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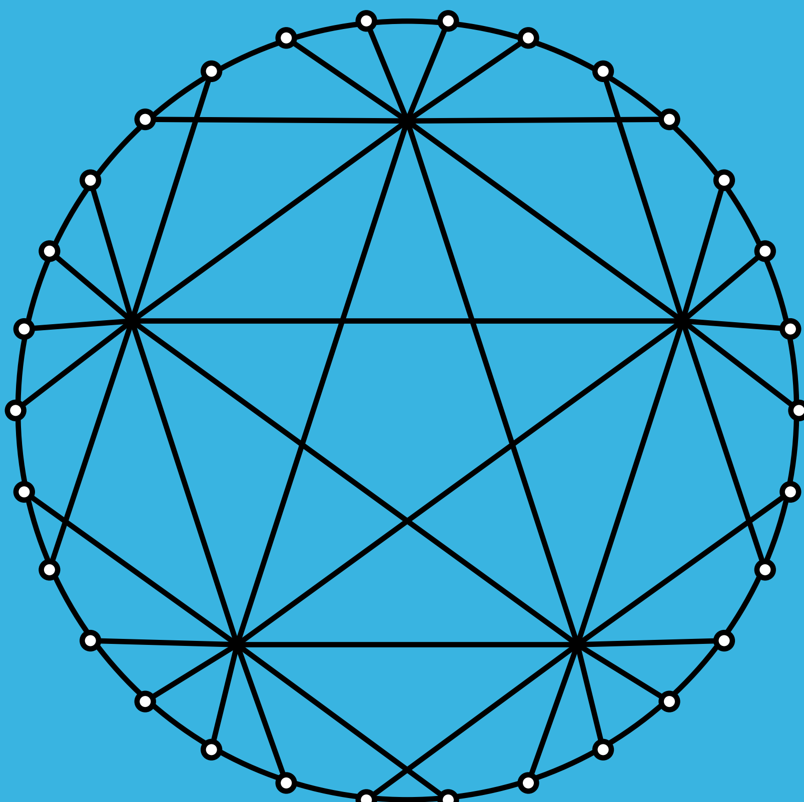

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Roman and Vatican crossover designs

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Abstract

Latin squares with a balance property among adjacent pairs of symbols—being "Roman" or "row-complete"—have long been used as uniform crossover designs with the number of treatments, periods and subjects all equal. This has been generalized in two ways: to crossover designs with more subjects and to balance properties at greater distances. We consider both of these simultaneously, introducing and constructing *Vatican designs*: these have ℓt subjects, t periods and treatments, and, for each d in the range $1 \leq d < t$, the number of times that any subject receives treatment j exactly d periods after receiving treatment i is at most ℓ. Results include showing the existence of Vatican designs when $4 \leq t \leq 14$ and $\ell > 1$, and when $t \in \{3, 15\}$ and ℓ is even.

1 Introduction

In the theory of experimental designs, a *crossover design* is one in which the experimental subjects each receive a test treatment in

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each of multiple periods. Suppose there are n subjects, t treatments and p periods. We shall display such a design as a $n \times p$ array D in which D_{ij} represents the treatment received by subject i in period j.

We shall limit ourselves to *uniform* crossover designs: those in which each treatment occurs the same number of times in each row and the same number of times in each column, and hence the number of rows and number of columns are each multiples of the number of treatments. We further limit our investigation to those in which $p = t$, so each subject receives each treatment exactly once.

In a uniform crossover design, for an ordered pair (x, y) of treatments define $o(x, y)$ to be the number of times y occurs immediately after x. If $o(x, y)$ is constant across all ordered pairs of distinct elements, then the design is *balanced.* Balance is desirable in situations where one treatment might have a "carry-over" effect to the next time period. A survey of the theory of such designs is [6].

A *Latin square* is a crossover design with $n = p = t$. If it is balanced, then it is *Roman* or *row-complete.* We extend the domain of this definition and call any balanced uniform crossover design *Roman.*

In the study of Roman squares, a stronger notion of balance was introduced by Etzion, Golomb and Taylor [9]. We also extend this to designs. Let $o_i(x, y)$ be the number of times that treatment y occurs exactly i time periods after x (so $o_1(x, y) = o(x, y)$ as defined above). A uniform design with $p = t$ is a Roman-k design if $o_i(x, y) \leq n/t$ for all $i \leq k$. For Latin squares, this says that each ordered pair of distinct treatments occurs at distance i at most once in the square for each $i \leq k$. Again mirroring the definitions for Latin squares, call a uniform design with $p = t$ *Vatican* if it is Roman-$(t-1)$ (that is, the balance property holds at all possible distances) and simply *Roman* if it is Roman-1.

Figure 1 shows a Roman (but not Roman-2) design with $n = t = 6$ and a Vatican design with $n = 2t = 10$. Clearly, there is some regularity to their construction; we explore this in the next two sections.

Figure 1: A Roman design and a Vatican design

0	5	1	4	2	3
1	0	2	5	3	4
2	1	3	0	4	5
3	2	4	1	5	0
4	3	5	2	0	1
5	4	0	3	1	2

0	1	3	4	2
1	2	4	0	3
2	3	0	1	4
3	4	1	2	0
4	0	2	3	1
0	4	2	1	3
1	0	3	2	4
2	1	4	3	0
3	2	0	4	1
4	3	1	0	2

Roman squares are a special case of a more general combinatotial object. A *Tuscan square* is a balanced crossover design with $n = p = t$ in which each treatment occurs once in each row. In other words, a Tuscan square is a Roman square, except that it need not have each treatment appearing exactly once in each column. Tuscan-k squares are similarly analogous to Roman-k squares and a Tuscan-$(t-1)$ square is called *Florentine.* For more on these objects, see [7]. Our notions of Roman-k and Vatican designs could be naturally generalised to Tuscan-k and Vatican designs in the same way, which might be of particular interest for parameters where Roman-k and Vatican designs do not exist. We do not pursue this line of enquiry here.

In the next section we show how to build designs from sequences of group elements and in Section 3 we employ and expand the theory of "terraces" to build these sequences. Ultimately, we are able to prove:

Theorem 1.1. *There is an $\ell t \times t$ Vatican design in each of the following cases:*

- $4 \leq t \leq 14$ *and* $\ell > 1$,
- $t \in \{3, 15\}$ *and* ℓ *is even,*

We also give various stronger results than the second item for some prime numbers t with $t \leq 281$. Existing results, which are described in the next section, imply the existence of an $\ell t \times t$ Vatican design when $t+1$ is prime (for any ℓ). Also, Williams proves directly that there is a Vatican design when t is prime and ℓ is a multiple of $t-1$ [18, pp. 156–7].

2 From groups to designs

The general method of construction is to form the desired $n \times t$ design, where $n = \ell t$, by taking ℓ Latin squares of order t. Each of the Latin squares is the Cayley table of a group. Most of the results can be achieved with cyclic groups, which we write as $\mathbb{Z}_t = \{0, 1, \ldots, t-1\}$ with the operation of addition modulo t, but we need the more general theory for some orders.

The following result means that we can limit our attention to small values of ℓ.

Lemma 2.1. *If there is an $\ell t \times t$ Roman-k design for each $\ell \in \{\ell_1, \ldots, \ell_m\}$, then there is an $nt \times t$ Roman-k design for $n = c_1\ell_1 + \cdots c_m\ell_m$ for any choice of non-negative integers $c_1, \ldots, c_m$.*

Proof. Simply stack c_i copies of the $\ell_i t \times t$ Roman-k design for each i. □

Thus Roman-k and Vatican squares are the ideal building block.

Existing results for these objects give many orders of Roman-k and Vatican designs:

Theorem 2.2. *There is an $\ell t \times t$ Roman-k design for all $\ell \in \mathbb{N}$ in each of the following cases:*

- $k = t - 1$ *(i.e. the design is Vatican) and* $t + 1$ *is prime,*
- $k = 2$ *and* $t = 2q$ *for some prime* q *with* $q \equiv 7 \pmod{12}$ *or* $q \equiv 5 \pmod{24}$*,*
- $k = 2$ *and* t *is even with* $t \leq 50$*,*
- $k = 2$ *and* $t = 21$*,*
- $k = 1$ *and* t *is composite.*

Proof. In each case there is a Roman-k square of order t [3, 8, 10, 12, 15, 18]. Applying Lemma 2.1 gives the result. □

The existence of an $\ell t \times t$ Vatican design when $t+1$ is prime follows from the first item of Theorem 2.2 for any ℓ.

Rather than trying to construct more Roman-k or Vatican squares, which seems to be a difficult problem, we take a different approach. Observe that for $\ell > 1$ we may write $\ell = 2c_1 + 3c_2$ for some $c_1, c_2 \geq 0$ and so to construct an $\ell t \times t$ Roman-k design for all $\ell > 1$ it suffices to construct them for $\ell = 2$ and $\ell = 3$. This is the essence of how the following result is proved (by Williams [18] for even ℓ and Prescott [17] for odd ℓ, using different terminology). We describe the proof early in the next section, using the terminology of this paper, when we have more machinery available.

Theorem 2.3. [17, 18] *There is an $\ell t \times t$ Roman design for $\ell > 1$ and $t \neq 3$.*

There is no $3\ell \times 3$ Roman design when ℓ is odd, and hence no $3\ell \times 3$ Vatican design when ℓ is odd.

Definition 2.4. Let G be a group of order t and let $\mathbf{a} = (a_1, \ldots, a_t)$ be an ordering of the elements of G. Let $g\mathbf{a} = (ga_1, \ldots, ga_t)$ and define $L(\mathbf{a})$ to be a Latin square with rows $\{g\mathbf{a} : g \in G\}$ (the order of the rows does not concern us). The squares we use to build design all have this form.

Given such a sequence $\mathbf{a}$, define its *quotient triangle*

$$(T_1, T_2, \ldots, T_{t-1})$$

by:

$$\begin{array}{rlllll}
T_1: & a_1^{-1}a_2, & a_2^{-1}a_3, & a_3^{-1}a_4, & \ldots, & a_{t-1}^{-1}a_t \\
T_2: & & a_1^{-1}a_3, & a_2^{-1}a_4, & \ldots, & a_{t-2}^{-1}a_t \\
T_3: & & & a_1^{-1}a_4, & \ldots, & a_{t-3}^{-1}a_t \\
 & & & & \vdots & \\
T_{t-1}: & & & & & a_1^{-1}a_t
\end{array}$$

When G is abelian, we usually use additive notation and call the quotient triangle the *difference triangle.*

These quotients control the neighbor properties we are interested in. For each occurrence of x in the ith line T_i of a quotient triangle, an ordered pair (g, h) with $g^{-1}h = x$ appears once at distance i among the rows of $L(\mathbf{a})$. This motivates the following definition.

Definition 2.5. Let G be a group of order t. Let $\mathbf{A} = (\mathbf{a_1}, \ldots, \mathbf{a}_\ell)$ where each $\mathbf{a_i} = (a_{i1}, a_{i2}, \ldots, a_{i,t})$ is an arrangement of the elements of G. Let T_i be the quotient triangle for $\mathbf{a_i}$, with lines

$$T_{i1}, T_{i2}, \ldots T_{i,t-1}$$

and let U_i be the concatenation of the ith lines of the quotient triangles $T_1, \ldots, T_\ell$. If each non-identity element of G appears at most ℓ times in each U_i for $1 \leq i \leq k$, then A is a *Roman-k ℓ-tuple.* Call a Roman-$(t-1)$ ℓ-tuple a *Vatican ℓ-tuple.* We refer to 1-, 2- and 3-tuples, with which we will mostly be working, as *singletons*, *pairs* and *triples* respectively.

A Roman-k singleton is known in the literature as a *directed T_k-terrace.*

Example 2.6. *A Vatican singleton for $\mathbb{Z}_6$:*

$\mathbf{a_1}$:	0	4	5	2	1	3
T_1 :		4	1	3	5	2
T_2 :			5	4	2	1
T_3 :				2	3	4
T_4 :					1	5
T_5 :						3

Theorem 2.7. *If a group of order t has a Roman-k ℓ-tuple then there is a $\ell t \times t$ Roman-k design.*

Proof. Let $(\mathbf{a_1}, \ldots, \mathbf{a}_\ell)$ be a Roman-k ℓ-tuple for G and consider the design D obtained by stacking $L(\mathbf{a_1})$, …, $L(\mathbf{a}_\ell)$. We have an occurrence of the ordered pair (g, h) of distinct elements of G at distance i in a row of D exactly once for every occurrence of $g^{-1}h$ in the ith line of the quotient triangle. Hence there are at most ℓ occurrences of each such pair (g, h) at distance i. $\square$

The challenge now is to construct these ℓ-tuples.

3 Constructing Roman-k and Vatican ℓ-tuples

In order to construct $2t \times t$ Roman designs, Williams introduced an example of what came to be known as a "terrace." We generalize this approach to create single sequences from which all of the sequences of an ℓ-tuple can in some cases be constructed.

Definition 3.1. Let G be a group of order t with an automorphism α of order ℓ. For $g \in G$ define the *cycle* of g under α as $\bar{g} = \{\alpha^r(g) : 1 \le r \le \ell\}$. Let $\mathbf{a} = (a_1, \ldots, a_t)$ be an ordering of the elements of G with quotient triangle $(T_1, \ldots, T_{t-1})$. For each non-identity element g, if the number of times an element of $\bar{g}$ occurs in T_i is at most $|\bar{g}|$ for $1 \le i \le k$ then $\mathbf{a}$ is an *ℓ-fold Roman-k pseudoterrace* with respect to α.

Lemma 3.2. *If a group G has an ℓ-fold Roman-k pseudoterrace then G has a Roman-k ℓ-tuple.*

Proof. Let $\mathbf{a}$ be an ℓ-fold Roman-k pseudoterrace for G where α is the appropriate automorphism of order ℓ. Set $\mathbf{A} = (\mathbf{a_1}, \ldots, \mathbf{a_\ell})$, where $\mathbf{a_r} = (\alpha^r(a_1), \ldots, \alpha^r(a_t))$ for each r. Then $\mathbf{A}$ is a Roman-k ℓ-tuple as U_i, the concatenation of the jth lines of the difference triangles, consists of the elements of the form $\alpha^r(a_i^{-1}a_{i+j})$ for $1 \leq r \leq \ell$. □

Hence an ℓ-fold Roman-k pseudoterrace for a group of order t implies the existence of an $\ell t \times t$ Roman-k design.

If α is given by $x \mapsto x^{-1}$ (in which case the group is abelian) then a Roman pseudoterrace is the same as a *terrace* as defined in [5]. The sequence

$$(0, t-1, 1, t-2, \ldots)$$

is a terrace for $\mathbb{Z}_t$, a construction first given by Walecki for even t (in which case it is a directed terrace or, in the vocabulary of this paper, a Roman singleton) and Williams for odd t [1, 18]. (Historical note: Williams and others do not apply the inverse automorphism to construct a Roman pair from a terrace. Instead they use that when $\mathbf{a}$ is a terrace then $\mathbf{a}$ along with the reverse of $\mathbf{a}$ is a Roman pair.)

Given that the Walecki construction gives a Roman singleton (and hence also a pair and a triple) for even t and a Roman pair when t is odd, a Roman triple for odd t is sufficient to complete the proof of Theorem 2.3 via application of Theorem 2.7 and Lemma 2.1.

For odd $t > 3$ Prescott [17] provides the Roman triple $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ given by

$$\mathbf{a_1} = \Bigg(0; 1, t-2, 3, t-4, \ldots, \frac{t+(-1)^{\lfloor t/2 \rfloor}}{2}; \\ \frac{t+(-1)^{\lfloor t/2 \rfloor}}{2} + 1, \ldots, t-3, 4, t-1, 2\Bigg)$$

$$\mathbf{a_2} = \Big(0; 2, t-1, 4, t-3, \ldots, \frac{t+(-1)^{\lfloor t/2 \rfloor}}{2} + 1; \\ \frac{t+(-1)^{\lfloor t/2 \rfloor}}{2}, \ldots, t-4, 3, t-2, 1\Big)$$

$$\mathbf{a_3} = \Big(0, t-1; t-2, 2, t-4, 4, \ldots, \frac{t-1+2(-1)^{\lfloor t/2 \rfloor}}{2}; \\ \frac{t-2-(-1)^{\lfloor t/2 \rfloor}}{2}; \frac{t-3+2(-1)^{\lfloor t/2 \rfloor}}{2}, \ldots, 3, t-5, 1, t-3\Big).$$

and thus we have all of the ingredients required to prove Theorem 2.3.

Additionally, Prescott [17] shows that each of the sequences $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$ are as close to Roman as possible, in the sense that one non-zero element appears twice among the differences, another does not appear at all, and the rest appear exactly once each.

Example 3.3. *Multiplication by* 2 *is an automorphism of* $\mathbb{Z}_7$ *of order* 3 *with cycles* $\{1, 2, 4\}$ *and* $\{3, 5, 6\}$*. A* 3*-fold Vatican pseudoterrace with respect to this automorphism:*

$\mathbf{a}$:	0	1	5	4	2	3	6
T_1:		1	4	6	5	1	3
T_2:			5	3	4	6	4
T_3:				4	1	5	2
T_4:					2	2	1
T_5:						3	5
T_6:							6

Hence

$$(0, 1, 5, 4, 2, 3, 6), (0, 2, 3, 1, 4, 6, 5), (0, 4, 6, 2, 1, 5, 3)$$

is a Vatican triple.

We now provide a number theoretic construction for pseudoterraces that are sometimes Roman-k for $k > 1$, and sometimes even Vatican (although these tend to be ℓ-fold with large ℓ). Given a prime p and a primitive root ρ of p. define the *primitive root construction* to be

$$0, \rho, \rho^2, \ldots, \rho^{p-1}.$$

Theorem 3.4. *The primitive root construction for a prime p with primitive root ρ is a Roman pseudoterrace with respect to multiplication by $r = \rho/(\rho - 1)$.*

Proof. The elements

$$\rho - 1, \rho^2 - \rho, \ldots, \rho^{p-1} - \rho^{p-2}$$

of $\mathbb{Z}_p$ are distinct. The differences of the primitive root construction are exactly these elements, with the exception that $\rho-1$ is replaced by ρ. As $r(\rho - 1) = \rho$, these two elements are in the same cycle with respect to r and the primitive root construction is a Roman pseudoterrace. □

When $\rho = (p+1)/2$ is a primitive root of p we find that $r = -1$ and so $\ell = 2$. In this case the primitive root construction is the "halving terrace" described in [16, Section 5], derived from [4, Theorem 2.1].

We are especially interested in determining when the primitive root construction gives a Roman-k pseudoterrace for $k > 1$. Trivially, if the pseudoterrace is ℓ-fold for $\ell = p - 1$ then it is a Vatican pseudoterrace (indeed, any arrangement of the elements of $\mathbb{Z}_p$ is a $(p - 1)$-fold pseudoterrace for any automorphism of order $p - 1$ [18]).

In Table 1 we compile the characteristics of pseudoterraces for all $\ell \mid p - 1$ that give the most neighbor-balance for primes up to $p = 61$. In Table 2 we give examples of Roman k pseudoterraces with $k > 1$ from the primitive root construction for primes p in the range $61 < p \leq 257$. In each case, Vatican pseudoterraces are bolded. For p in the range $258 < p < 1000$, here is a list of primitive root constructions $(p; \ell, k, \rho, r)$ that give Roman-k pseudoterraces with $k > 1$ and $\ell \leq 40$:

$(\mathbf{281};\mathbf{35},\mathbf{280},\mathbf{187},\mathbf{211})$,	$(281;40,3,3,142)$,	$(307;34,3,45,8)$,
$(331;11,2,3,167)$,	$(337;21,3,46,16)$,	$(337;28,2,154,164)$,
$(401;16,2,3,202)$,	$(419;22,2,6,85)$,	$(431;5,2,189,95)$,
$(443;13,2,136,339)$,	$(463;14,2,3,233)$,	$(521;40,2,41,509)$,
$(541;36,2,409,302)$,	$(601;30,2,254,583)$,	$(613;9,2,163,474)$,
$(701;35,2,39,536)$,	$(751;30,2,39,337)$,	$(757;28,3,206,710)$,
$(829;36,2,321,444)$,	$(991;22,4,22,237)$,	$(991;33,2,89,733)$.

Limiting to single-digit values of ℓ, in the range $1000 < p < 10000$, there is just one ℓ-fold Roman-k pseudoterrace with $k > 1$ and $2 \le \ell < 10$ (same format as previous list): $(2017;9,2,1032,1525)$.

The bolded entries in the table and lists above give rise to many new families of Vatican designs. Theorem 3.5 collects those that are better in the sense that $(p-1)/\ell$ is larger (in particular, it collects the instances with $(p-1)/\ell \ge 5$).

Theorem 3.5. *There is an $\ell t \times t$ Vatican design for*

$$(t,\ell) \in \left\{ \begin{array}{l} (61,10),(71,14),(79,13),(101,20),(109,18), \\ (113,16),(131,26),(151,25),(157,26),(181,36), \\ (191,38),(211,42),(239,34),(251,50),(281,35) \end{array} \right\}$$

Proof. The required ℓ-fold Vatican pseudoterraces may be found as bolded entries in Table 1 (for $t = 61$), the list of Roman-k pseudoterraces with $k > 1$ and $\ell < 40$ above (for $t = 281$), and in Table 2 (all remaining cases). □

We now turn to pseudoterraces in small groups. Let

$$D_{2m} = \langle u, v : u^m = e = v^2, vu = u^{-1}v \rangle$$

be the dihedral group of order $2m$, and let

$$Q_8 = \langle u, v : u^4 = e, v^2 = u^4, vu = u^{-1}v \rangle$$

be the quaternion group of order 8.

Table 1: Achieving the highest value of k in an ℓ-fold Roman-k pseudoterrace for $\mathbb{Z}_p$ for non-trivial divisors ℓ of $p-1$ with the primitive root method. For each prime p we give quadruples (ℓ, k, ρ, r). Vatican pseudoterraces are given in bold. Values of ℓ for which there is no successful primitive root construction are given as singletons (ℓ).

$p = t$	(ℓ, k, ρ, r)
5	$\mathbf{(2, 4, 3, 4)}$
7	$(2), (3)$
11	$(2, 1, 6, 10)$, $\mathbf{(5, 10, 8, 9)}$
13	$(2, 1, 7, 12), (3, 1, 6, 9), (4, 1, 11, 5), (6)$
17	$(2), (4, 2, 7, 4)$, $\mathbf{(8, 16, 12, 15)}$
19	$(2, 1, 10, 18), (3, 2, 3, 11), (6)$, $\mathbf{(9, 18, 15, 16)}$
23	(2), $\mathbf{(11, 22, 21, 16)}$
29	$(2, 1, 15, 28), (4, 1, 21, 17)$, $\mathbf{(7, 28, 27, 20)}$, $\mathbf{(14, 28, 19, 22)}$
31	$(2), (3), (5, 1, 22, 4), (6)$, $\mathbf{(10, 30, 21, 15)}$, $\mathbf{(15, 30, 24, 28)}$
37	$(2, 1, 19, 36), (3), (4, 1, 22, 31), (6), (9, 1, 15, 9)$, $\mathbf{(12, 36, 17, 8)}$, $\mathbf{(18, 36, 24, 30)}$
41	$(2), (4), (5, 1, 12, 16), (8, 1, 11, 38)$, $\mathbf{(10, 40, 29, 23)}$, $\mathbf{(20, 40, 35, 36)}$
43	$(2), (3), (6), (7, 1, 20, 35)$, $\mathbf{(14, 42, 18, 39)}$, $\mathbf{(21, 42, 34, 31)}$
47	(2), $\mathbf{(23, 46, 45, 32)}$
53	$(2, 1, 27, 52), (4, 1, 12, 30)$, $\mathbf{(13, 52, 51, 36)}$, $\mathbf{(26, 52, 35, 40)}$
59	$(2, 1, 30, 58)$, $\mathbf{(29, 58, 56, 45)}$
61	$(2, 1, 31, 60), (3), (4, 1, 6, 50), (5), (6)$, $\mathbf{(10, 60, 30, 41)}$, $(12, 1, 59, 21), (15, 2, 51, 12), (20, 1, 26, 23)$, $\mathbf{(30, 60, 54, 39)}$

Table 2: Achieving values of $k>1$ in an ℓ-fold Roman-k pseudoterrace for $\mathbb{Z}_p$ for non-trivial divisors ℓ of $p-1$ with the primitive root method. For each prime p we give quadruples (ℓ,k,ρ,r). Vatican pseudoterraces are given in bold.

$p=t$	(ℓ,k,ρ,r)
67	$\mathbf{(22,66,50,27)}, \mathbf{(33,66,46,4)}$
71	$\mathbf{(14,70,55,26)}, \mathbf{(35,70,67,15)}$
73	$(9,2,34,32), \mathbf{(24,72,33,17)}, \mathbf{(36,72,59,35)}$
79	$\mathbf{(13,78,75,64)}, \mathbf{(26,78,74,14)}, \mathbf{(39,78,70,72)}$
83	$\mathbf{(41,82,80,63)}$
89	$\mathbf{(22,88,30,44)}, \mathbf{(44,88,76,20)}$
97	$(16,2,15,8), \mathbf{(24,96,59,93)}, \mathbf{(32,96,87,45)}, \mathbf{(48,96,56,31)}$
101	$\mathbf{(20,100,48,44)}, \mathbf{(25,100,99,68)}, \mathbf{(50,100,94,64)}$
103	$\mathbf{(34,102,84,37)}, \mathbf{(51,102,96,91)}$
107	$\mathbf{(53,106,104,81)}$
109	$\mathbf{(18,108,14,43)}, \mathbf{(27,108,70,80)}, \mathbf{(36,108,103,32)}, \mathbf{(54,108,99,100)}$
113	$\mathbf{(16,112,92,78)}, \mathbf{(28,112,76,111)}, \mathbf{(56,112,80,104)}$
127	$(9,2,6,52), (18,3,12,105), \mathbf{(42,126,114,10)}, \mathbf{(63,126,112,120)}$
131	$\mathbf{(26,130,95,47)}, \mathbf{(65,130,127,27)}$
137	$\mathbf{(34,136,125,22)}, \mathbf{(68,136,114,98)}$
139	$\mathbf{(46,138,119,87)}, \mathbf{(69,138,134,24)}$
149	$\mathbf{(37,148,137,127)}, \mathbf{(74,148,147,100)}$
151	$\mathbf{(25,150,134,110)}, \mathbf{(50,150,146,26)}, \mathbf{(75,150,141,97)}$
157	$\mathbf{(26,156,21,56)}, \mathbf{(39,156,104,126)}, \mathbf{(52,156,123,149)},$ $\mathbf{(78,156,96,120)}$
163	$(27,3,19,155), \mathbf{(54,162,112,48)}, \mathbf{(81,162,148,113)}$
167	$\mathbf{(83,166,165,112)}$
173	$\mathbf{(43,172,166,109)}, \mathbf{(86,172,171,116)}$
179	$\mathbf{(89,178,176,135)}$
181	$\mathbf{(36,180,171,149)}, \mathbf{(60,180,76,71)}, \mathbf{(90,180,163,20)}$
191	$(19,4,58,125), \mathbf{(38,190,148,14)}, \mathbf{(95,190,189,128)}$
193	$(16,2,53,27), \mathbf{(48,192,174,165)}, \mathbf{(64,192,188,33)}, \mathbf{(96,192,155,95)}$
197	$\mathbf{(49,196,195,132)}, \mathbf{(98,196,185,107)}$
199	$(33,2,38,157), \mathbf{(66,198,176,59)}, \mathbf{(99,198,195,160)}$
211	$(5,2,3,107), (30,2,48,10), \mathbf{(42,210,155,38)},$ $\mathbf{(70,210,118,102)}, \mathbf{(105,210,187,136)}$
223	$\mathbf{(74,222,149,111)}, \mathbf{(111,222,205,177)}$
227	$\mathbf{(113,226,224,171)}$
229	$\mathbf{(57,228,201,151)}, \mathbf{(76,228,205,175)}, \mathbf{(114,228,223,99)}$
233	$\mathbf{(58,232,166,210)}, \mathbf{(116,232,213,123)}$
239	$(17,2,42,36), \mathbf{(34,238,156,203)}, \mathbf{(119,238,237,160)}$
241	$(60,2,66,90), \mathbf{(80,240,227,17)}, \mathbf{(120,240,204,20)}$
251	$\mathbf{(50,250,29,10)}, \mathbf{(125,250,248,189)}$
257	$(16,2,86,128), \mathbf{(64,256,217,95)}, \mathbf{(128,256,252,215)}$

Table 3: Some ℓ-fold Vatican pseudoterraces for small groups; $\ell \in \{2,3\}$.

Group	ℓ	Aut.	Pseudoterrace
$\mathbb{Z}_3$	2	$1 \mapsto 2$	$(0,1,2)$
$\mathbb{Z}_4$	2	$1 \mapsto 3$	$(0,1,3,2)$
$\mathbb{Z}_2^2$	3	$01 \mapsto 10$	$(00,01,10,11)$
		$10 \mapsto 11$	
$\mathbb{Z}_5$	2	$1 \mapsto 4$	$(0,1,3,2,4)$
$\mathbb{Z}_6$	2	$1 \mapsto 5$	$(0,1,4,2,3,5)$
D_6	2	$u \mapsto u^2$	(e,u^2v,u^2,u,v,uv)
		$v \mapsto v$	
	3	$u \mapsto u$	(e,u,v,u^2,u^2v,uv)
		$v \mapsto u^2v$	
$\mathbb{Z}_7$	2	$1 \mapsto 6$	$(0,1,3,6,4,5,2)$
	3	$1 \mapsto 2$	$(0,1,3,2,5,6,4)$
$\mathbb{Z}_8$	2	$1 \mapsto 3$	$(0,1,3,5,2,6,7,4)$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	2	$10 \mapsto 30$	$(00,01,10,21,31,11,30,20)$
		$01 \mapsto 01$	
$\mathbb{Z}_2^3$	2	$100 \mapsto 101$	$(000,010,100,011,111,110,101,001)$
		$010 \mapsto 010$	
		$001 \mapsto 001$	
D_8	2	$u \mapsto u^3$	$(e,v,u^3v,u^2v,u,u^3,uv,u^2)$
		$v \mapsto v$	
Q_8	2	$u \mapsto u$	$(e,u,v,u^2v,u^3v,u^3,uv,u^2)$
		$v \mapsto u^2v$	
	3	$u \mapsto v$	$(e,u,u^2,v,uv,u^3v,u^2v,u^3)$
		$v \mapsto$	u^3v
$\mathbb{Z}_9$	2	$1 \mapsto 8$	$(0,1,4,2,7,5,6,3,8)$
$\mathbb{Z}_{10}$	2	$1 \mapsto 9$	$(0,1,2,8,6,3,5,9,4,7)$
D_{10}	2	$u \mapsto u^4$	$(e,v,u,u^2,u^3,u^4v,u^2v,u^4,uv,u^3v)$
		$v \mapsto v$	
$\mathbb{Z}_{11}$	2	$1 \mapsto 10$	$(0,1,3,6,10,7,5,4,9,2,8)$

Table 4: More ℓ-fold Vatican pseudoterraces for small groups; $\ell \in \{2, 3\}$.

Group	ℓ	Aut.	Pseudoterrace
$\mathbb{Z}_{13}$	2	$1 \mapsto 12$	$(0, 1, 3, 4, 9, 6, 10, 2, 8, 11, 7, 5, 12)$
	3	$1 \mapsto 3$	$(0, 1, 2, , 3, 10, 4, 8, 7, 12, 5, 9, 11, 6)$
$\mathbb{Z}_{14}$	2	$1 \mapsto 13$	$(0, 1, 5, 2, 8, 6, 12, 3, 10, 9, 11, 7, 4, 13)$
	3	$1 \mapsto 9$	$(0, 1, 2, 3, 5, 13, 9, 12, 4, 11, 10, 6, 8, 7)$
$\mathbb{Z}_{15}$	2	$1 \mapsto 14$	$(0, 1, 4, 10, 8, 2, 7, 12, 9, 13, 11, 3, 14, 6, 5)$

Table 3 gives 2- and 3-fold Vatican pseudoterraces for all groups of order up to 11 in which they exist.

Table 4 extends Table 3 up to order 15, except that for brevity entries are limited to at most one 2- and 3-fold Vatican pseudoterrace for each order. When $n = 12$ a Vatican singleton exists, so this order is omitted from the table.

As 11 is prime, there is a 1-fold Vatican pseudoterrace for $\mathbb{Z}_{10}$, and so the lack of a 3-fold Vatican pseudoterrace for a group of order 10 is not detrimental to the construction of Vatican designs. However, for completeness, here is a 5-fold Vatican pseudoterrace for D_{10} with automorphism $u \mapsto u,\ v \mapsto u^4v$:

$$(e, v, u^4v, u^2v, u^2, uv, u, u^4, u^3, u^3v).$$

This implies the existence of a $10\ell \times 10$ Vatican design built from Cayley tables of D_{10} when $\ell \notin \{1, 3\}$.

We run into the same issue at $t = 11$, except that here we do not have that $t + 1$ is prime to construct the desired Vatican designs. The primitive root construction gives a 5-fold Vatican pseudoterrace for $\mathbb{Z}_{11}$ using the primitive root $\rho = 8$, which is sufficient to show that there is an $11\ell \times 11$ Vatican design when $\ell \notin \{1, 3\}$. The $\ell = 3$ case is covered below.

The gaps in the tables are genuine. There is no 2-fold Vatican pseudoterrace for $\mathbb{Z}_3 \times \mathbb{Z}_3$ despite this group having an automor-

Table 5: Some Vatican triples

Group	Triple
$\mathbb{Z}_5$	$(0,1,2,4,3),(0,2,1,4,3),(0,2,3,1,4)$
$\mathbb{Z}_9$	$(0,1,2,3,6,8,5,4,7),(0,4,6,3,2,7,5,1,8),(0,7,1,5,2,6,8,4,3)$
$\mathbb{Z}_{11}$	$(0,1,2,3,5,7,4,10,9,8,6),(0,2,9,5,10,6,3,8,1,4,7),$
	$(0,5,8,1,7,6,4,2,10,3,9)$

phism of order 2 (the same is true for $\mathbb{Z}_{17}$). Similarly, there are no 3-fold Vatican pseudoterraces for $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. There is no 3-fold Vatican pseudoterrace (or even an ℓ-fold one for any odd ℓ) for $\mathbb{Z}_3$ or $\mathbb{Z}_{15}$ as these groups have no automorphisms of odd order.

Collecting the various constructions of the paper together we can now prove Theorem 1.1: there is an $\ell t \times t$ Vatican design when $4 \le t \le 14$ and $\ell > 1$, and when $t \in \{3, 15\}$ and ℓ is even.

Proof of Theorem 1.1. When ℓ is even, we require a 2-fold (or 1-fold) Vatican pseudoterrace. There is a 1-fold Vatican pseudoterrace for $t \in \{4, 6, 10, 12\}$ as $t+1$ is prime; for other t with $3 \le t \le 12$ there is a 2-fold Vatican pseudoterrace for $\mathbb{Z}_t$ in Table 3; and for $13 \le t \le 15$ there is a 2-fold Vatican pseudoterrace for $\mathbb{Z}_t$ in Table 4.

When ℓ is odd, we require a 3-fold (or 1-fold) Vatican pseudoterrace in addition to the 2-fold one. We again have a 1-fold Vatican pseudoterrace for $t \in \{4, 6, 10, 12\}$; for $t \in \{7, 8\}$ there is a 3-fold Vatican pseudoterrace for a group of order t in Table 3; for $t \in \{13, 14\}$, there is a 3-fold Vatican pseudoterrace for $\mathbb{Z}_t$ in Table 4; and for $t \in \{5, 9, 11\}$ there is a direct construction of a Vatican triple in Table 5.

These pseudoterraces are sufficient to construct the claimed designs. $\square$

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