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On (k, l)-radii of wheels

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Abstract

To determine the (k, l)-radius of a graph we have to find a set L of l vertices, such that the maximum k-distance of a set K, where $|K| = k$ and $L \subseteq K$, attains the minimum value in a graph. This notion generalizes the radius, diameter and k-diameter. In this contribution the (k, l)-radius of the wheel W_n is determined for all possible values of parameters k and l.

1 Introduction

We consider connected, undirected graphs G of order n with the vertex set $V(G)$. By *distance* between two vertices in G we mean the minimum length of a path connecting them. The *eccentricity* $e(v)$ of v is the distance to a vertex farthest from v, $e(v) = \max\limits_{u \in V(G)}(d(u, v))$. Then the *radius* $r(G)$ of the graph G is the minimum eccentricity, $r(G) = \min\limits_{v \in V(G)}(e(v))$, while the *diameter* $\mathrm{diam}(G)$ is the maximum eccentricity, $\mathrm{diam}(G) = \max\limits_{v \in V(G)}(e(v))$. More information related to basic distance concepts can be found in [2].

Definition 1 *Let G be a graph on n vertices and let k be an integer, $k \le n$. The* distance of k vertices *(k-distance), $d_k(v_1, v_2, \dots, v_k)$, is the sum of distances between all pairs of vertices from $\{v_1, v_2, \dots, v_k\}$.*

We remark that the n-distance is called the *transmission* of the graph (see [7]), while the maximum k-distance in a graph is known as a *k-diameter* (see [1]).

Definition 2 *Let G be a connected graph with the vertex set $V(G)$, $|V(G)| = n$, and let k, l be integers, $0 \le l \le k \le n$ and $k > 0$. The* (k, l)-eccentricity *of the set*

$L \subseteq V(G)$ of l vertices, $e_{k,l}(L)$, is the maximum distance of k vertices $u_1, u_2, \ldots, u_k$, such that $L \subseteq \{u_1, u_2, \ldots, u_k\}$. That is,

$$e_{k,l}(L) = \max_{K}\{d_k(K);\ |K| = k, L \subseteq K \subseteq V(G)\}.$$

Observe that $(2,1)$-eccentricity is the usual eccentricity of a vertex.

Definition 3 *The* (k,l)-radius, $\mathrm{rad}_{k,l}(G)$, *is the minimum (k,l)-eccentricity in G,*

$$\mathrm{rad}_{k,l}(G) = \min_{L}(e_{k,l}(L)) = \min_{L}\big(\max_{L\subseteq K\subseteq V(G)} d_k(K)\big),$$

where $|L| = l$ and $|K| = k$.

Thus the usual radius of a graph G equals its $(2,1)$-radius, while the diameter is its $(2,0)$-radius. Moreover, the k-diameter is the $(k,0)$-radius in our notation.

Definition 4 *Let k, l be integers, $1 \le l \le k \le n$, where $|V(G)| = n$. A set $C = \{u_1, u_2, \ldots, u_l\} \subseteq V(G)$ is a* (k,l)-central set *of the graph G if $e_{k,l}(C) = \mathrm{rad}_{k,l}(G)$.*

The determination of the $(3,l)$-radius for some classes of graphs can be found in [4]. To find the (k,l)-radius for all values of k and l is not an easy task even for very simple classes of graphs. Up to now this problem has been succesfully solved only for complete graphs K_n (which is trivial, see [3]), the Petersen graph [6] and complete bipartite graphs K_{n_1,n_2} [5]. In this paper we present a complete solution for wheels W_n.

Definition 5 *The wheel W_n is a graph on n vertices with the vertex set $V(W_n) = \{s, u_0, u_1, \ldots, u_{n-2}\}$ and with $n-1$ edges su_i, $0 \le i \le n-2$ and $n-1$ edges $u_i u_{(i+1) \bmod (n-1)}$, $0 \le i \le n-2$.*

In the following statements we assume $n \ge 5$, because for $n < 5$ the wheel W_n is a complete graph K_n for which $\mathrm{rad}_{k,l}(K_n) = k \cdot (k-1)/2$ for each $l \le k$ (see [3]).

Theorem 1 *Let k, l and n be integers, $2 \le l \le k \le n$ and $n \ge 5$. Then for the (k,l)-radius of the wheel W_n we have:*

1. *If $k - l \le \lfloor \frac{n-l-1}{2} \rfloor$ then $rad_{k,l}(W_n) = k^2 - 2\cdot k - l + 3$.*
2. *If $k - l > \lfloor \frac{n-l-1}{2} \rfloor$ then $rad_{k,l}(W_n) = k^2 - 4\cdot k + n + 2$.*

Observe that if $k - l > \lfloor \frac{n-l-1}{2} \rfloor$ then the (k,l)-radius of a wheel does not depend on the parameter l.

The cases $l = 1$ and $l = 0$ are considered separately.

Assertion 1 *Let k and n be integers, $1 \le k \le n$, $n \ge 5$. Then for the $(k,1)$-radius of the wheel W_n the following hold:*

1. *If $k-2 \le \lfloor \frac{n-3}{2} \rfloor$ then* $\mathrm{rad}_{k,1}(W_n) = k^2 - 2 \cdot k + 1$.
2. *If $k-2 > \lfloor \frac{n-3}{2} \rfloor$ then* $\mathrm{rad}_{k,1}(W_n) = k^2 - 4 \cdot k + n + 2$.

Assertion 2 *Let k and n be integers, $1 \le k \le n$, $n \ge 5$. Then for the $(k,0)$-radius of the wheel W_n we have:*

1. *If $k \le \lfloor \frac{n-1}{2} \rfloor$ then* $\mathrm{rad}_{k,0}(W_n) = k^2 - k$.
2. *If $n > k \ge \lfloor \frac{n-1}{2} \rfloor$ then* $\mathrm{rad}_{k,0}(W_n) = k^2 - 3 \cdot k + n - 1$.
3. *If $k = n$ then* $\mathrm{rad}_{k,0}(W_n) = n^2 - 3 \cdot n + 2$.

Proofs of Theorem 1 and Assertions 1 and 2 are postponed to the next section.

2 Proofs

Proof of Theorem 1. By the definition of a wheel, the vertex set $V(W_n)$ contains one vertex s of degree $n-1$ and $n-1$ vertices $u_0, \ldots, u_{n-2}$ of degree 3. We have $d_2(u_i, s) = 1$ and $d_2\left(u_i, u_{(i+1) \bmod (n-1)}\right) = 1$, for $i = 0, 1, \ldots, n-2$. Thus the mutual distance of two vertices is at most 2.

The proof is done in two steps:

1. We find a (k,l)-central set of W_n.
2. For the (k,l)-central set we determine the value of its eccentricity, i.e., the (k,l)-radius of W_n.

We prove that for $l > 0$ the l-set $L = \{s, u_0, u_1, \ldots u_{l-2}\}$ is the (k,l)-central set of W_n. We do not say that it is the only (k,l)-central set.

At first we prove that there is a (k,l)-central set containing the vertex s. Suppose that there is a (k,l)-central set L' such that $s \notin L'$. Let $u_i \in L'$. Denote $L = L' \setminus \{u_i\} \cup \{s\}$. We show that $e_{k,l}(L) \le e_{k,l}(L')$. Let K be a set, $L \subset K$, on which $d_k(K)$ attains its maximum. If $u_i \in K$ then $L' \subseteq K$, so that

$$e_{k,l}(L') \ge d_k(K) = e_{k,l}(L).$$

On the other hand if $u_i \notin K$, then

$$e_{k,l}(L') \ge d_k(K \setminus \{s\} \cup \{u_i\}) \ge d_k(K) = e_{k,l}(L).$$

Hence there is a (k,l)-central set containing s.

Now we prove that if $L = \{s, u_0, u_1, \dots u_{l-2}\}$, then for any l-set $L' = \{s, u_{i_0}, u_{i_1}, \dots, u_{i_{l-2}}\} \subset V(W_n)$, the following holds:

$$e_{k,l}(L) \leq e_{k,l}(L'). \tag{1}$$

To determine $e_{k,l}(L)$ ($e_{k,l}(L')$) we find the set U (U') of $(k-l)$ vertices such that $d_k(L \cup U)$ ($d_k(L' \cup U')$) has the maximum possible value. It means we find a set of $(k-l)$ vertices such that $L \cup U$ ($L' \cup U'$) contains the minimum possible number of pairs of adjacent vertices (p.a.v.). In what follows the number of p.a.v. does not involve pairs containing the vertex s.

Using the term "*p.a.v.*" inequality (1) can be rewritten as

$$|p.a.v.\ of\,(L \cup U)| \geq |p.a.v.\ of\,(L' \cup U')|.$$

By the construction of the set L there are $(l-2)$ p.a.v. in L and all vertices of $W_n \setminus L$ are on a path P_{n-l} on $n-l$ vertices (recall that $|V(W_n)| = n$ and $|L| = l$).

Let p' denote the number of p.a.v. in the set L'. As $p' \leq l-2$, vertices of $W_n \setminus L'$ are on $(l-p'-1)$ paths. Sorting the paths of $W_n \setminus L'$ according to their lengths we get

$$\begin{aligned} &n - l = \\ &(l-p'-1) + d_2 + 3 \cdot d_4 + \dots + (n_e - 1) \cdot d_{n_e} + 2 \cdot d_3 + 4 \cdot d_5 + \dots + (n_o - 1) \cdot d_{n_o}, \end{aligned} \tag{2}$$

where d_i is the number of paths P_i in $W_n \setminus L'$ and n_e (n_o) is the maximum order of a path in $W_n \setminus L'$ on an even (odd) number of vertices. Let x_e (x_o) denote the total number of these paths of even (odd) order. Then $x_e = d_2 + d_4 + \dots + d_{n_e}$, $x_o = d_1 + d_3 + \dots + d_{n_o}$ and

$$x_e + x_o = l - p' - 1. \tag{3}$$

Now we define a set M (M') of maximum cardinality, such that $M \subset V(W_n) \setminus L$ ($M' \subset V(W_n) \setminus L'$) and the vertices of M (M') are mutually nonadjacent and are adjacent to no vertex of $L \setminus \{s\}$ ($L' \setminus \{s\}$). Vertices of such sets do not increase the number of p.a.v..

At first we determine the maximum possible cardinality of the set M'. Consider the set $M'_q = M' \cap V(P_q)$, where P_q is one of the paths of $W_n \setminus L'$. Since M'_q is an independent set of vertices of a path, obtained from P_q by deleting the terminal vertices, we have

$$|M'_q| = \frac{q-1}{2} \tag{4}$$

if q is odd and

$$|M'_q| = \frac{q-2}{2} \tag{5}$$

if q is even. The set M' is a union of M'_q determined for each of $l-1-p'$ paths in $W_n \setminus L'$. Analogously, the maximum possible cardinality of the set M is $|M| = \frac{n-l-2}{2}$ if $n-l$ is even and $|M| = \frac{n-l-1}{2}$ otherwise.

By the construction of sets M and M', the vertices $V(W_n) \setminus (L \cup M)$ $(V(W_n) \setminus (L' \cup M'))$ induce paths of lengths 1 or 2. Furthermore, the number of paths P_2 in $W_n \setminus (L' \cup M')$ equals the number of even paths in $W_n \setminus L'$. If $n-l$ is even then vertices of $V(W_n) \setminus (L \cup M)$ induce $\frac{n-l-2}{2}$ paths P_1 and one path P_2. If $n-l$ is odd then there are $\frac{n-l+1}{2}$ paths P_1 in $V(W_n) \setminus (L \cup M)$.

Our requirement that $d_k(U \cup L)$ $(d_k(U' \cup L'))$ is maximum implies that, depending on the value of $k-l$, the following hold:

1. If $k-l \leq |M|$ then $U \subseteq M$, analogously if $k-l \leq |M'|$ then $U' \subseteq M'$.
2. If $k-l > |M|$ then $M \subset U$ (if $k-l > |M'|$ we have $M' \subset U'$). It means that, besides vertices of M (M'), U (U') contains also vertices increasing the number of p.a.v. by 1 (vertices of paths P_2) or by 2 (vertices of paths P_1).

Now we focus on these cases in detail. As $|M'| \leq |M|$ and $p' \leq l-2$, the case

$$k-l \leq |M'|$$

is trivial because then $U' \subseteq M'$ $(U \subseteq M)$ and the number of p.a.v. in $(L' \cup U')$ is p'. The number of p.a.v. in $(L \cup U)$ remains $l-2$, so that $e_{k,l}(L) \leq e_{k,l}(L')$ as we require.

Now we consider the other case, namely

$$k-l > |M'|.$$

Then the set U' contains also vertices increasing the number of p.a.v..

In what follows we use a set U'_0 which has the following properties:

1. $M' \subset U'_0$.
2. The set $L' \cup U'_0$ attains the maximum possible distance.
3. In $L' \cup U'_0$ the number of p.a.v. is $l-2$ (as in the set L) or $l-1$ (a special case explained below).

To prove (1) these situations have to be solved:

1. $|M'| < k-l < |U'_0|$. Then $U' \subset U'_0$ and the number of p.a.v. in $L' \cup U'$ is at most $l-2$, which implies (1).
2. $k-l \geq |U'_0|$. Then $|U'_0| \leq |U'|$. In the following part we show that $|U'_0| = |M|$ which means that we have $|U'_0|$ vertices in $V(W_n) \setminus L$ that do not increase the number of p.a.v.. Furthermore, we show that remaining vertices are on the

paths of order 1 and at most one path of order 2. As vertices of $V(W_n)\setminus(L'\cup U_0')$ induce paths of the same types, adding of vertices to form U' (from the set U_0') and U causes the same increasing of the number of p.a.v.. These imply $e_{k,l}(L) = e_{k.l}(L')$, which is a special case of (1).

Since in the set L' there are p' p.a.v., to form the set U_0' we must increase the number of p.a.v. by $l-2-p'$. We have to consider the following cases:

(a)
$$l - 2 - p' \le x_e,$$
which means that $l-2-p'$ is less than or equal to the number of even paths in $W_n \setminus L'$.

Then the set U_0' involves, besides vertices of M', only vertices from paths P_2. As each vertex from P_2 increases the number of p.a.v. by 1 we have $|U_0'| = |M'| + l - 2 - p'$. The number of paths P_2 in $W_n \setminus (L' \cup U_0')$ is $x_e - (l-2-p')$. This means that if $x_e = l-2-p'$ then there remains no path P_2. Else $x_e > l-2-p'$ and using equality $x_e + x_o = l-1-p'$, see (3), we have
$$l-2-p' < x_e \le l-1-p',$$
which implies that $x_e - (l-2-p') = 1$.

Now we focus on the set L. We know that vertices of $W_n\setminus L$ are on the path P_{n-l}. We show that in $V(W_n)\setminus L$ there are more than $2\cdot(|M'|+l-2-p')$ vertices, which means that there are $|M'|+l-2-p'$ vertices which do not increase the number of p.a.v.. Let U_0 denote the hypothetical set of these vertices. Then $|U_0| = |U_0'|$. Suppose that vertices of $V(W_n)\setminus(L\cup U_0)$ induce paths P_1 and one path P_z. Our aim is to show that $1 \le z \le 2$. By equalities (2), (4) and (5) we have

$$z = n - l - 2\cdot|M'| - 2\cdot(l-2-p') =$$

$$\underbrace{l-1-p'+d_2 + 3\cdot d_4 + \cdots + (n_e-1)\cdot d_{n_e} + 2\cdot d_3 + 4\cdot d_5 + \cdots + (n_o-1)\cdot d_{n_o}}_{n-l,\ \textit{see (2)}}$$

$$-2\cdot\underbrace{\left(d_4 + 2\cdot d_6 + \cdots + \frac{n_e-2}{2}\cdot d_{n_e} + d_3 + 2\cdot d_5 + \cdots + \frac{n_o-1}{2}\cdot d_{n_o}\right)}_{|M'|,\ \textit{see (4) and (5)}}$$

$$-2\cdot(l-2-p')$$

$$= l-1-p' + \underbrace{d_2 + d_4 + \cdots + d_{n_e}}_{x_e} - 2\cdot(l-2-p') = x_e + p' - l + 3. \quad (6)$$

By equalities $x_e + x_o = l-1-p'$ and $l-2-p' \le x_e$ the following holds.
$$l-2-p' \le x_e \le (l-2-p') + 1.$$

Then
$$1 \le x_e - (l-2-p') + 1 = x_e + p' - l + 3 = z \le 2.$$

Specially, if $x_e = l-2-p'$ then $z = x_e+p'-l+3 = 1$ and if $l-2-p' < x_e$ then $1 < z = x_e+p'-l+3 \leq 2$ which implies $z = x_e+p'-l+3 = 2$. Hence $|M| = |U_0| = |U_0'|$, and all vertices in $W_n \setminus (L \cup U_0)$ and $W_n \setminus (L' \cup U_0')$ are on paths of order 1 (if $x_e = l-2-p'$) or on paths of order 1 and one path of order 2 ($x_e > l-2-p'$). Then any other adding of vertices to U_0' and U_0 implies the same increasing of the number of p.a.v. in $L' \cup U_0'$ and $L \cup U_0$, as we require.

(b) Otherwise

$$l - 2 - p' > x_e$$

and two types of situation must be solved. We proceed similarly as in the previous case:

i. $(l-2-p'-x_e)$ is even. Then $|U_0'| = |M'| + x_e + (\frac{l-2-p'-x_e}{2})$. (Recall that $|M'|$ is the number of vertices that do not increase the number of p.a.v., x_e is the number of vertices that increase the number of p.a.v. by 1 and $\frac{l-2-p'-x_e}{2}$ vertices increase it by 2.) Setting U_0 so that $|U_0| = |U_0'|$, the equality

$$\begin{aligned} z &= n - l - 2 \cdot |M'| - 2 \cdot x_e - 2 \cdot \frac{l-2-p'-x_e}{2} \\ &= \underbrace{l - 1 - p' + x_e}_{\textit{see also (6)}} - 2 \cdot x_e - (l - 2 - p' - x_e) \\ &= 1 \end{aligned}$$

implies that $|M| = |U_0|$ and all vertices of $W_n \setminus (L \cup U_0)$ are on paths P_1 as we require.

ii. $(l-2-p'-x_e)$ is odd. Then $\frac{l-2-p'-x_e}{2}$ is not an integer. This is the case when we cannot construct the set U_0' such that in $L' \cup U_0'$ there are $l-2$ p.a.v. (see the definition of the set U_0'). Let us stop the construction of U_0' at the moment when the number of p.a.v. in $L' \cup U_0'$ is $l-3$. Then $|U_0'| = |M'| + x_e + (\frac{l-2-p'-x_e-1}{2})$. So for $|U_0| = |U_0'|$ all vertices of $W_n \setminus (L \cup U_0)$ are on paths P_1 and one path P_z, where

$$\begin{aligned} z &= n - l - 2 \cdot |M'| - 2 \cdot x_e - 2 \cdot \frac{l-2-p'-x_e-1}{2} = \\ &= \underbrace{l - 1 - p' + x_e}_{\textit{see also (6)}} - 2 \cdot x_e - (l - 2 - p' - x_e - 1) = 2. \end{aligned}$$

It means that in $W_n \setminus (L \cup U_0)$ there exists a path P_2. On the other hand all vertices of $W_n \setminus (L' \cup U_0')$ are on paths P_1. So if we add a next vertex to the sets U_0' and U_0, there will be $l-1$ p.a.v. in $L' \cup U_0'$ and $l-1$ p.a.v. in $L \cup U_0$, as we require.

We determined the (k,l)-central set L; now it remains to find the (k,l)-radius, $\mathrm{rad}_{k,l}(W_n)$. To do this we have to use $k-l$ vertices at maximum k-distance from the (k,l)-central set $L = \{s, u_0, u_1, \ldots, u_{l-2}\}$. As above let U denote the set of $k-l$ vertices on which $d_k(L \cup U)$ attains its maximum.

1. Suppose that $k-l \le \lfloor \frac{n-l-1}{2} \rfloor$. Then the vertices in U are mutually nonadjacent; they are at distance 1 from s and at distance 2 from other vertices of the (k,l)-central set. Then

$$\mathrm{rad}_{k,l}(W_n) = d_k(L \cup U) = \underbrace{l^2 - 3 \cdot l + 3}_{d_l(L)} + \underbrace{(k-l)}_{\substack{\text{distance of } s \\ \text{to vertices of } U}}$$

$$+ \underbrace{\binom{k-l}{2}}_{\text{(k-l)-distance of } U} \cdot 2 + \underbrace{(l-1) \cdot (k-l) \cdot 2}_{\substack{\text{from vertices of } U \\ \text{to those of } L \setminus \{s\}}} = k^2 - 2 \cdot k - l + 3.$$

2. Otherwise $k - l > \lfloor \frac{n-l-1}{2} \rfloor$. Then the set U contains $\lfloor \frac{n-l-1}{2} \rfloor$ vertices from $V(W_n \setminus L)$ that do not increase the number of p.a.v. in $d_k(L \cup U)$ and $k - l - \lfloor \frac{n-l-1}{2} \rfloor$ vertices that increase it. The pattern of these vertices was explained above. Since

$$\underbrace{n-l}_{|W_n \setminus L|} - \lfloor \frac{n-l-1}{2} \rfloor \cdot 2$$

attains only the value 1 (if $n-l$ is odd) or 2 (if $n-l$ is even) it follows that besides $\lfloor \frac{n-l-1}{2} \rfloor$ vertices that do not increase the number of p.a.v., U contains also one vertex that increases the number of p.a.v. by 2 or 1, respectively, and $k - l - \lfloor \frac{n-l-1}{2} \rfloor - 1$ vertices that increase the number of p.a.v. by 2.

Thus we have

$$\mathrm{rad}_{k,l}(W_n) = \underbrace{k^2 - 2 \cdot k - l + 3}_{\substack{d_k(L \cup U) \text{ if no vertex of } U \\ \text{increases the number of p.a.v.}}}$$

$$- \left[\underbrace{\lceil \frac{n-l}{2} \rceil - \lfloor \frac{n-l}{2} \rfloor + 1}_{\substack{\text{one vertex that increases} \\ \text{the number of p.a.v. by 1 or 2}}} + 2 \cdot \underbrace{(k - l - \lfloor \frac{n-l-1}{2} \rfloor - 1)}_{\substack{\text{the number of vertices that} \\ \text{increase the number of p.a.v. by 2}}} \right]$$

$$= k^2 - 4 \cdot k + l + 4 + 2 \cdot \lfloor \frac{n-l-1}{2} \rfloor - \lceil \frac{n-l}{2} \rceil + \lfloor \frac{n-l}{2} \rfloor.$$

Since

$$2 \cdot \lfloor \frac{n-l-1}{2} \rfloor - \lceil \frac{n-l}{2} \rceil + \lfloor \frac{n-l}{2} \rfloor = n - l - 2$$

for odd $n-l$ as well as for even $n-l$, the previous equality can be simplified to

$$\mathrm{rad}_{k,l}(W_n) = k^2 - 4 \cdot k + n + 2. \ \square$$

Proof of Assertion 1. Let $l = 1$. Analogously as above it can be shown that there is a $(k,1)$-central set containing the vertex s. For $k = 1$ we have $\mathrm{rad}_{1,1}(W_n) = 0$.

Now let $k > 1$. By the construction used in the proof of Theorem 1, one $(k,2)$-central set contains the vertex s and u_0. As all $u_0, u_1, \dots, u_{n-2}$ belong to the same orbit of $\mathrm{out}(W_n)$, to determine the $(k,1)$-radius we can choose the first vertex of the set U arbitrarily (recall that $U \subset V(W_n \setminus \{s\})$. Hence, suppose that this vertex is u_0. Then the set U_1 of $k-1$ vertices, such that $d_k(\{s\} \cup U_1)$ has the maximum possible value, coincides with the set $u_0 \cup U_2$ ($|U_2| = k-2$), for which $d_k(\{s, u_0\} \cup U_2)$ has the maximum possible value. I.e., $\mathrm{rad}_{k,1}(W_n) = \mathrm{rad}_{k,2}(W_n)$. The rest follows from Theorem 1. □

Proof of Assertion 2. If $k = n$ then the $(n,0)$-radius is the transmission of the graph and by Theorem 1 $\mathrm{rad}_{n,0}(W_n) = n^2 - 3 \cdot n + 2$. Now we consider $k < n$. Then the set K on which $\mathrm{rad}_{k,0}(W_n)$ is attained does not contain the vertex s. Thus $\mathrm{rad}_{k,0}(W_n) = \mathrm{rad}_{k+1,1}(W_n) - k$. Hence, by Assertion 1 we have:

1. If $k \le \lfloor \frac{n-1}{2} \rfloor$ then
$$\mathrm{rad}_{k,0}(W_n) = (k+1)^2 - 2 \cdot (k+1) + 1 - k = k^2 - k.$$
2. If $n > k \ge \lfloor \frac{n-1}{2} \rfloor$ then
$$\mathrm{rad}_{k,0}(W_n) = (k+1)^2 - 4 \cdot (k+1) + n + 2 - k = k^2 - 3 \cdot k + n - 1. \ \square$$

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