Shooter Location Problems

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1 Introduction

The transversal or stabbing problem is well-studied in computational geometry and has applications in image processing, robotics, hidden surface removal in graphics, etc. Given a number of objects distributed on a two- or three-dimensional region, the objective is to check whether a common transversal i.e., a straight line segment passing through all the objects exists, and if so, determine the same. Edelsbrunner, Overmars and Wood [5] developed a method for computing the transversal in E^2, if it exists, in $O(n^2 \log n)$ time and $O(n)$ space, where n is the number of objects. However an $O(n \log n)$ time algorithm exists for line segments of arbitrary orientations [4], for isothetic rectangles of arbitrary aspect ratio[8], for circles of unequal radii [1], and for a set of arbitrary polygons [3]. In E^3, a stabbing plane is obtained in $O(n^2)$ time for n line segments, and in $O(n^2\alpha(n))$ time for a set of polyhedra with n total vertices, where $\alpha(n)$ represents the extremely slowly growing functional inverse of Ackermann's function. In general, the hyperplane stabber for n lines in E^d can be reported in $O(n^d)$ time [2]. Other variations of shooting problem have been considered in [9, 10, 11].

This paper outlines a variant of the classical stabbing problem which is called the *shooter location problem.* A shooter can fire or emit rays along straight lines in arbitrary directions. The position of the shooter and/or the locus of the shooter's motion are given. A ray stabs all the objects in its linear path of motion from its origin upto infinity. Our objective is to minimize the number of shots necessary to exhaust (hit) all the objects. Thus, instead of finding a transversal (if it exists) through all the objects, we are interested in determining the minimum number of shots hitting all objects, subject to certain constraints the position or the locus of the shooter. In this paper, we shall discuss the case where the targets are straight line segments. The technique can easily be extended to account for any arbitrary shaped polygonal objects on the 2-D plane. If multiple shooters are available and each shooter is allowed to fire a single ray in a fixed direction from a given shooting line then the minimum number of shooters required and their positions can trivially be obtained in $O(n\log n)$ time using interval graph. If two shooting lines are given, the problem becomes NP-complete [6].

2 Formulation

We consider a finite set S of n straight line segments called sticks, each with finite length but arbitrary orientation, within a rectangular bounding box. A *shooting line* L passing through the box is given, and the shooter can be positioned anywhere on the line within the box and can emit multiple rays in arbitrary directions. The problem is to locate the position p of the shooter on L so that the number of shots required to exhaust all the objects is minimum. First we introduce the more primitive problem, called the *fixed shooter problem.* The formulation of the original *shooter location problem* will follow subsequently.

2.1 Fixed shooter problem

Consider a single shooter, positioned at a fixed point inside the bounding rectangle. The shooter is allowed to take multiple shots in arbitrary directions. The objective is to find the different firing angles, so that number of shots is minimized.

To solve this problem, we proceed as follows. For each member ℓ of S, join both the end points of ℓ with the specified point p, and extend these two lines upto the boundary of the floor to get its projection. Thus we obtain a set of arcs around the boundary of the floor. Now consider a graph whose nodes correspond to the arcs thus obtained; between two nodes, there is an edge if the corresponding arcs have a common point along the peripheri of the boundary. This graph is refered as *circular-arc graph* [7].

Definition : A set of nodes $\{ v_i$, i = 1,2,... ,k$\}$ of the circular-arc graph is said to form a *linear clique* if all the arcs represented by them have a common point along the boundary of the circle, and if the set is maximal, i.e., it is not subsumed by any other bigger set [7].

The minimum number of shots required to hit all the line segments from point p, is thus equal to the size of minimum linear clique covering all nodes of the above circular-arc graph. The number and directions of the shots can be obtained in $O(n)$ time if the end points of the arcs are sorted along the boundary of the circumscribing circle [7].

2.2 Shooter location problem

In this problem, a *shooting line* L is given. The shooter can sit at any point on L and can fire multiple shots in arbitrary directions. The goal is to locate the position p of the shooter on line L within the bounding rectangle, such that the number of shots required to hit all the objects is minimized. Notice that, each point on line L might serve as the position of the shooter; thus the solution space is infinite. We first reduce the search domain into finitely many classes, and then select the best position of the shooter.

To solve this problem, let us consider each pair of line segments (ℓ_i, ℓ_j) from the set S. On L, there are certain intervals from which both ℓ_i and ℓ_j can be hit by a single ray, fired in an appropriate direction (we call this region as '+' region for the pair (ℓ_i, ℓ_j)) in L; also there will be some regions on L from which the shooter needs two rays to hit ℓ_i and ℓ_j (we call it '$-$' region on L). We will now show that it is sufficient to consider onle the '+' regions for all pairs of sticks.

Consider a pair of non-intersecting sticks (ℓ_i, ℓ_j), and let us define a *separator* line τ_{ij} such that ℓ_i and ℓ_j lie in the opposite side of the straight line τ_{ij}. It is easy to observe that there exist an infinite number of separators of (ℓ_i, ℓ_j). A separator τ_{ij} is said to be *extremal* if it touches the end points of ℓ_i and ℓ_j. Needless to say, there exists exactly two extremal separators for every pair of non-intersecting sticks (as shown in Fig. 1). Let these two extremal separators meet the shooting line L at points μ_1 and μ_2.

Lemma 1 *For every pair of non-intersecting sticks (ℓ_i, ℓ_j), the nature of the region changes from '+' to '$-$' or from '$-$' to '+' at a point q on the shooting line L, if and only if an extremal separator passes through q.*

Proof : Clear.

Thus, the shooting line L is divided into at most three regions at the points μ_1 and μ_2 with either one '+' region between two '−' regions, or one '−' region between two '+' regions (Fig. 1). For all pair of sticks $\{(\ell_i, \ell_j) \mid i \neq j, i, j = 1, 2, \ldots n\}$, we find the '+' regions. If a pair (ℓ_i, ℓ_j) is intersecting, the entire span of L is marked '+', because, from any point on L, the intersection point of ℓ_i and ℓ_j can be hit by a single ray (see Fig. 2). Thus, in our algorithm, we shall ignore the '+' regions of all pairs of intersecting sticks and consider the '+' regions of non-intersecting sticks only. The number of '+' regions generated by all pairs of non-intersecting sticks is $O(n^2)$ in the worst case, since for each such pair, at most two '+' regions are generated.

We now consider an interval graph G^+ with the intervals that are labeled '+' on the shooting line L, and let us consider the set of all cliques of G^+. Each clique in the interval graph defines a *zone*, which is the maximum interval on L, common to all participating members of the clique (see Fig. 3). Let Z^+ be the set of zones corresponding to all cliques of G^+.

Lemma 2 *The zones in Z^+ are mutually disjoint.*

Proof : Clear.

Lemma 3 *Let x and y be any two points on the line L such that both of them lie in the same zone $z \in Z^+$. Let $S(x)$ denote the subset of sticks that can be shot by any single ray from x; then all members of $S(x)$ can be shot by a single ray fired from y also.*

Proof : Let L split the bounding box in two parts F_1 and F_2. Since the set of sticks $S(x)$ can be shot by a single ray from x, some portion of them must lie in a specific side, say F_1. Let $S = S_1 \bigcup S_2 \bigcup S_3$, where S_1 is the set of sticks which lie entirely on the side F_1; S_2 is the set of sticks which intersect L and S_3 is the set of sticks which lie entirely on the side F_2. We truncate each member of S_2 such that the portion lying in F_1, remains present. Let S_2' denote the set of truncated sticks. Now for each stick in $S_1 \bigcup S_2'$, consider an arc on the circumscribing circle of the bounding rectangle; each arc is obtained by joining its both end points with the point x and extending it upto the peripheri of the bounding circle. The family of arcs will thus produce an interval graph I_x. Similarly, for the point y, consider the interval graph I_y for the same set of sticks $S_1 \bigcup S_2'$. These two interval graphs are isomorphic as both x and y belong to the same zone z. The set of sticks $S(x)$ $(S(y))$ must correspond to a clique of I_x (I_y). Since I_x and I_y are isomorphic, all the members of $S(x)$ can be hit by a single ray from y also. □

Lemma 4 *Let x and y be any two points in a zone z on L. Then the minimum number of rays fired from the point x, and that fired from the point y to hit all members of S, are equal.*

Proof : Consider the minimum set of rays that are required to hit all members of S, fired from point x. Lemma 2 implies that for each ray fired from x (y), that hits the set of sticks $S(x)$ $(S(y))$, one can get a ray fired from y (x) that hits every member of the set $S(x)$ $(S(y))$. □

Lemma 5 *Let z be a zone on the line L and R be an adjacent interval which is not a zone (see Fig. 3). Then for each point in R, the minimum number of rays required to fire all sticks is greater than or equal to the minimum number of rays required from a shooting point in z.*

Proof : Let us consider two points x and y on L such that x lies in a zone z, and y lies in a neighboring interval R outside the zone z. Let $G_x(V_x, E_x)$ and $G_y(V_y, E_y)$ denote the circular-arc graphs formed by the projections of the members in S on the circle that circumscribes the bounding box as in the *fixed shooter problem*. Clearly, $V_x = V_y$, but $E_x \supseteq E_y$. Since the minimum nnumber of rays required to hit all the members of S from x (y) is equal to the minimum linear-clique cover of the graph G_x (G_y), the lemma follows. □

Thus, in order to find the desired position on L from which the number of shots required to hit all sticks is minimum, it suffices to consider only one representative shooting point from each zone. Finally, we select the one whose linear clique cover is minimum by solving the *fixed shooter problem* once for each zone. We now present our algorithm for solving this problem.

Algorithm

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for each pair of sticks {(ℓ_i, ℓ_j) | i ≠ j, i,j = 1,2,...n} do
begin
      if the pair (ℓ_i, ℓ_j) is intersecting then ignore it
        elsebegin
                  find two extremal separators of (ℓ_i, ℓ_j);
                  determine the region(s) marked '+' on the shooting line defined
                  by these two separators;
           end;
      (* Let I+ be the set of all intervals marked '+' on the line L *)
      locate the set (Z+) of zones by finding all cliques in the interval graph G+
      formed by the family of intervals I+;
      for each zone z_i ∈ Z+ do
      begin
        choose any representative point p_i in z_i on L;
        find the minimum number of shots M(i) fired from p_i, to hit all the sticks
        (* by solving fixed shooter problem *);
      end;
      report the point p_i on L for which M(i) is minimum, and
      the corresponding firing angles
end;
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Theorem 1 *The worst case time complexity of this algorithm is* $O(n^3)$.

Proof : On the basis of lemmata 2 to 5, we find that one has to consider just one representative point from each zone $z_i \in Z^+$. For each such point, we calculate the minimum number of rays necessary to hit all the objects by the algorithm described for the **fixed shooter problem**, and choose the point for which it is minimum amongst them. The time complexity for constructing Z^+ is $O(n^2)$, which is determined by its cardinality in the worst case. Initially, the end points of each stick are joined with the end points of all other sticks to get the extremal separators. These joining lines as well as the given sticks are extended (if necessary) to intersect the shooting line L. The number of such intersections inside the bounding box may be $(n^2 + n)$ in the worst case. The relative order of the end points of the projections of sticks along the boundary will be different for these intersection points on L. So the circular-arc graph is to be updated for each of these $(n^2 + n)$ positions on L. It can be easily shown that when a shooter moves from one intersection point to the other, the circular arc graph formed by the projected intervals can be updated just by swapping two appropriate points of projection. Furthermore, these two projection points are consecutive to each other in their circular order along the boundary. The minimum clique cover is to be obtained for a representative point in each zone of Z^+ by the linear time algorithm suggested in [7]. Thus, the total time complexity of our algorithm is $O(n^3)$ in worst case.

3 Conclusion

In this paper, the *shooter location problem* is introduced. A set of line segments, called sticks, are distributed inside a rectangular floor in 2-D, and a single shooter is available which is allowed to move on a given shooting line and can fire in any arbitrary direction. The objective is to minimize the number of shots. We have presented an $O(n^3)$ time algorithm for solving this problem. A linear time preprocessing step is suggested in the original paper to handle the case where the targets are arbitrary polygons. Next, the same techniques as that of sticks is adopted to solve this problem. The problem of finding the desired position of a shooter in the floor for firing all objects with minimum shots, is still open.

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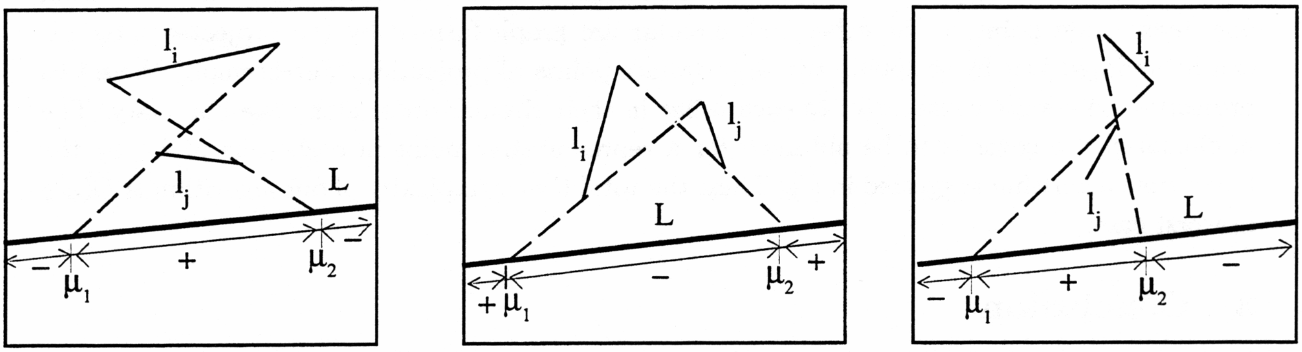

Fig. 1 : Extremal separators of a pair of non-intersecting sticks generating '+' and '–' regions

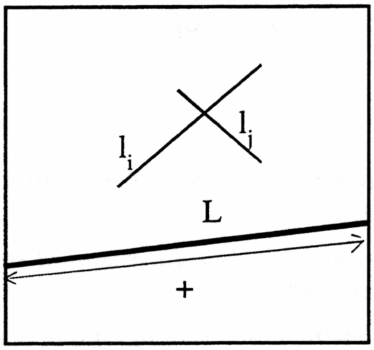

Fig. 2 : '+' region for intersecting sticks

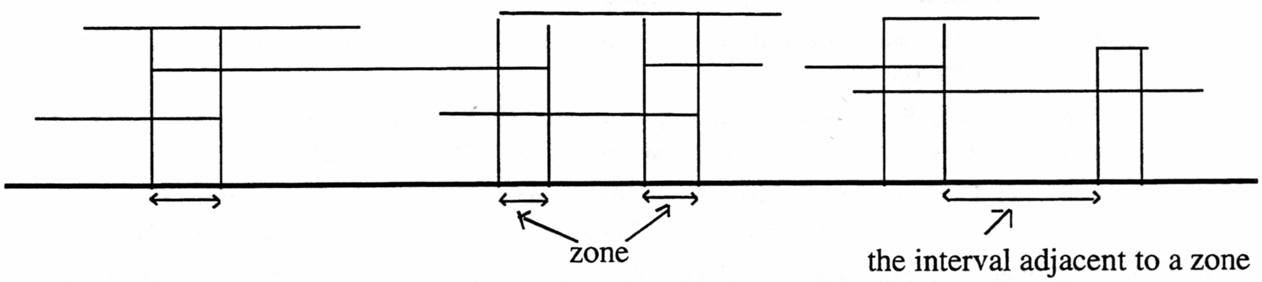

Fig. 3 : Zones of '+' regions on L