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Regular Turán numbers

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Abstract

The regular Turán number of a graph F, denoted by $\mathrm{rex}(n, F)$, is the largest number of edges in a regular graph G of order n such that G does not contain subgraphs isomorphic to F. Giving a partial answer to a recent problem raised by Gerbner et al. [arXiv:1909.04980] we prove that $\mathrm{rex}(n, F)$ asymptotically equals the (classical) Turán number whenever the chromatic number of F is at least four; but it is substantially different for some 3-chromatic graphs F if n is odd.

1 Introduction

Let F be a fixed 'forbidden' graph. We denote by

- $\mathrm{ex}(n, F)$ the maximum number of edges in a graph of order n that does not contain F as a subgraph — the classical *Turán number*;
- $\mathrm{rex}(n, F)$ the maximum number of edges in a *regular* graph of order n that does not contain F as a subgraph — the *regular Turán number*.

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Of course $\mathrm{rex}(n, F) \leq \mathrm{ex}(n, F)$ holds for every F by definition.

The Turán number of graphs is one of the most famous functions of graph theory. The celebrated theorem of Turán [13] states

$$\mathrm{ex}(n, K_{r+1}) = \binom{n}{2} - \sum_{i=0}^{r-1} \binom{\lfloor \frac{n+i}{r} \rfloor}{2}.$$

Moreover, the unique extremal graph for K_{r+1} (the *Turán graph*, often denoted by $T_{n,r}$) is obtained by partitioning the n vertices into r classes as equally as possible (each class has $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ vertices), and two vertices are adjacent if and only if they belong to distinct classes. Erdős and Stone [7] proved in general that

$$\mathrm{ex}(n, F) = (1 + o(1))\, \mathrm{ex}(n, K_{\chi(F)})$$

holds for all graphs F with chromatic number $\chi(F) \geq 3$.

The regular Turán number was introduced recently by Gerbner, Patkós, Vizer, and the second author in [8], motivated by the study of singular Turán numbers introduced in [4]. We quote the following results from [8], where the first displayed formula is derived from a theorem of Andrásfai [2]. Later we shall present this theorem in detail, and also some recent progress motivated by it.

- $\mathrm{rex}(n, K_3)$ is not a monotone function of n because $\mathrm{rex}(n, K_3) = \mathrm{ex}(n, K_3) = n^2/4$ if n is even, while $\mathrm{rex}(n, K_3) \leq n^2/5$ if n is odd.
- There exists a quadratic lower bound on $\mathrm{rex}(n, F)$ whenever $\chi(F) \geq 3$, namely

  $$\mathrm{rex}(n, F) \geq n^2/(g + 6) - O(n),$$

  where g is the length of a shortest *odd* cycle in F (that is, the *odd girth* of F).
- If $\chi(F) = r + 1 \geq 3$ and n is a multiple of r, then

  $$\mathrm{rex}(n, F) = (1 + o(1))\, \mathrm{ex}(n, F)$$

  as $n \to \infty$, by the regularity of the Turán graph.
- If F is a tree on $p + 1$ vertices and $\mathrm{ex}(n, F) \leq (p - 1)n/2$, then $\mathrm{rex}(n, F) = \mathrm{ex}(n, F)$ for every n divisible by p. In this case the extremal graph is a disjoint union of copies of K_p.

In spite of many similarities, the case of K_3 already indicates that there are substantial differences between the regular Turán function and the classical one. It should be noted that in every Turán graph the vertex degrees differ by at most 1, hence relaxing the condition of regularity to 'nearly regular' we would obtain a function whose behavior is very different from that of $\mathrm{rex}(n, F)$. Even more generally, one may introduce $\mathrm{nrex}_t(n, F)$ as the maximum number of edges in an F-free graph of order n such that all vertex degrees differ by at most t. This yields the inequality chain

$$\mathrm{rex}(n, F) = \mathrm{nrex}_0(n, F) \le \mathrm{nrex}_1(n, F) \le \mathrm{nrex}_2(n, F) \le \cdots \le \mathrm{ex}(n, F)$$

where, for any two of the functions involved, there is a graph F showing that the functions are not identical.

Problem 4.2 of [8] asks for the determination of $\liminf \mathrm{rex}(n, F)/n^2$ for non-bipartite graphs F. The goal of our present note is to solve this problem for a large class of graphs F, as expressed in the following results.

We begin with the study of $\mathrm{rex}(n, K_{r+1})$ for $r \ge 3$, and more generally $\mathrm{rex}(n, F)$ for non-3-colorable graphs F. Although its first part is obvious by Turán's theorem, we include it for the sake of completeness.

Theorem 1 *Let $r \ge 3$.*

(*i*) *If n is a multiple of r, then* $\mathrm{rex}(n, K_{r+1}) = \mathrm{ex}(n, K_{r+1})$.

(*ii*) *If n is not a multiple of r, then* $\mathrm{rex}(n, K_{r+1}) = \mathrm{ex}(n, K_{r+1}) - \Theta(n)$ *as $n \to \infty$.*

(*iii*) *More precisely, if $n = qr + s$ with $1 \le s \le r - 2$, and at least one of $r - s$ and q is even, then*

$$\mathrm{rex}(n, K_{r+1}) = \mathrm{ex}(n, K_{r+1}) - \frac{(r-s)q}{2}.$$

(*iv*) *If F is any graph with $\chi(F) \ge 4$, then* $\mathrm{rex}(n, F) = (1 - o(1))\, \mathrm{ex}(n, F)$.

(*v*) *For $F = K_4$,*

$$\mathrm{rex}(n, K_4) = \begin{cases} 1 & \text{if } n = 2, \\ 20 & \text{if } n = 8, \\ n \cdot \lfloor \frac{n}{3} \rfloor & \text{otherwise.} \end{cases}$$

Some combinations of n and r with $r \ge 4$ are not covered in this theorem. These cases are settled in the follow-up paper [9].

The next result deals with graphs whose chromatic number is equal to 3.

Theorem 2 *Let F be a 3-chromatic graph.*

(*i*) *If n is even, then* $\mathrm{rex}(n, F) = (1 - o(1))\, \mathrm{ex}(n, F) = n^2/4 + o(n^2)$; *moreover,* $\mathrm{rex}(n, K_3) = \mathrm{rex}(n, K_4 - e) = n^2/4$.

(*ii*) *If n is odd, and $F = K_3$ or $F = K_4 - e$ or F is a unicyclic graph with C_3 as its cycle, then* $\mathrm{rex}(n, F) = n^2/5 - O(n)$.

(*iii*) *If $F = K_3$ and n is odd, then* $\mathrm{rex}(n, F) = n \cdot \lfloor \frac{n}{5} \rfloor$.

We also study the relation between odd girth and regular Turán numbers.

Theorem 3 *Let n be odd, and $\chi(F) = 3$. If F has odd girth g, then*

$$\mathrm{rex}(n, F) \geq n^2/(g + 2) - O(n)\,.$$

Moreover, if $F = C_5$ or F is a unicyclic graph with C_5 as its cycle, then the following holds:

$$\mathrm{rex}(n, F) = n^2/7 - O(n)\,.$$

More precisely, if n is sufficiently large with respect to F, then $\mathrm{rex}(n, F) = n \cdot \lfloor \frac{n}{7} \rfloor$.

The study of $\mathrm{rex}(n, F)$ for bipartite graphs F was started in [8] and further developed in [12], but only a few results are available so far. Moreover, the following problem remains widely open for 3-chromatic graphs.

Problem 1 *Determine* $\mathrm{rex}(n, F)$*, or its asymptotic growth as $n \to \infty$, for graphs F with $\chi(F) = 3$ for odd n.*

As a particular case, we expect that the exact results on K_3 and C_5 extend to a unified formula for every odd cycle.

Conjecture 1 *If both n and g are odd, with $n \geq g+2$ and $g \geq 3$, then* $\mathrm{rex}(n, C_g) = n \cdot \lfloor \frac{n}{g+2} \rfloor$.

The unicyclic extensions given in Theorems 2 and 3 are consequences of the following principle. It has an analogous implication also for those graphs whose unique non-trivial block is $K_4 - e$.

Proposition 1 *If the growth of* $\mathrm{rex}(n, F)$ *is superlinear in n, and F^+ is a graph obtained from F by inserting a pendant vertex, then* $\mathrm{rex}(n, F^+) = \mathrm{rex}(n, F)$ *for every sufficiently large n.*

At the end of this introduction let us recall the full statement of Andrásfai's theorem, which plays an essential role in the current context.

Theorem 4 ([2]) *If G is a triangle-free graph on n vertices and with minimum degree $\delta(G) > 2n/5$, then G is bipartite.*

This theorem is the source of motivation for recent research which is also related and relevant to our Theorem 3 and Conjecture 1; see [1, 6, 10, 11]. We explicitly quote the following very useful generalization, proved by Andrásfai, Erdős, and Sós, as read out from the combination of their Theorem 1.1 and Remark 1.6.

Theorem 5 ([3]) *For each odd integer $k \geq 5$ and each integer $n \geq k$, if G is a simple n-vertex graph with no odd cycles of length less than k and with minimum degree $\delta(G) > 2n/k$, then G is bipartite.*

2 Forbidden graphs with chromatic number at least 4

In this section we prove Theorem 1. We assume throughout that $n = qr + s$, where q is an integer and $0 \le s \le r - 1$ holds.

Proof of Part (*i*)

Whenever n is a multiple of r, the equality $\mathrm{rex}(n, K_{r+1}) = \mathrm{ex}(n, K_{r+1})$ is clear because the r-partite Turán graph $T_{n,r}$ is regular in all such cases.

Proof of Part (*ii*)

(1) First we argue that $\mathrm{ex}(n, K_{r+1}) - \mathrm{rex}(n, K_{r+1})$ is at least a linear function of n if r does not divide n. We see from Turán's theorem that the degree of regularity cannot exceed $(1 - 1/r)\, n$ if the graph is K_{r+1}-free. For $n = qr + s$ with $0 < s < r$ it means that the degrees are at most $n - (n + r - s)/r$. This value is the degree of vertices in the s larger classes of the Turán graph $T_{n,r}$; but the $r - s$ smaller classes consist of vertices of degree $n - (n + r - s)/r + 1$. Hence the degree sum in a regular graph is smaller by at least $\frac{r-s}{r}\, n - O(1)$.

(2) Next, we construct r-chromatic (hence also K_{r+1}-free) regular graphs to show that the difference between $\mathrm{ex}(n, K_{r+1})$ and $\mathrm{rex}(n, K_{r+1})$ is at most $O(n)$. For $n = qr + s$ with $0 < s < r$ the Turán graph has s classes of size $q + 1$ and $r - s$ classes of size q. Putting this in another way, the vertices in s classes have degree $n - q - 1$, and in $r - s$ classes have degree $n - q$.

We are going to delete $O(n)$ edges from $T_{n,r}$ and obtain a regular graph. For this purpose we shall use Dirac's theorem [5], which states that if a graph H has minimum degree at least half of its order, then H contains a Hamiltonian cycle.

(2.a) Assume first that both s and $r - s$ are at least 2. Since the degree sum is even, the number of vertices in odd-degree classes is also even, and the subgraph induced by them is Hamiltonian, due to Dirac's theorem. Hence the union of these classes admits a 1-factor, which we remove. If this is the larger degree, then we are done. Otherwise the subgraph induced by the larger degrees also has a Hamiltonian cycle, whose removal leaves a regular graph of degree $n - q - 2$.

(2.b) Assume next that $s = 1$; i.e., only one class contains vertices of low degree. If the number $(r - 1) \cdot q$ of vertices in the $r - 1$ high-degree classes is even, we remove a 1-factor from the subgraph induced by these vertices, and we are done. On the other hand, if their number is odd, then both q and $r - 1$ are odd; in particular, $r - 1 \ge 3$ holds. Moreover, the high degree must be even. In this situation our plan is to delete two edges from each such vertex, and one edge from each vertex of the low-degree class.

We begin with the single class, which has even size. We omit one edge from each of its vertices — mutually disjoint edges — in such a way that the other ends of those $q + 1$ edges are distributed as equally as possible among the $r - 1 \ge 3$ high-degree classes. Then Dirac's condition holds for the subgraph induced by the high-degree ends of the omitted matching, and also for the subgraph induced by the vertices that are not incident with edges omitted so far. Hence both parts are Hamiltonian. Since $q+1$ is even, we can omit a perfect matching from the former, and a Hamiltonian cycle from the latter, thus obtaining a regular graph.

(2.c) Finally, consider the case $s = r-1$; i.e., only one class contains vertices of high degree. We must remove edges from the high-degree class, which means that also some low-degree vertices will decrease their degree. Similarly to the previous case, here again, the parity of classes will matter.

If the high-degree class has even size q, we decrease the degrees of its vertices by 3, distributing the neighbors equally among the low-degree classes. Hence the decrease of low degrees is either 0 and 1, or 1 and 2 (or only 1 or only 2). In either case the number of 1-decreases is even (it has the same parity as q). For 2-decrease we do nothing, in the subgraph induced by the vertices of 1-decrease we remove a 1-factor, and in the subgraph induced by the vertices of 0-decrease we remove a Hamiltonian cycle. The only objection against this plan would be if the number of vertices with 0-decrease was exactly 2. However, this would require that the number of low-degree vertices is $(r-1)(q+1) = 3q+2 = 3(q+1)-1$, implying $r = 4 - \frac{1}{q+1}$, which is not an integer.

Suppose that the high-degree class has odd size q. Then the number of low-degree vertices is even because each such class has even size $q+1$. In particular, n is odd. We now delete two edges from each high-degree vertex, the other ends of deleted edges being distributed equally among the $r-1$ other classes. This decreases $2q$ of the low degrees by 1. From the other $n-3q$ (in particular, even number of) vertices we delete a 1-factor which exists because either the subgraph induced by them is Hamiltonian or we have $n-3q = 2$ and the single edge has to be deleted that joins the two vertices. This modification yields a regular graph, and completes the proof of (*ii*).

Proof of Part (*iii*)

Recall that the number of classes of high-degree vertices in the Turán graph is $r-s$, and these classes have cardinality q each. Note further that their union induces a Hamiltonian subgraph whenever $s \le r-2$. Under the assumption that at least one of $r-s$ and q is even, the length $(r-s)\,q$ of a corresponding Hamiltonian cycle is even, hence contains a perfect matching, say M. Removing M from $T_{n,r}$ we obtain a K_{r+1}-free regular graph, and the degree is largest possible, according to the first part of the proof of (*ii*) as given above.

Proof of Part (*iv*)

The r-colorable construction given above for (*ii*) proves that $\mathrm{rex}(n, F)$ is at least $\mathrm{ex}(n, K_{r+1}) - O(n)$ if $\chi(F) = r+1$. On the other hand, as mentioned already, $\mathrm{ex}(n, F) = (1+o(1))\,\mathrm{ex}(n, K_{r+1})$ is valid by the Erdős–Stone theorem, implying that $\mathrm{rex}(n, F)$ cannot be larger. Thus, $\mathrm{rex}(n, F) = (1+o(1))\,\mathrm{ex}(n, K_{r+1})$ also holds.

Proof of Part (*v*)

(1) The following list is a summary of optimal constructions according to n (mod 3). Optimality is clear in the first two cases, and it will be proved for the third case afterwards.

- $n = 3k$: the complete 3-partite graph with equal classes, i.e. $T_{n,3}$, is regular of degree $2k$.
- $n = 3k + 1$: here $T_{n,3}$ has vertex classes of respective sizes $k, k, k + 1$ and vertex degrees $2k + 1, 2k + 1, 2k$; it can be made regular by removing a perfect matching between the two classes of size k.
- $n = 2$: obviously K_2 is the unique extremal graph.

  $n = 8$: the complement of $C_3 \cup C_5$ is 5-regular, and also K_4-free, because the independence number of $C_3 \cup C_5$ is 3. This graph is extremal, since the unique 6-regular graph of order 8 is $K_8 - 4K_2$ (omitting a perfect matching), which contains many copies of K_4.

  $n = 3k + 2$, $n \notin \{2, 8\}$: here $T_{n,3}$ has vertex classes of respective sizes $k, k + 1, k + 1$ and vertex degrees $2k + 2, 2k + 1, 2k + 1$; it can be made regular by removing a matching of k edges between the class of size k and each class of size $k + 1$ (hence removing kP_3, that is, k vertex-disjoint paths of length two, from $T_{n,3}$), moreover deleting the edge that joins the two vertices whose degree has not been decreased by the removal of the two matchings.

(2) Let G be a K_4-free regular graph of order $n = 3k + 2$. Turán's theorem implies that the vertex degrees are smaller than $2k + 2$. Moreover, if n is odd, then G cannot be $(2k+1)$-regular, and hence the construction described above is extremal in this case.

Now let n be even, say $n = 6t + 2$, and suppose that G is $(4t + 1)$-regular. Then the complement $H = \overline{G}$ of G is $2t$-regular and has independence number 3. It follows that $\chi(H) = 2t+1$; moreover H has a connected component $H' \cong K_{2t+1}$, by Brooks's theorem.

The other component $H - H'$ of H has $4t + 1$ vertices, independence number 2, and is regular of degree $2t$. Hence its complement, say G', is a K_3-free $2t$-regular non-bipartite graph of order $4t + 1$. Andrásfai's theorem implies

$$2t \leq \frac{2}{5}\,(4t + 1)\,.$$

Thus, $t \leq 1$, which is not the case for $n > 8$, hence completing the proof.

3 3-chromatic forbidden graphs

Let us begin this section with the proof of Proposition 1, as it is applicable for Theorems 2 and 3 as well.

Proof of Proposition 1

Certainly we have $\mathrm{rex}(n, F^+) \geq \mathrm{rex}(n, F)$. Suppose that G is a regular graph of order n, which is extremal for F^+. If G is F-free, then the reverse inequality $\mathrm{rex}(n, F^+) \leq \mathrm{rex}(n, F)$ also holds and the assertion follows immediately. On the other hand, if $F \subset G$ but $F^+ \not\subset G$, the degree of regularity in G must be smaller than $|V(F^+)|$, for otherwise it would be possible to extend F to F^+ in G. This fact puts a $O(n)$ upper bound on $|E(G)|$, contradicting the assumption (in the statement of Proposition 1) on the superlinear growth of $\mathrm{rex}(n, F)$. Thus the largest F^+-free regular graphs are F-free, too, as n gets large.

Proof of Theorem 3, general lower bound

We construct a graph of order n and odd girth $g+2$, hence it will not contain F as a subgraph. Let us write n in the form $n = (g+2) \cdot a + 2b$ where b is an integer in the range $0 \leq b \leq g+1$. Such a and b exist because $g+2$ is odd. We start with a blow-up of $C_{g+2} = v_1 v_2 \dots v_{g+2}$ by substituting independent sets $A_1, A_2, \dots, A_{g+2}$ into its vertices, completely joining A_i with A_{i+1} for $i = 1, \dots, g+2$ (where $A_{g+3} := A_1$). We let $|A_1| = |A_2| = a + b$, and $|A_i| = a$ for all $3 \leq i \leq g+2$. The degree of a vertex v in this graph is $2a + b$ if $v \in A_1 \cup A_2 \cup A_3 \cup A_{g+2}$, and it is $2a$ otherwise. The graph is regular if $b = 0$, and it will be made regular by the removal of

$$2ab + b^2 = b\left(\frac{2n - 4b}{g+2} + b\right) = \frac{b \cdot (2n + (g-2)b)}{g+2} \leq 2n - \frac{2n - (g-2)(g+1)^2}{g+2}$$

edges otherwise.

Between A_{g+2} and A_1 we remove a bipartite graph H such that all vertices of H in A_{g+2} have degree b, and all degrees in A_1 are $\lfloor \frac{ab}{a+b} \rfloor$ or $\lceil \frac{ab}{a+b} \rceil$. Such H clearly exists. We also remove a bipartite graph isomorphic to H between A_2 and A_3, such that the degree-b vertices are in A_3.

If ab is a multiple of $a+b$, then the current vertex degrees in $A_1 \cup A_2$ are $2a + b - \frac{ab}{a+b} = 2a + \frac{b^2}{a+b}$. Then removing a regular bipartite graph of degree $\frac{b^2}{a+b}$, hence with b^2 edges, yields a $(2a)$-regular graph of order n.

Otherwise, if ab is not divisible by $a+b$, we first remove a perfect matching between the vertices of degree $2a + b - \lfloor \frac{ab}{a+b} \rfloor$ in $A_1 \cup A_2$. After that, the bipartite graph induced by $A_1 \cup A_2$ is regular, hence deleting a regular subgraph of degree $b - \lceil \frac{ab}{a+b} \rceil$ from it, we obtain a $(2a)$-regular graph of order n.

Proof of Theorem 3 for asymptotics of C_5 and unicyclic graphs

We have to prove that $\mathrm{rex}(n, F) \leq n^2/7$, if n is large enough. By Proposition 1 it is enough to deal with the case $F = C_5$. Let G be a C_5-free regular graph with $\mathrm{rex}(n, C_5)$ edges. If G is triangle-free, the proof is done by Theorem 5, because the odd girth cannot be exactly 5, and every regular bipartite graph must contain an even number of vertices, which is not the case here. On the other hand it was proved in [10, Lemma 33] that if G contains a triangle and the degrees are greater than

$(1/6 + \epsilon)\, n$, for any $\epsilon > 0$ and sufficiently large n, then G also has a C_5. Hence in C_5-free graphs with triangles we cannot have more than $n^2/12 + o(n^2)$ edges.

We postpone the proof of the exact formula for $\mathrm{rex}(n, C_5)$ to the end of this paper, due to its similarity to the argument concerning $\mathrm{rex}(n, C_3)$.

Proof of Theorem 2, Parts (*i*) and (*ii*)

If F is 3-chromatic and n is even, the lower bound of $n^2/4$ is shown by the complete bipartite graph $K_{n/2,n/2}$, while an asymptotic upper bound follows by the Erdős–Stone theorem as

$$\mathrm{rex}(n, F) \leq \mathrm{ex}(n, F) = (1 - o(1))\, \mathrm{ex}(n, K_3) = n^2/4 + o(n^2).$$

The tight results $\mathrm{rex}(n, K_3) = \mathrm{rex}(n, K_4 - e) = n^2/4$ follow from the facts that the Turán number of K_3 and also of $K_4 - e$ is $n^2/4$.

Assume that n is odd. The lower bound $\mathrm{rex}(n, K_3) \geq n^2/5 - O(n)$ is a particular case of the previous construction, putting $g = 3$. Moreover, in triangle-free regular graphs we cannot have more than $n^2/5$ edges, due to Theorem 4 and by the fact that every regular bipartite graph has an even order. This already settles the case of K_3 (and also of the unicyclic graphs having a 3-cycle, by Proposition 1). For $K_4 - e$ assume that x, y, z induce a triangle. If this triangle cannot be extended to $K_4 - e$, then the degree of x, y, z is at most $2 + (n-3)/3 = n/3 + 1$, thus by the condition of regularity the number of edges is at most $n^2/6 + n/2$, which is much less than $n^2/5$ if n is large.

Proof of Theorem 2, Part (*iii*)

Let G be a triangle-free regular graph on n vertices, with $|E(G)| = \mathrm{rex}(n, K_3)$. Assume that $n = 5k + s$, where $s = 0, 1, 2, 3, 4$ and n is odd. As a consequence, $k + s$ is odd as well; however, the degree d of regularity must be even. From Theorem 4 we also know that $d \leq \lfloor 2n/5 \rfloor = 2k + \lfloor 2s/5 \rfloor \leq 2k + 1$; hence $d \leq 2k$ by the parity of d. Thus, $|E(G)| \leq kn$.

It remains to show that for every odd $n = 5k + s$ there exists a K_3-free graph of order n which is $2k$-regular. The general principle of the construction is to substitute independent sets $A_1, \ldots, A_5$ into the vertices of C_5, where each edge of C_5 becomes a complete bipartite graph between the corresponding two sets A_i, A_{i+1} cyclically; and then delete some edges so that a regular graph is obtained. We are going to describe these constructions for each s one by one, specifying the sequences $(|A_1|, \ldots, |A_5|)$ as follows.

- $s = 0:\quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k, k, k, k, k)$

  This graph is $2k$-regular.

- $s = 1:\quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k+1, k+1, k, k-1, k)$

  This graph becomes $2k$-regular after the deletion of a perfect matching from the induced subgraph $G[A_1 \cup A_2]$.

- $s = 2: \quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k+1, k+1, k, k, k)$

  This graph becomes $2k$-regular after the deletion of a matching of size k from $G[A_1 \cup A_5]$ and from $G[A_2 \cup A_3]$, and the edge between the two unmatched vertices of $A_1 \cup A_2$.

- $s = 3: \quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k+1, k+1, k+1, k, k)$

  This graph becomes $2k$-regular after the deletion of a perfect matching from each of the induced subgraphs $G[A_1 \cup A_2]$, $G[A_2 \cup A_3]$, $G[A_4 \cup A_5]$.

- $s = 4: \quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k+2, k+2, k, k, k)$

  This graph can be made $2k$-regular in the following way. Specify two vertices $a'_1, a''_1 \in A_1$ and $a'_2, a''_2 \in A_2$; set $A'_1 = A_1 \setminus \{a'_1, a''_1\}$ and $A'_2 = A_2 \setminus \{a'_2, a''_2\}$. Delete a 2-factor from $G[A'_1 \cup A_5]$ and from $G[A'_2 \cup A_3]$; and delete the edges of the 4-cycle $a'_1 a'_2 a''_1 a''_2$.

Proof of Theorem 3 for exact rex(n, C_5)

Since the proof is very similar to that of the exact formula for rex(n, K_3), we give a more concise description here. Let $n = 7k + s$, where $0 \le s \le 6$. We have already seen that the degree d of regularity satisfies $d \le \lfloor 2n/7 \rfloor = 2k + \lfloor 2s/7 \rfloor \le 2k+1$; and d must be even; thus $d \le 2k$. It remains to give suitable substitutions of sets $A_1, \dots, A_7$ into the vertices of C_7 in such a way that the graphs can be made $2k$-regular by the deletion of some edges. Below we define a sequence $|A_1|, |A_2|, |A_3|, |A_4|, |A_5|, |A_6|, |A_7|$ similar to the case of C_5, now for each $s = 0, 1, \dots, 6$.

- $s = 0: \quad k, k, k, k, k, k, k$

  Nothing to delete.

- $s = 1: \quad k+1, k, k, k+1, k, k-1, k$

  Delete a 1-factor from $G[A_2 \cup A_3]$.

- $s = 2: \quad k+1, k+1, k, k, k, k, k$

  Delete a matching of size k from $G[A_1 \cup A_7]$ and from $G[A_2 \cup A_3]$, and the edge between the two unmatched vertices of $A_1 \cup A_2$.

- $s = 3: \quad k+1, k+1, k, k, k+1, k, k$

  Delete a 1-factor from $G[A_1 \cup A_2]$, from $G[A_3 \cup A_4]$, and from $G[A_6 \cup A_7]$.

- $s = 4: \quad k+2, k+2, k, k, k, k, k$

  Delete the edges of a $H \cong C_4$ in $A_1 \cup A_2$, and a 2-factor from $G[(A_1 \cup A_7) \setminus V(H)]$ and from $G[(A_2 \cup A_3) \setminus V(H)]$.

- $s = 5 : \quad k+2, k+2, k, k, k+1, k, k$

  Delete a C_4 from $G[A_1 \cup A_2]$, and a matching of size k from each consecutive pair of A_i, A_{i+1} along the cycle (also including A_7, A_1 as cyclically consecutive), except $G[A_4 \cup A_5]$ and $G[A_5 \cup A_6]$.

- $s = 6 : \quad k+2, k+2, k, k, k+2, k, k$

  Delete a 2-factor from $G[A_1 \cup A_2]$, from $G[A_3 \cup A_4]$, and from $G[A_6 \cup A_7]$.

After the deletions, all graphs are $2k$-regular, completing the proof of the theorem.

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References

[1] P. Allen, J. Böttcher, S. Griffiths, Y. Kohayakawa and R. Morris, The chromatic thresholds of graphs, *Adv. Math.* 235 (2013), 261–295.

[2] B. Andrásfai, Graphentheoretische Extremalprobleme, *Acta Math. Acad. Sci. Hungar.* 15 (1964), 413–418.

[3] B. Andrásfai, P. Erdős and V. T. Sós, On the connection between chromatic number, maximal clique and minimal degree of a graph, *Discrete Math.* 8 (1974), 205–218.

[4] Y. Caro and Zs. Tuza, Singular Ramsey and Turán numbers, *Theory Appl. Graphs* 6 (2019), 1–32.

[5] G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc. (3)* 2 (1952), 69–81.

[6] O. Ebsen and M. Schacht, Homomorphism thresholds for odd cycles, arxiv:1712.07026

[7] P. Erdős and A. H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* 52 (1946), 1087–1091.

[8] D. Gerbner, B. Patkós, Zs. Tuza and M. Vizer, Singular Turán numbers and WORM-colorings, *Discuss. Math. Graph Theory* (to appear); doi:10.7151/dmgt.2335; arXiv:1909.04980.

[9] D. Gerbner, B. Patkós, Zs. Tuza and M. Vizer, Some exact results for regular Turán problems, arXiv:1912.10287.

[10] S. Letzter and R. Snyder, The homomorphism threshold of $\{C_3, C_5\}$-free graphs, *J. Graph Theory* 90 (2019), 83–106.

[11] S. Messuti and M. Schacht, On the structure of graphs with given odd girth and large minimum degree, *J. Graph Theory* 80 (2015), 69–81.

[12] M. Tait and C. Timmons, Regular Turán numbers of complete bipartite graphs, arXiv:2005.02907.

[13] P. Turán, Egy gráfelméleti szélsőértékfeladatról (On an extremal problem in graph theory), *Mat. Fiz. Lapok* 48 (1941), 436–452. (In Hungarian)

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