h_k over any house in H'_V (or k would not have been activated), ℓ then exchanges her initial endowment house for h_k. This starts a chain of parents obtaining their child's house directly after it becomes vacant. At the end of this chain, i is activated again directly after h becomes vacant (as j obtained her child's vacated house), while h' remains the only originally vacant house that is no longer vacant. □

Lemma 2. *For agent i and any given set of her neighbors* r'_i*, submitting her true preference* $\succ_i$ *is a dominant strategy.*

Proof. First, observe that an agent cannot affect the timing when she is selected first because it is determined by the priority order and the preferences of her ancestors.

By Lemma 1, if the set of currently vacant houses is H'_V when agent i is first selected, then, a house that she has a chance to obtain in the current selection or any future selections, is (i) any element of H'_V, (ii) a house owned by her ancestor (assuming her ancestor is still in the market), (iii) her initial endowment house, or (iv) a house owned by her child. If her most preferred house is among (i), (ii), or (iii), she can obtain it right now, so modifying her preference is useless. If her most preferred house is in (iv), by Lemma 1, either she can obtain it or she will be re-selected with the same set of choices. Thus, modifying her preference is useless. □

Lemma 3. *For agent i and any given preference* $\succ'_i$*, submitting all her children* r_i *is a dominant strategy.*

Proof. Let us compare two cases: (a) agent i invites $r'_i \subsetneq r_i$, (b) agent i invites $r'_i \cup \{j\}$, where $j \in r_i \setminus r'_i$. Assume i obtains h in case (a). Then, she can also clearly obtain h in case (b) if it is (i) a currently vacant house, (ii) a house owned by her ancestor (if her ancestor is still in the market) or (iii) her own initial endowment house. Finally, if h is (iv) a house owned by another child $k \in r'_i$, then k will be activated and leave the market before j is first activated. Whether j is invited therefore cannot influence whether k's house is attainable for i. Thus, inviting one more child is never harmful; she is guaranteed to obtain h. Furthermore, if she prefers the initial endowment house of j over h, she might be able to obtain it. As a result, inviting r_i, i.e., all of her children, is a dominant strategy. □

We also show that YRMH-IGYT for TSSNs satisfies weak NW, IR, and SC4N for any tree, and runs in polynomial-time $O(n \cdot n')$.

Theorem 5. *YRMH-IGYT for TSSNs is weakly NW, IR, and SC4N for any tree.*

Proof. For weak NW, by way of contradiction, assume $\exists i \in A$, $\exists h \in m(s) \cap \widehat{m}(s)$, $h \succ_i m(i)$ holds. When agent i is selected, i must have requested h, since h is initially vacant and remains vacant, and $h \succ_i m(i)$ holds. Also, h must be assigned to i since h is vacant. This contradicts our assumption that $m(i)(\neq h)$ is allocated to i.

IR is obvious since agent i can request her initial endowment house $\widehat{m}(i)$ anytime. The fact that she obtains $m(i)(\neq \widehat{m}(i))$ implies that she requested $m(i)$ instead of $\widehat{m}(i)$ and further implies that $m(i) \succ_i \widehat{m}(i)$ holds.

For SC4N, by way of contradiction, assume there exists a blocking pair (i, j), where i is the parent of j. There are two cases: (i) $\widehat{m}(j) \succ_i m(i)$ and $\widehat{m}(i) \succeq_j m(j)$ hold, or (ii) $\widehat{m}(j) \succeq_i m(i)$ and $\widehat{m}(i) \succ_j m(j)$ hold. Since i is the parent of j, i must be selected before j. Then, for either case, i must eventually point to j, since $\widehat{m}(j)$ is at least weakly better than $m(i)$. Then, j is selected, and she must eventually point to i, since $\widehat{m}(i)$ is at least weakly better than $m(j)$. Then, they must exchange their initial endowment houses; $m(i) = \widehat{m}(j)$ and $m(j) = \widehat{m}(i)$ hold. This contradicts our assumption that either $\widehat{m}(j) \succ_i m(i)$ or $\widehat{m}(i) \succ_j m(j)$ holds. □

Theorem 6. *For given A, H, $\rhd$, θ', the time complexity of YRMH-IGYT for TSSNs is $O(n \cdot n')$, where n is the number of agents and n' is the number of houses.*

Proof. First, the procedure within the while loop (from line (3) to line (27)), where we create the list of houses that she can request (H'), and find the most preferred house within it can be done $O(n')$ time. Further, the while loop is executed at most $2n - 1$ times. This is because, agent i with $|r_i|$ children can be selected at most $1 + |r_i|$ times (the worst case is that she requests all her children's houses and fails, and finally obtains something else). Summing this over all agents, we obtain $\sum_{i \in A}(1+|r_i|) = n+\sum_{i \in A} |r_i| \leq 2n-1$, where the last inequality follows because any agent has exactly one parent (which can be another agent or the moderator). Hence, the overall time complexity is $O(n \cdot n')$. □

As described in Section 4, the original YRMH-IGYT is equivalent to the variant of TTC where each agent points to her favorite house, each occupied house points to its owner, and each vacant house points to the remaining top agent according to $\rhd$. This equivalence is not preserved when restricting whom agents can point to. If we modify TTC such that each agent can point to only (i) any currently vacant house, (ii) the houses of her ancestors, (iii) her initial endowment house, or (iv) a house owned by her child, then the resulting mechanism might look similar to YRMH-IGYT for TSSN, but in fact fails to satisfy SP. This is illustrated by the following example.

Example 3. *There exist three agents in the market, tenants 1, 2, and 3. The houses initially owned by 1, 2, 3 are h_1, h_2, h_3, while h_4 is a vacant house. Agents' preferences are given as follows:*

$$\begin{array}{lcl} \succ_1 & : & h_2 \succ h_1 \succ \cdots \\ \succ_2 & : & h_4 \succ h_1 \succ \cdots \\ \succ_3 & : & h_3 \succ \cdots \end{array}$$

The social network is given in Fig. 1b. A distance-based priority is given as $\rhd := (2, 1, 3)$. The allocation decided by YRMH-IGYT for TSSNs should be $m(1) = h_2$, $m(2) = h_4$ and $m(3) = h_3$. On the other hand, in the TTC-based mechanism, agents 2 and 3 can point to their favorite house, i.e., h_4 and h_3. However, for agent 1, since her favorite house h_2 is not vacant, she cannot point to it. As a result, agent 1 points to h_1. Vacant house h_4 points to agent 2, and each occupied house h_i points to agent i. There are three cycles and each agent obtains the house to which she points. As a result, the allocation should be $m(1) = h_1$, $m(2) = h_4$, and $m(3) = h_3$, which is different from the allocation obtained by YRMH-IGYT for TSSNs. Assume agent 1 misreports her preference as $h_3 \succ_1 h_2 \succ_1 h_1 \succ_1 \cdots$. Then, in the TTC-based mechanism, agent 1 first points to h_3. As a result, agent 1 is not included in any cycle. Agent 2 and 3 obtains h_4 and h_3 respectively and leave the market. h_2 becomes vacant and points to agent 1 (i.e., the only agent remaining in the market). Then, agent 1 points to h_2. A cycle is formed and agent 1 obtains h_2 and leaves the market. Thus, this misreport is profitable for agent 1. In short, an agent may have an incentive to delay the timing that she is included in a cycle until her favorite house becomes available. Such an incentive does not exist in YRMH-IGYT for TSSNs since an agent cannot control the timing when she is first selected. Also, as shown in Lemma 1, pointing to her child such that she will be selected later is useless.

6 COMPARISON WITH BASELINES

Assuming there are no newcomers or vacant houses, Kawasaki mechanism [9] is also SP and IR for trees. It is defined based on the original TTC algorithm, where each agent points to the agent who owns her preferred house. The only difference is that each agent is only allowed to point to herself, her parent, and her descendants. In this section, we present two extensions of the Kawasaki mechanism to work in our model, as well as another simple mechanism called Take-it-or-leave-it with distance-based priority. Although these mechanisms are SP, they fail to satisfy some of our desiderata.

Kawasaki mechanism with market split: Apply Kawasaki mechanism among existing tenants (who can exchange their initial endowment houses), while initially vacant houses are allocated only to newcomers. We apply a serial dictatorship mechanism with distance-based priority for all the newcomers.

Kawasaki mechanism with random endowments: First randomly assign vacant houses to newcomers, and then apply Kawasaki mechanism to all the agents.

Take-it-or-leave-it with the distance-based priority: Choose agents sequentially based on the distance-based priority. For agent i, if $i \in A_N$, assign i her most preferred house within the currently vacant houses. If $i \in A_E$, assign her most preferred house within the currently vacant houses and h_i (i.e., her initial endowment house). If a currently vacant house is allocated to i, then h_i becomes vacant.

The following examples show that these mechanisms fail to satisfy some of our desiderata.

Example 4. *Assume there are two existing tenants, 1 and 2, that initially own h_1 and h_2 respectively. There is also a newcomer 3. The moderator owns two vacant houses, h_3 and h_4. The social network is given in Fig. 1b, and the agents' preferences are given as follows:*

$$\begin{array}{lcl} \succ_1 & : & h_3 \succ h_4 \succ h_1 \\ \succ_2 & : & h_4 \succ h_3 \succ h_1 \succ h_2 \\ \succ_3 & : & h_4 \succ h_3 \succ h_1 \succ h_2 \end{array}$$

In both TTC-based mechanisms, either h_3 or h_4 is assigned to newcomer 3, while one of them remains vacant. In both extensions, agent 1 (as well as 2) obtains her initial endowment house. Therefore, she prefers the vacant house (which is either h_3 or h_4) over her assignment. Thus, these mechanisms fail to satisfy weak NW.

Example 5. *Assume there are three existing tenants, 1, 2, and 3, each of whom initially owns h_1, h_2, and h_3, respectively. The moderator owns one vacant house, h_4. The social network is given in Fig. 1b,* *and the agents' preferences are given as follows:*

$$\begin{array}{lcl} \succ_1 & : & h_3 \succ h_1 \succ \cdots \\ \succ_2 & : & h_4 \succ h_2 \succ \cdots \\ \succ_3 & : & h_1 \succ h_3 \succ \cdots \end{array}$$

In Take-it-or-leave-it mechanism with distance-based priority $\rhd := (1, 2, 3)$, first, agent 1 is chosen and assigned h_1, since the only currently vacant house is h_4 and $h_1 \succ_1 h_4$ holds. Then, agent 2 is chosen and assigned h_4, i.e., her most preferred house. Finally, agent 3 is chosen and assigned h_3, since she prefers h_3 over the currently vacant house h_2. This assignment satisfies weak NW and this mechanism even satisfies weak NW in general. However, $(1, 2)$ is a blocking pair since they would be happier if they exchange their initial endowment houses. Thus, Take-it-or-leave-it mechanism does not satisfy WC4N.

Weak NW fails in the two extensions of Kawasaki mechanism since the number of vacant houses ultimately allocated to the agents is exactly the same as the number of newcomers. In YRMH-IGYT for TSSNs, as many new vacant houses as possible can be assigned if agents prefer them; giving it more flexibility to achieve weakly NW allocations. Also, Take-it-or-leave-it mechanism fails to satisfy WC4N, since agents cannot exchange their initial endowments.

We can also consider the following simple extension of Kawasaki mechanism to handle vacant houses, i.e., each agent is also allowed to point to any currently vacant house. However, we can show that this simple extension does not satisfy SP, since it becomes equivalent to the modified TTC in the setting of Example 3. Furthermore, if we modify YRMH-IGYT for TSSNs such that an agent can request a house owned by her descendant, it is no longer SP. This is because for agent i, her child j can be selected before her (and can take away her favorite house) if her ancestor requests j's house.

7 RESILIENCE TO GROUP MANIPULATIONS

In the literature of house allocations and housing market, discussing the resilience to manipulations by groups of agents is another important direction.

Definition 11 (Strong Group Strategy-Proofness). *Mechanism f is weakly manipulable by coalition $S \subseteq A$ if $\exists \theta'_S \in R(\theta_S)$ s.t. (a) $\forall i \in S, f_i(\theta'_S, \theta_{-S}) \succeq_i f_i(\theta_S, \theta_{-S})$, and (b) $\exists j \in S, f_j(\theta'_S, \theta_{-S}) \succ_j f_j(\theta_S, \theta_{-S})$. A mechanism satisfies* strong group strategy-proofness *(strong GSP) if it is not weakly manipulable by any coalition.*

We can analogously define *weak group strategy-proofness* (weak GSP) by replacing conditions (a) and (b) with (c) $\forall i \in S, f_i(\theta'_S, \theta'_{-S}) \succ_i f_i(\theta_S, \theta'_{-S})$. Here, we require that all members in the coalition must be strictly better off by manipulation.

Example 6 below shows that YRMH-IGYT for TSSNs violates strong GSP, while Theorem 7 shows that it satisfies weak GSP.

Example 6. *There exist four agents in the market: existing tenants 1, 2, 3 and newcomer 4. The social network is given in Fig. 1c and their preferences are given as follows:*

$$\begin{array}{lcl} \succ_1 & : & h_5 \succ h_1 \succ \cdots \\ \succ_2 & : & h_3 \succ h_2 \succ \cdots \\ \succ_3 & : & h_1 \succ h_2 \succ h_3 \succ \cdots \\ \succ_4 & : & h_2 \succ h_4 \succ \cdots \end{array}$$

There are three occupied houses h_1, h_2, h_3, and two vacant houses h_4 and h_5. Assume $\rhd := \{4, 1, 2, 3\}$. Then, the allocation decided by YRMH-IGYT for TSSNs should be $m(1) = h_5, m(2) = h_3, m(3) = h_2$, and $m(4) = h_4$. Assume agents 3 and 4 collude; agent 4 modifies her preference as $h_4 \succ_4 \cdots$, while agent 3 submits her true preference. Then, the allocation becomes $m(1) = h_5$, $m(2) = h_2$, $m(3) = h_1$, $m(4) = h_4$. Here, the assignment of agent 4 is the same, while the assignment of agent 3 becomes h_1 instead of h_2, while she prefers h_1 over h_2. Hence, YRMH-IGYT for TSSNs does not satisfy strong GSP.

Theorem 7. *YRMH-IGYT for TSSNs satisfies weak GSP.*

Proof. Assume a group misreport by a set of agents $S \subseteq A$ exists, where each agent in S obtains a strictly better house than with truthful reporting. Consider agent $i \in S$ that is selected first within S. Note that agent i can be selected several times before she leaves the market. If S contains agents who are selected first after i leaves the market, these agents cannot affect the outcome of the rest of S (who were selected before i leaves). Thus, we can assume w.l.o.g. that all agents in $S \setminus \{i\}$ are selected and leave the market before i leaves the market. By Lemma 1, an agent that is selected can only obtain one of the houses that are currently vacant at the time of their first selection H'_V, her initial endowment house, the house owned by her ancestor (assuming the ancestor is still in the market), or the houses owned by her children. If agent i is the first selected agent that misreports, the misreports of the other agents cannot change H'_V. Therefore, for i to obtain a better outcome than with truthful reporting, she has to obtain the house of one of her children j that she does not obtain if everyone reports truthfully. Denote this house by h'. The only misreports of i that have any effect on the mechanism (and do not make i leave the market with a house in H'_V) is the order she gives over the houses of her children. However, by Lemma 1, any child that i points to before i points to j either does not change H'_V or causes i to leave the market (and therefore, would not result in her obtaining h'). Thus, no matter what misreport i makes, when she points to j, the set of vacant houses is H'_V. Then, for misreporting agents except i, the situation is identical to the case that i acts truthfully. This fact implies $S \setminus \{i\}$ can also make a group-misreport that is profitable for each member. By repeating the same argument, we obtain a situation where only one agent remains in the group. However, this contradicts the fact that YRMH-IGYH for TSSNs is SP. □

8 CONCLUDING REMARKS

To the best of our knowledge, our work is the first attempt to consider house allocation with existing tenants over social networks. Clarifying some other network structures where we can obtain a mechanism that satisfies desirable properties is an important future direction. Discussing other resource allocation problems (e.g., two-sided matching, combinatorial auctions) from the perspective of mechanism design over social networks would also be interesting.

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