

Note on edge-colored graphs and digraphs without properly colored cycles

GREGORY GUTIN

*Department of Computer Science
Royal Holloway, University of London
Egham, Surrey, TW20 0EX
U.K.*

Gutin@cs.rhul.ac.uk

Abstract

We study the following two functions: $d(n, c)$ and $\vec{d}(n, c)$; here $d(n, c)$ ($\vec{d}(n, c)$) is the minimum number k such that every c -edge-colored undirected (directed) graph of order n and minimum monochromatic degree (out-degree) at least k has a properly colored cycle. Abouelaoualim et al. (2007) stated a conjecture which implies that $d(n, c) = 1$. Using a recursive construction of c -edge-colored graphs with minimum monochromatic degree p and without properly colored cycles, we show that $d(n, c) \geq \frac{1}{c}(\log_c n - \log_c \log_c n)$ and, thus, the conjecture does not hold. In particular, this inequality significantly improves a lower bound on $\vec{d}(n, 2)$ obtained by Gutin, Sudakov and Yeo in 1998.

1 Introduction

All directed and undirected graphs considered in this paper are simple, i.e., have no loops or parallel edges. We consider only directed cycles in digraphs; the term cycle (in a digraph) will always mean a directed cycle.

Let $G = (V, E)$ be a directed or undirected graph, and let $\chi : E \rightarrow \{1, 2, \dots, c\}$ be a fixed (not necessarily proper) edge-coloring of G with c colors, $c \geq 2$. With given χ , G is called a c -edge-colored (or, edge-colored) graph. A subgraph H of G is called *properly colored* if χ defines a proper edge-coloring of H , i.e., no vertex of H is incident to a pair of edges of the same color. For a vertex of a c -edge-colored graph G , $d_i(x)$ denotes the number of edges of color i incident with x . Let $\delta_{mon}(G) = \min\{d_i(x) : x \in V(G), i \in \{1, 2, \dots, c\}\}$. If G is directed, $d_i^+(x)$ denotes the number of edges of color i in which x is tail. Let $\delta_{mon}^+(G) = \min\{d_i^+(x) : x \in V(G), i \in \{1, 2, \dots, c\}\}$.

The authors of [2] stated the following:

Conjecture 1.1 *Let G be a c -edge-colored undirected graph of order n with $\delta_{\text{mon}}(G) = d \geq 1$. Then G has a properly colored cycle of length at least $\min\{n, cd\}$. Moreover, if $c > 2$, then G has a properly colored cycle of length at least $\min\{n, cd + 1\}$.*

In the next section, using a recursive construction of c -edge-colored graphs with minimum monochromatic degree d and without properly colored cycles, we show that this conjecture does not hold. Moreover, for every $d \geq 1$ there exists an edge-colored graph G with $\delta_{\text{mon}}(G) \geq d$ and with no properly colored cycle.

We will study the following two functions: $d(n, c)$ and $\vec{d}(n, c)$; $d(n, c)$ ($\vec{d}(n, c)$) is the minimum number k such that every c -edge-colored graph (digraph) of order n and minimum monochromatic degree (out-degree) at least k has a properly colored cycle. Gutin, Sudakov and Yeo [5] proved the following bounds for $\vec{d}(n, 2)$

$$\frac{1}{4} \log_2 n + \frac{1}{8} \log_2 \log_2 n + \Theta(1) \leq \vec{d}(n, 2) \leq \log_2 n - \frac{1}{3} \log_2 \log_2 n + \Theta(1) \quad (1)$$

Using our construction, we prove that $\vec{d}(n, 2) \geq \frac{1}{2}(\log_2 n - \log_2 \log_2 n)$. This improves the lower bound in (1). (The lower bound in (1) was obtained using significantly more elaborate arguments.) This bound on $\vec{d}(n, 2)$ follows from lower and upper bounds on $d(n, c)$ and $\vec{d}(n, c)$ obtained for each value of c . The bounds imply that $d(n, c) = \Theta(\log_2 n)$ and $\vec{d}(n, c) = \Theta(\log_2 n)$ for each fixed $c \geq 2$.

Properly colored cycles have been studied in several papers, for a survey, see Chapter 11 in [3]. Properly colored cycles in 2-edge-colored undirected graphs generalize cycles in digraphs and are of interest in genetics [3]. More recent papers on properly colored cycles include [1, 2, 4]. Interestingly, the problem to check whether an edge-colored undirected graph has a properly colored cycle is polynomial time solvable (we can even find a shortest properly colored cycle in polynomial time [1]), but the same problem for edge-colored digraphs is NP-complete [5].

2 Results

Theorem 2.1 *For each $d \geq 1$ there is an edge-colored graph G with $\delta_{\text{mon}}(G) = d$ and with no properly colored cycle.*

Proof: Let (p_1, p_2, \dots, p_c) be a vector with nonnegative integral coordinates p_i . For an arbitrary (p_1, p_2, \dots, p_c) , $G(p_1, p_2, \dots, p_c)$ is recursively defined as follows: take a new vertex x and graphs $H_1 = G(p_1 - 1, p_2, p_3, \dots, p_{c-1}, p_c)$ if $p_1 > 0$, $H_2 = G(p_1, p_2 - 1, p_3, \dots, p_{c-1}, p_c)$ if $p_2 > 0$, ..., $H_c = G(p_1, p_2, p_3, \dots, p_{c-1}, p_c - 1)$ if $p_c > 0$ and add an edge of color i between x and every vertex of H_i for each i for which $p_i > 0$. In particular, $G(0, 0, \dots, 0) = K_1$. (Fig. 1 depicts $G(1, 0)$, $G(0, 1)$ and $G(1, 1)$.)

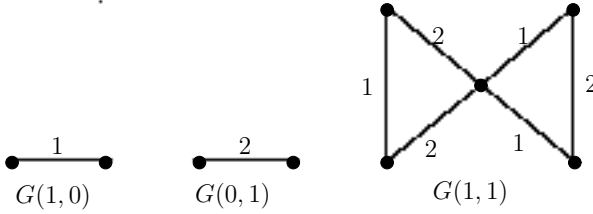


Figure 1: Edge-coloured graphs with no PC cycles.

It is easy to see, by induction on $p_1 + p_2 + \dots + p_c$, that $G = G(p_1, p_2, \dots, p_c)$ has no properly colored cycle and $\delta_{mon}(G) = \min\{p_i : i = 1, 2, \dots, c\}$. \square

In fact, for each $d \geq 1$ there are infinitely many edge-colored graphs G with $\delta_{mon}(G) = d$ and with no properly colored cycle. Indeed, in the construction of $G(p_1, p_2, \dots, p_c)$ above we may assume that $G(0, 0, \dots, 0)$ is an edgeless graph of arbitrary order.

Lemma 2.2 *Let $n(p_1, p_2, \dots, p_c)$ be the order of $G(p_1, p_2, \dots, p_c)$ and let $n_c(p) = n(p_1, \dots, p_c)$ for $p = p_1 = \dots = p_c$. Then $n(p_1, \dots, p_c) \leq s2^s$, where $s = p_1 + p_2 + \dots + p_c$, provided $s > 0$ and $p \geq \frac{1}{c}(\log_c n_c(p) - \log_c \log_c n_c(p))$.*

Proof: We first prove $n(p_1, \dots, p_c) \leq s2^s$ by induction on $s \geq 1$. The inequality clearly holds for $s = 1$. By induction hypothesis, for $s \geq 2$, we have

$$\begin{aligned} n(p_1, \dots, p_c) &\leq 1 + \sum_{i=1}^c \{n(p_1, \dots, p_{i-1}, p_i - 1, p_{i+1}, \dots, p_c) : p_i > 0\} \\ &\leq 1 + c(s - 1)c^{s-1} \leq sc^s \end{aligned}$$

Thus, $n_c(p) \leq cp \cdot c^p$. Observe that $n_c(p) > ac^a$ provided $a = \log_c n_c(p) - \log_c \log_c n_c(p)$ and, thus, $cp \geq \log_c n_c(p) - \log_c \log_c n_c(p)$. \square

Corollary 2.3 *We have $\vec{d}(n, c) \geq d(n, c) \geq \frac{1}{c}(\log_c n - \log_c \log_c n)$.*

Proof: Let H be a c -edge-colored undirected graph and H^* be a digraph obtained from H by replacing every edge $e = xy$ with arcs xy and yx both of color $\chi(e)$. Clearly, H has a properly colored cycle if and only if H^* has a properly colored cycle. Thus, $\vec{d}(n, c) \geq d(n, c)$. The inequality $d(n, c) \geq \frac{1}{c}(\log_c n - \log_c \log_c n)$ follows from Lemma 2.2 and the fact that graphs $G(p, p, \dots, p)$ have no properly colored cycles. \square

We see that $\vec{d}(n, 2) \geq \frac{1}{2}(\log_2 n - \log_2 \log_2 n)$. This is an improvement over the lower bound on $\vec{d}(n, 2)$ in (1). Using the upper bound in (1), we will obtain an upper bound on $\vec{d}(n, c)$ and, thus, $d(n, c)$.

Proposition 2.4 *We have $\vec{d}(n, c) \leq \frac{1}{\lfloor c/2 \rfloor}(\log_2 n - \frac{1}{3} \log_2 \log_2 n + \Theta(1))$.*

Proof: Let D be a c -edge-colored digraph of order n with $\delta_{\text{mon}}(D) \geq \frac{1}{\lfloor c/2 \rfloor}(\log_2 n - \frac{1}{3} \log_2 \log_2 n + \Theta(1))$. Let D' be the 2-edge-colored digraph obtained from D by assigning color 1 to all edges of D of color $1, 2, \dots, \lfloor c/2 \rfloor$ and color 2 to all edges of D of color $\lfloor c/2 \rfloor + 1, \lfloor c/2 \rfloor + 2, \dots, c$. It remains to observe that $\delta_{\text{mon}}(D') \geq \log_2 n - \frac{1}{3} \log_2 \log_2 n + \Theta(1)$ and every properly colored cycle in D' is a properly colored cycle in D . \square

Corollary 2.5 *For every fixed $c \geq 2$, we have $d(n, c) = \Theta(\log_2 n)$ and $\vec{d}(n, c) = \Theta(\log_2 n)$.*

3 Open Problems

We believe that there are functions $s(c), r(c)$ dependent only on c such that $d(n, c) = s(c) \log_2 n(1 + o(1))$ and $\vec{d}(n, c) = r(c) \log_2 n(1 + o(1))$. In particular, it would be interesting to determine $s(2)$ and $r(2)$.

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