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Minimum order of r-regular bipartite graphs of pair length k

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Abstract

The concepts of k-pairable graphs and the pair length of a graph were introduced by Chen [*Discrete Math.* 287 (2004), 11–15] to generalize an elegant result of Graham et al. [*Amer. Math. Monthly* 101 (1994), 664–667] from hypercubes and graphs with antipodal isomorphisms to a much larger class of graphs. A graph G is *k-pairable* if there is a positive integer k such that the automorphism group of G contains an involution ϕ with the property that the distance between x and $\phi(x)$ is at least k for any vertex x of G. The pair length of a graph G, denoted by $p(G)$, is the maximum positive integer k such that G is k-pairable; and $p(G) = 0$ if G is not k-pairable for any positive integer k.

The aim of this paper is to answer an open question posted in our previous paper [*Discrete Math.* 310 (2010), 3334–3350]; that is, the question of determining the minimum order of a graph in the set of r-regular bipartite graphs of pair length k. We solve the problem for all positive integers k and r except for the case when both $k \geq 5$ and $r \geq 3$ are odd. For the case that is still open, we provide bounds on the minimum order concerned. Also we post a conjecture on the minimum order of a cubic bipartite graph of pair length k for any odd number $k > 1$.

1 Introduction

In 1994, Graham et al. [15] proved that for any spanning tree T of a hypercube Q_n, there is an edge of Q_n outside T whose addition to T forms a cycle of length at least $2n$. They also generalized this result from hypercubes to the connected graphs with antipodal isomorphisms. Ten years later, Chen [6] further generalized the above elegant result to a much larger class of connected graphs called k-pairable graphs. These graphs have a special kind of symmetry which is different from the well-studied types of symmetry such as vertex-transitive, edge-transitive or distance-transitive. The set of k-pairable graphs contains many graphs of theoretical and practical importance, such as hypercubes, Hamming graphs, antipodal graphs, diametrical graphs and S-graphs, etc. (cf. [1], [2], [4], [12], [13], [16], [18] and [20]).

Interesting results on k-pairable graphs have been obtained since they were introduced in [6]. Christofides [11] showed that the pair length of a Cartesian product graph is the sum of pair lengths of its factors, which answered an open question raised by Chen [6]. Though the problem on characterizing graphs of pair length k raised in [6] is still open, it has motivated many results in literature. For example, a characterization of trees of pair length k was given in [7], and a characterization of uniquely k-pairable graphs in terms of the prime factor decomposition with respect to the Cartesian product was given in [8]. We also provided a necessary and sufficient condition for graphs of pair length k in [9].

When we study a class of connected graphs with specific property and related parameters, it is natural to ask: What are the bounds on the orders of those graphs with the given specific parameters? What are the bounds on the sizes of those graphs with the given order and specific parameters? In [10], we provided sharp bounds on the size of a connected graph G of even order n and pair length k for both cases when G is bipartite and when G is not restricted to be bipartite. In [9], we gave the minimum order of an r-regular connected graph of pair length k and raised an open question: What is the minimum order of an r-regular connected bipartite graph of pair length k? In this paper, we first give some properties of k-pairable bipartite graphs, and then answer the above question for all positive integers k and r except the case when both $k \geq 5$ and $r \geq 3$ are odd. For the case that is still open, we provide bounds on the concerned minimum order and post a conjecture on the minimum order of a cubic bipartite graph of pair length k for any odd number $k > 1$.

2 Preliminaries

All graphs considered in this paper are finite simple graphs. We use [5] and [21] as reference books for basic terminologies. The set of all vertices of a graph G is denoted by $V(G)$, and its cardinality $|V(G)|$ is called the order of G. The set of all edges of a graph G is denoted by $E(G)$, and its cardinality $|E(G)|$ is called the size of G. An *involution* ϕ of the vertex set of a graph G is a permutation on $V(G)$ such that $\phi^2(x) = x$ for any vertex x of G. An *automorphism* of a graph G is a permutation on $V(G)$ such that two vertices of G are adjacent if and only if their images under the permutation are adjacent. The set of all automorphisms of G forms a group and is called the automorphism group of G. The concepts of a k-pairable graph and a pair partition of a graph introduced in [6] can be easily extended from the set of connected graphs to the set including both connected and disconnected graphs.

Definition 2.1 Let k be a positive integer. A graph G is called *k-pairable* if its automorphism group contains an involution ϕ with the property that the distance between x and $\phi(x)$ is at least k for any vertex x of G. If such an involution ϕ exists, then it is called a *k-pair partition* of G, and $\phi(x)$ is the mate of x (respectively, x is the mate of $\phi(x)$) under ϕ.

By definition, any k-pair partition of G is a fixed-point-free involution of $V(G)$, and so any k-pairable graph has even order. If there is no need to specify the positive integer k, then a k-pairable graph is called a *pairable graph* and a k-pair partition is called a *pair partition* briefly. Note also that, by definition, any k-pair partition may be considered a 1-pair partition.

Let C_{2n} be an even cycle with the vertex set $\{v_i | 1 \leq i \leq 2n\}$ and the edge set $\{e_i | 1 \leq i \leq 2n\}$ where $e_i = v_i v_{i+1}$ for $1 \leq i \leq 2n-1$ and $e_{2n} = v_{2n}v_1$. For each $1 \leq i \leq n$, recall that v_{i+n} is the antipodal vertex of v_i, and vice versa; e_{i+n} is the antipodal edge of e_i, and vice versa. The *antipodal automorphism* of C_{2n} is an involution on the vertex set of C_{2n} that sends each vertex to its antipodal vertex. To describe a k-pair partition of C_{2n} easily, we assume that vertices v_i $(1 \leq i \leq 2n)$ of C_{2n} are ordered evenly clockwise around a circle. Then C_{2n} has exactly two types of k-pair partitions ϕ: either ϕ is the antipodal automorphism of C_{2n} and $k = n$, or ϕ is a reflection about an axis through midpoints of two antipodal edges of C_{2n} and $k = 1$.

Definition 2.2 [6] The *pair length* of a connected graph G, denoted by $p(G)$, is the maximum positive integer k such that G is k-pairable, and $p(G) = 0$ if G is not pairable.

If we drop the requirement that the automorphism be an involution, then the above definition for the pair length of a connected graph becomes the same as the absolute mobility of a connected graph, a concept defined in [19] by Potočnik, Šajna and Verret.

In the studies of k-pairable graphs, the concept of a strongly induced cycle plays a very important role. An induced cycle C of a graph G is called a *strongly induced cycle* if $d_C(x, y) = d_G(x, y)$ for any two vertices x, y of C. It is clear that if $n = 3, 4, 5$, then an induced n-cycle of G is a strongly induced n-cycle of G, but it is not necessarily true when $n > 5$. A necessary and sufficient condition for a connected graph to have its pair length equal to a positive integer k is given in the following theorem.

Theorem 2.3 [9] *Let G be a connected graph of pair length $p(G) > 0$. Then*

(i) $p(G) = 1$ if and only if G is 1-pairable and for any 1-pair partition ϕ of G there is an edge e_ϕ of G such that ϕ maps e_ϕ onto itself.

(ii) $p(G) = k(> 1)$ *if and only if* G *is* k*-pairable and for any* k*-pair partition* ϕ *of* G *there is a strongly induced* $2k$*-cycle* C_ϕ *of* G *such that* ϕ *maps* C_ϕ *onto itself.*

A graph G is *bipartite* if its vertex set can be partitioned into two independent sets (called the two color classes of G). The partition is unique when the bipartite graph is connected. We use B and W to represent the two color classes of G, and say a vertex is in color black (respectively, in color white) if it is contained in B (respectively, in W). The pair length of a complete bipartite graph $K_{m,n}$ is given in [10] as follows:

$$p(K_{m,n}) = \begin{cases} 2, & \text{if both } m \text{ and } n \text{ are even;} \\ 1, & \text{if } m = n \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

Due to the uniqueness of the 2-color vertex partition of a connected bipartite graph G, the two color classes are preserved setwise by any automorphism of G. Thus, Theorem 2.2 in [10] can be somewhat generalized as follows.

Theorem 2.4 [10] *Let* G *be a connected bipartite graph with a* k*-pair partition* ϕ*. Assume that* $d_G(v, \phi(v)) = k$ *for some vertex* v *of* G*. Then*

(i) If k *is even,* x *and* $\phi(x)$ *must be in the same color class of* G *for any* $x \in V(G)$*.*

(ii) If k *is odd,* x *and* $\phi(x)$ *must be in distinct color classes of* G *for any* $x \in V(G)$*.*

When bipartite graphs are regular, they have regularity not only in vertex degrees but also in the vertex numbers of two color classes. It is well known that the regular bipartite graphs have nice properties involving perfect matchings and edge colorings. Naturally, one may ask: Is every connected regular bipartite graph pairable? The answer is "No", which can be seen from the following counterexample given in Fig. 1. Note that the vertex v is the unique vertex which is not on any induced cycle of length greater than 4, and so the graph is not pairable. In fact, using Lemma 2.7 and Lemma 2.8 below, one can further show that this is a counterexample with the smallest number of vertices. See Proposition 2.10 at the end of this section.

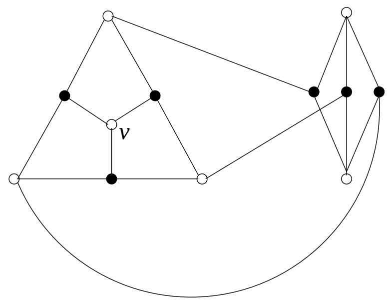

Figure 1: A 3-regular bipartite graph that is not pairable.

The main object of study in this paper is a special type of pairable bipartite graphs, that is, regular bipartite graphs of pair length $k > 0$, which are called *pairable regular bipartite graphs*. It is well known [3] that any connected r-regular bipartite graph H is a spanning subgraph of $K_{n,n}$ for an integer $n \geq r$, and H is 1-factorable. The following corollary can be derived from Theorem 2.4 immediately.

Corollary 2.5 *Let G be a connected r-regular bipartite graph of pair length $k > 0$. Then $|V(G)| = 2n$ for an integer $n \geq r$, and G is a spanning subgraph of $K_{n,n}$. Moreover,*

(i) If n is odd, then k must be odd; if n is even, then k may be either even or odd.

(ii) If k is even, then n must be even; if k is odd, then n may be either even or odd.

Let G be a connected regular bipartite graph of order $2n$ and color classes B and W. The graph with the same vertex set as G and the edge set $E(K_{n,n}) \setminus E(G)$ is called the *relative complement* of G in $K_{n,n}$. Note that the relative complement of G in $K_{n,n}$ may be disconnected, and so its two color classes are not necessary to be uniquely determined.

Now we give the following concept which will be used frequently in the paper.

Definition 2.6 Let G be a connected r-regular bipartite graph of order $2n$ with color classes B and W. The relative complement of G in $K_{n,n}$ is called the *bipartite complement* of G, and denoted as $\overline{G}^{bc}$, if it carries the same two color classes as G.

Lemma 2.7 *Let G be a connected regular bipartite graph, and $\overline{G}^{bc}$ be the bipartite complement of G. Then any pair partition of G is a pair partition of $\overline{G}^{bc}$. On the other hand, if ϕ is a pair partition of $\overline{G}^{bc}$ such that x and $\phi(x)$ are in the same color class (respectively, from distinct color classes) for all $x \in V(\overline{G}^{bc})$, then ϕ is a k-pair partition of G where $k > 0$ is even (respectively, odd) such that $d_G(v, \phi(v)) = k$ for some vertex v of G.*

Proof. Let ϕ be a pair partition of G. Then ϕ is also a fixed-point-free involution of $V(\overline{G}^{bc})$. It remains to show that ϕ is an automorphism of $\overline{G}^{bc}$. Assume that B and W are two color classes of $\overline{G}^{bc}$. Then they are also two color classes of G. Since G is connected, $d_G(v, \phi(v)) = \min_{u \in V(G)}\{d_G(u, \phi(u))\} = k$ for some vertex v of G and a positive integer k. We distinguish two cases based on the parity of k.

Case 1. k is even. By the definition of bipartite complement, two vertices $x \in B$ and $y \in W$ are adjacent in $\overline{G}^{bc}$ if and only if $x \in B$ and $y \in W$ are not adjacent in G. Since ϕ is an automorphism of G, x and y are not adjacent in G if and only if $\phi(x)$ and $\phi(y)$ are not adjacent in G. By Theorem 2.4, $\phi(x) \in B$ and $\phi(y) \in W$ since $x \in B$ and $y \in W$ and k is even. By the definition of bipartite complement, $\phi(x) \in B$ and $\phi(y) \in W$ are not adjacent in G if and only if $\phi(x) \in B$ and $\phi(y) \in W$ are adjacent in $\overline{G}^{bc}$.

Case 2. k is odd. Similarly, we can show that two vertices $x \in B$ and $y \in W$ are adjacent in $\overline{G}^{bc}$ if and only if $\phi(x) \in W$ and $\phi(y) \in B$ are adjacent in $\overline{G}^{bc}$.

Hence, we show that two vertices are adjacent in $\overline{G}^{bc}$ if and only if their images under ϕ are adjacent in $\overline{G}^{bc}$, and so ϕ is an automorphism of $\overline{G}^{bc}$. It follows that ϕ is a pair partition of $\overline{G}^{bc}$.

On the other hand, if ϕ is a pair partition of $\overline{G}^{bc}$ such that either x and $\phi(x)$ are in the same color class for all x of $\overline{G}^{bc}$, or x and $\phi(x)$ are from distinct color classes for all x of $\overline{G}^{bc}$, then similarly we can show that ϕ is a k-pair partition of G such that $d_G(v, \phi(v)) = k$ for some vertex v of G, and the parity of k is based on whether x and $\phi(x)$ are in same color class of G or not. $\square$

Lemma 2.8 *Assume that G is a connected regular bipartite graph and $\overline{G}^{bc}$ is the bipartite complement of G. If $\overline{G}^{bc}$ has a perfect matching M such that the involution switching end vertices of each edge in M is a 1-pair partition of $\overline{G}^{bc}$, then $p(G) \geq 3$. The converse is true if the pair length $p(G) \geq 3$ is an odd number.*

Proof. Assume that $\overline{G}^{bc}$ has a perfect matching $M = \{b_i w_i | 1 \leq i \leq n\}$ such that the involution ϕ satisfying $w_i = \phi(b_i)$ for $1 \leq i \leq n$ is 1-pair partition of $\overline{G}^{bc}$. Then without loss of generality, we may assume that $B = \{b_i | 1 \leq i \leq n\}$ and $W = \{w_i | 1 \leq i \leq n\}$ are two color classes of G. By Lemma 2.7, ϕ is a pair partition of G. Moreover, for each $1 \leq i \leq n$, $d_G(b_i, \phi(b_i)) = d_G(b_i, w_i)$ is some odd number at least 3 since $b_i \in B$ and $w_i \in W$ are not adjacent in G. It follows that $p(G) \geq 3$.

On the other hand, if $p(G) = k \geq 3$ is an odd number, then by Theorem 2.4, we may assume that $B = \{b_i | 1 \leq i \leq n\}$ and $W = \{w_i | 1 \leq i \leq n\}$ are two color classes of G and ϕ is a k-pair partition of G satisfying $\phi(b_i) = w_i$ for $1 \leq i \leq n$. Then $d_G(b_i, w_i) = d_G(b_i, \phi(b_i)) \geq k \geq 3$ for all $1 \leq i \leq n$. Hence, b_i and w_i are adjacent in $\overline{G}^{bc}$ for all $1 \leq i \leq n$. By Lemma 2.7, ϕ is a 1-pair partition of $\overline{G}^{bc}$. $\square$

Note that the converse is not necessarily true if $p(G) \geq 3$ is an even integer. For example, if $G = C_8$, then $p(G) = 4 > 2$ and the bipartite complement of G is $\overline{G}^{bc} = C_8$. But C_8 has exactly two perfect matchings and each of them is a maximum set of nonadjacent edges of C_8. So any involution switching the end vertices of each edge in a perfect matching of C_8 cannot be a 1-pair partition of C_8.

Corollary 2.9 *Assume that G is a pairable connected regular bipartite graph of order $2n$ and $\overline{G}^{bc}$ is the bipartite complement of G. If n is odd, then $p(G) = 1$ if and only if $\overline{G}^{bc}$ has no perfect matching M such that the involution switching end vertices of each edge in M is a 1-pair partition of $\overline{G}^{bc}$.*

Proof. By Corollary 2.5, the pair length $p(G)$ must be odd since n is odd. By Lemma 2.8, $p(G) \geq 3$ is an odd number if and only if $\overline{G}^{bc}$ has a perfect matching M such that the involution switching end vertices of each edge in M is a 1-pair partition of $\overline{G}^{bc}$. Hence, the corollary follows. $\square$

The next proposition shows that the graph of order 12 in Figure 1 is a smallest regular bipartite graph that is not pairable. For convenience, in the proof we will call a graph *non-pairable* if it is not pairable.

Proposition 2.10 *Let G be a regular bipartite graph that is not pairable. Then $|V(G)| \geq 12$, and the bound is sharp.*

Proof. First, we point out that a smallest non-pairable regular bipartite graph must be connected, since any non-pairable disconnected regular bipartite graph must have a non-pairable regular bipartite component. Next, note that the complete graph K_2 is the unique connected 1-regular bipartite graph and K_2 is pairable. Also note that any connected 2-regular bipartite graph is an even cycle and so is pairable. Then we only need to show that any connected r-regular bipartite graph H (where $r \geq 3$) of order less than 12 is pairable. Since $|V(H)| < 12$, we have $r < 6$, and so we may distinguish three cases according to the possible values of the degree r of H. Before discussing each case separately, recall the well-known fact [3] that any connected r-regular bipartite graph of order $2n$ (where $n \geq r$) is a spanning subgraph of $K_{n,n}$.

Case 1. $r = 3$. Then $|V(H)| = 2n \geq 6$. If $|V(H)| = 6$, then $H = K_{3,3}$ and so H has pair length 1 by [10]. If $|V(H)| = 8$, then H is the subgraph of $K_{4,4}$ obtained by removing a perfect matching, and so H has pair length 3 by Lemma 2.8. If $|V(H)| = 10$, then the bipartite complement $\overline{H}^{bc}$ is a union of disjoint even cycles, and so $\overline{H}^{bc}$ is pairable. Then by Lemma 2.7, H is also pairable.

Case 2. $r = 4$. Then $|V(H)| = 2n \geq 8$. If $|V(H)| = 8$, then $H = K_{4,4}$ and so H has pair length 2 by [10]. If $|V(H)| = 10$, then H is the subgraph of $K_{5,5}$ obtained by removing a perfect matching, and so H has pair length 3 by Lemma 2.8.

Case 3. $r = 5$. Then $|V(H)| = 2n \geq 10$. If $|V(H)| = 10$, then $H = K_{5,5}$ and so H has pair length 1 by [10].

Thus, we have shown that any connected r-regular bipartite graph (where $r \geq 3$) of order less than 12 is pairable. It follows that for any regular bipartite graph G that is not pairable, we must have $|V(G)| \geq 12$. The bound 12 is sharp, since the graph given in Figure 1 is a non-pairable connected regular bipartite graph of order 12. □

3 Main Results

Theorem 3.1 *Let $\mathbb{B}(r, k)$ be the set of connected r-regular bipartite graphs of pair length k where both k and r are positive integers. Then $\mathbb{B}(r, 1)$ is nonempty if and only if $r \neq 2$; and $\mathbb{B}(r, k)$ where $k > 1$ is nonempty if and only if $r > 1$.*

(i) If $k = 1$ and $r \geq 1$, then

$$\min_{G \in \mathbb{B}(r,1)} \{|V(G)|\} = \begin{cases} 2r, & \text{if } r \equiv 1 (mod\ 2), \\ 2(r+3) = 14, & \text{if } r = 4, \\ 2(r+2), & \text{if } r > 4 \text{ and } r \equiv 0 (mod\ 2). \end{cases}$$

(ii) If $k = 2$ and $r > 1$, then

$$\min_{G \in \mathbb{B}(r,2)} \{|V(G)|\} = \begin{cases} 2r, & \text{if } r \equiv 0 (mod\ 2), \\ 2(r+3), & \text{if } r \equiv 1 (mod\ 2). \end{cases}$$

(iii) If $k = 3$ and $r > 1$, then

$$\min_{G \in \mathbb{B}(r,3)} \{|V(G)|\} = 2(r+1).$$

(iv) If $k > 3$ and $r > 1$, and at least one of them is even, then

$$\min_{G\in\mathbb{B}(r,k)}\{|V(G)|\} = \begin{cases} kr, & \text{if } r \equiv 0(mod\ 2), \\ kr, & \text{if } r \equiv 1(mod\ 2) \text{ and } k \equiv 0(mod\ 4), \\ kr+2, & \text{if } r \equiv 1(mod\ 2) \text{ and } k \equiv 2(mod\ 4). \end{cases}$$

Proof. By Lemmas 3.2, 3.3, 3.4, 3.6, and 3.7 given in the following subsections. □

3.1 $\mathbb{B}(r, k)$ where $k \leq 3$

Lemma 3.2 *Let r be a positive integer. Then $\mathbb{B}(r, 1)$ is nonempty if and only if $r \neq 2$.*

(i) If $r \equiv 1(mod\ 2)$, then $\min_{G\in\mathbb{B}(r,1)}\{|V(G)|\} = 2r$.

(ii) If $r \equiv 0(mod\ 2)$, then $\min_{G\in\mathbb{B}(r,1)}\{|V(G)|\} = \begin{cases} 2(r+3) = 14, & \text{if } r = 4; \\ 2(r+2), & \text{if } r > 4. \end{cases}$

Proof. Let G be an arbitrary graph in $\mathbb{B}(r, 1)$. Then $r \neq 2$ since any connected 2-regular bipartite graph is an even cycle, and so has pair length > 1. By Corollary 2.5, G is a spanning subgraph of $K_{n,n}$ where $n \geq r$. Hence, $|V(G)| \geq 2r$.

(i) If $r \equiv 1(\text{mod } 2)$, then $r \geq 1$ since $p(K_{r,r}) = 1$ by [10] and, hence, $|V(G)| \geq 2r$ is sharp.

(ii) If $r \equiv 0(\text{mod } 2)$, then $r \geq 4$ since $r \neq 2$. Note that $|V(G)| \neq 2r$ since $p(K_{r,r}) = 2$ when r is even. Moreover, $|V(G)| \neq 2(r+1)$ since any r-regular spanning subgraph of $K_{r+1,r+1}$ has pair length 3 by Lemma 2.8. Hence, $|V(G)| \geq 2(r+2)$. We distinguish two cases based on $r = 4$ and $r > 4$ respectively.

Case 1. $r = 4$. We show that $|V(G)| \geq 2(r+3) = 14$ is a sharp bound.

First, $|V(G)| \neq 2(r+2) = 12$. Otherwise, if $|V(G)| = 12$, then G is a 4-regular spanning subgraph of $K_{6,6}$ of pair length $p(G) = 1$. Let $\overline{G}^{bc}$ be the bipartite complement of G. Then $\overline{G}^{bc}$ is a 2-regular spanning subgraph of $K_{6,6}$, and so a disjoint union of even cycles. There are four possibilities for $\overline{G}^{bc}$: $\overline{G}^{bc} = C_{12}$, or $\overline{G}^{bc} = C_4 \cup C_8$, or $\overline{G}^{bc} = C_4 \cup C_4 \cup C_4$, or $\overline{G}^{bc} = C_6 \cup C_6$. Suppose that $\overline{G}^{bc} = C_{12}$, or $\overline{G}^{bc} = C_4 \cup C_8$, or $\overline{G}^{bc} = C_4 \cup C_4 \cup C_4$. Then $\overline{G}^{bc}$ has a pair partition ϕ such that the restriction of ϕ on each even cycle is the antipodal automorphism of the even cycle. Note that any two antipodal vertices of an even cycle C_{2n} have the same color if n is even. Then v and $\phi(v)$ are in the same color class of $\overline{G}^{bc}$ for any vertex v of $\overline{G}^{bc}$. By Lemma 2.7, G has a k-pair partition ϕ for some even integer $k \geq 2$. This is a contradiction to the assumption that $p(G) = 1$. Suppose that $\overline{G}^{bc} = C_6 \cup C_6$. Let B and W be two color classes of $\overline{G}^{bc}$ and so of G. We denote one $C_6 = b_1w_1b_2w_2b_3w_3$ and the other $C_6 = b_1'w_1'b_2'w_2'b_3'w_3'$ where $b_i, b_i' \in B$ and $w_i, w_i' \in W$ for $1 \leq i \leq 3$. Then the involution ϕ satisfying $\phi(b_i) = b_i'$ and $\phi(w_i) = w_i'$ for $1 \leq i \leq 3$ is a pair partition of $\overline{G}^{bc}$ such that v and $\phi(v)$ are in same color class for any vertex v of $\overline{G}^{bc}$. By Lemma 2.7, G has a k-pair partition ϕ for some even integer $k \geq 2$. This is again a contradiction to the assumption that $p(G) = 1$. Therefore, $|V(G)| \neq 12$.

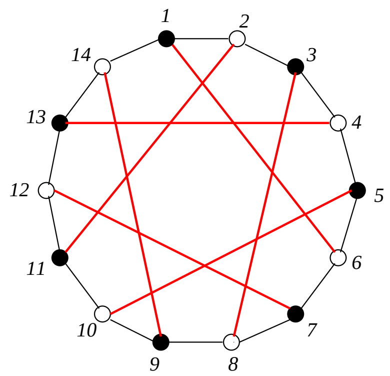

Figure 2: Heawood graph.

Next, we construct a 4-regular bipartite graph of order 14 and pair length 1. Let H be the Heawood graph, see Fig. 2. Then H is a 3-regular spanning subgraph of $K_{7,7}$. Let $G = \overline{H}^{bc}$ be the bipartite complement of H. Then G is an $r(= 4)$-regular bipartite graph of order $2n(= 14)$. Since $r > n/2$, it is easily seen that G is connected. So, both G and H are connected pairable graphs. By Corollary 2.5, each of $p(G)$ and $p(H)$ must be an odd number since G and H are spanning subgraphs of $K_{7,7}$. The Heawood graph H is 2-factor hamiltonian, that is, all its 2-factors are Hamilton cycles [14]. So for each perfect matching M of the Heawood graph H, the set of edges not in M forms a hamiltonian cycle. Every two perfect matchings and every two Hamiltonian cycles of the Heawood graph H can be transformed into each other by an automorphism of H [17]. Without loss of generality, we can consider the perfect matching M of H marked in red. It is easily seen that the involution switching the end vertices of each edge in M is not a 1-pair partition of H. By Corollary 2.9, $p(G) = 1$ where $G = \overline{H}^{bc}$.

Case 2. $r > 4$. We show that $|V(G)| \geq 2(r+2)$ is a sharp bound by constructing an r-regular bipartite graph G of order $2(r + 2)$ and pair length 1 as the bipartite complement of the following graph H.

Subcase 2.1. If $r = 4t$ where $t > 1$, then $2(r+2) = 2(4t+2) = 2(2t-1)+2(2t+3)$. Let H be a disjoint union of two even cycles: $H = C_{2(2t-1)} \cup C_{2(2t+3)}$. Then any pair partition of H maps $C_{2(2t-1)}$ (respectively, $C_{2(2t+3)}$) onto itself.

Subcase 2.2. If $r = 4t + 2$ where $t \geq 1$, then $2(r + 2) = 2(4t + 4) = 2(2t + 1) + 2(2t+3)$. Let H be a disjoint union of two even cycles: $H = C_{2(2t+1)} \cup C_{2(2t+3)}$. Then any pair partition of H maps $C_{2(2t+1)}$ (respectively, $C_{2(2t+3)}$) onto itself.

Since each pair partition of H preserves a cycle at least as big as C_6, there is always a pair of mates that is not adjacent in H.

Let $G = \overline{H}^{bc}$ be the bipartite complement of H in $K_{r+2,r+2}$. Then G is an r-regular bipartite graph. It is easy to check that G is connected from its specific structure, since $\overline{G}^{bc} = H$ is a disjoint union of two even cycles and $|V(G)| = 2(r+2)$ with $r \geq 6$.

It is clear that G is pairable by Lemma 2.7. Additionally, by Lemma 2.7, any pair partition of G is a pair partition of H. By the construction, any pair partition of H has the property that any pair of mates must be from distinct color classes of H, and at least one pair of them cannot be adjacent in H. It follows that any pair partition of G has a pair of mates adjacent in G. Hence, $p(G) = 1$. □

Lemma 3.3 *Let r be a positive integer. Then $\mathbb{B}(r, 2)$ is nonempty if and only if $r > 1$.*

(i) If $r \equiv 0(mod\ 2)$, then $\min_{G\in\mathbb{B}(r,2)}\{|V(G)|\} = 2r$.

(ii) If $r \equiv 1(mod\ 2)$, then $\min_{G\in\mathbb{B}(r,2)}\{|V(G)|\} = 2(r + 3)$.

Proof. Let G be an arbitrary graph in $\mathbb{B}(r, 2)$. Then $r > 1$ since any connected 1-regular graph is an edge of pair length 1. By Corollary 2.5, G is a spanning subgraph of $K_{n,n}$ where $n \geq r$. Moreover, n is an even integer since $p(G) = 2$ is even.

(i) If $r \equiv 0(\text{mod } 2)$, then $|V(G)| = 2n \geq 2r$ is a sharp bound since $p(K_{r,r}) = 2$ when r is even.

(ii) If $r \equiv 1(\text{mod } 2)$, then we claim that $n \geq r + 3$ as follows. First, $n \neq r, r + 2$ since r is odd and n is even. Secondly, $n \neq r + 1$ since any r-regular spanning subgraph of $K_{r+1,r+1}$ has pair length 3, but $p(G) = 2$. Hence, $n \geq r + 3$. We show that $|V(G)| = 2n \geq 2(r + 3)$ is sharp by constructing a desired bipartite graph of order $2(r + 3)$ and pair length 2 as follows.

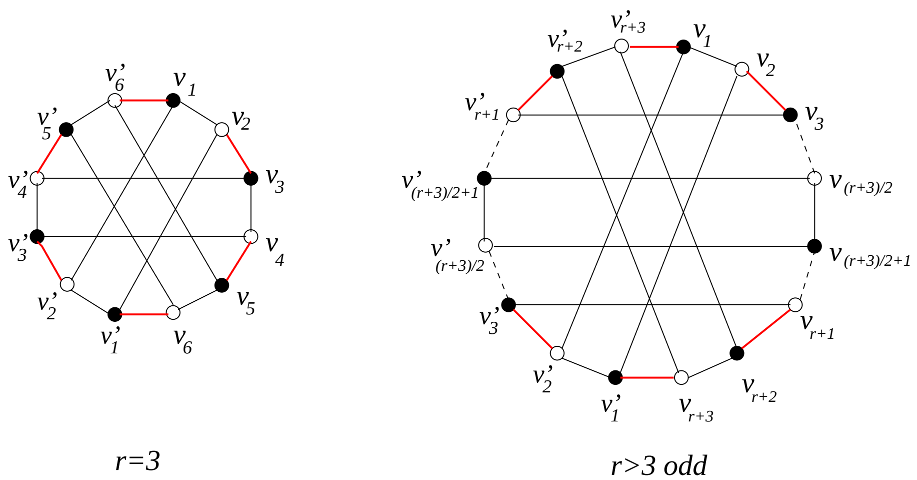

Figure 3: A 3-regular spanning subgraph H of $K_{r+3,r+3}$ when $r > 1$ is odd.

Let H be the 3-regular bipartite graph of order $2(r + 3)$ shown in Fig. 3. Let $G = \overline{H}^{bc}$ be the bipartite complement of H in $K_{r+3,r+3}$. Then G is an r-regular bipartite graph. Again, it is easy to check the connectivity of G by the construction. By Lemma 2.7, any partition of G is a pair partition of H and vice versa since both of them are connected. Let ϕ be an arbitrary pair partition of H. We show that ϕ is a k-pair partition of G for some integer k at most 2. By Theorem 2.4, there are two possible cases.

Case 1. $x, \phi(x)$ are from different color classes of H for each vertex x of H. Note that H has exactly six edges (marked in red) that are not contained in any 4-cycles of H. Any pair partition of H must map these six red edges among themselves as an automorphism of H. Moreover, any automorphism of H cannot switch the two end vertices of each edge for all six of these edges. It follows that there exists a vertex x_0 such that $x_0, \phi(x_0)$ are not adjacent in H, and so $x_0, \phi(x_0)$ are adjacent in G. Hence, ϕ is a 1-pair partition of G.

Case 2. $x, \phi(x)$ are in the same color class of H for each vertex x of H. Then $x, \phi(x)$ have a common neighbor in G as follows. If $r = 3$, it is easy to check that $x, \phi(x)$ have a common neighbor in G; If $r > 3$, then $r > \frac{n}{2} = \frac{r+3}{2}$, and so any two vertices in the same color class of G has a common neighbor in G. Hence, ϕ is a 2-pair partition of G. Existence of such a ϕ can be seen by the example, shown in Fig. 3, whose restriction to the boundary cycle C of H is the antipodal automorphism of C.

Therefore, $p(G) = 2$. □

Lemma 3.4 *Let r be a positive integer. Then $\mathbb{B}(r, 3)$ is nonempty if and only if $r > 1$. Moreover,* $\min_{G\in\mathbb{B}(r,3)} \{|V(G)|\} = 2(r + 1)$.

Proof. Let G be an arbitrary graph in $\mathbb{B}(r, 3)$. Then $r > 1$. By Corollary 2.5, G is a spanning subgraph of $K_{n,n}$ where $n \geq r$. It is clear that $n \neq r$ since the only r-regular spanning subgraph of $K_{r,r}$ is itself which has pair length 1 or 2 based on the parity of r. Then $n \geq r + 1$ and $|V(G)| \geq 2(r + 1)$. This bound is sharp since any bipartite graph obtained from $K_{r+1,r+1}$ by removing a perfect matching is a connected r-regular bipartite graph of order $2(r+1)$ and pair length 3. Therefore, $\min_{G\in\mathbb{B}(r,3)} \{|V(G)|\} = 2(r + 1)$. □

3.2 $\mathbb{B}(r, k)$ where $k > 3$ and at least one of k, r is even

Proposition 3.5 *For any integers $k > 3$ and $r > 1$, $\min_{G\in\mathbb{B}(r,k)} \{|V(G)|\} \geq kr$. In particular, if both k and r are odd, then $\min_{G\in\mathbb{B}(r,k)} \{|V(G)|\} \geq kr + 1$.*

Proof. If $r = 2$, then it holds trivially since the only graph in $\mathbb{B}(2, k)$ is the $2k$-cycle. Assume that $r > 2$ and G is a graph in $\mathbb{B}(r, k)$ for $k > 3$. By Theorem 2.3, G contains a strongly induced $2k$-cycle C. Since G is r-regular, each vertex of C is adjacent with $r - 2$ vertices of $G - C$. Hence, the number of edges between C and $G-C$ is $2k(r-2)$. On the other hand, each vertex of $G-C$ can be adjacent with at most two vertices of C since C is a strongly induced $2k$-cycle where $k > 3$ and G is bipartite. So the number of edges between C and $G - C$ is at most $2(|V(G)| - 2k)$. Then $2k(r - 2) \leq 2(|V(G)| - 2k)$. It follows that $|V(G)| \geq kr$. In particular, if both r and k are odd, then $\min_{G\in\mathbb{B}(r,k)} \{|V(G)|\} \geq kr + 1$ since G has even order. □

Lemma 3.6 *Let $k > 3$ and $r > 1$ be integers. If r is even, then $\min_{G\in\mathbb{B}(r,k)} \{|V(G)|\} = kr$.*

Proof. By Proposition 3.5, we only need to show that $\min_{G\in\mathbb{B}(r,k)}\{|V(G)|\} \geq kr$ is a sharp bound. Construct a desired bipartite graph of order kr as follows. Start with

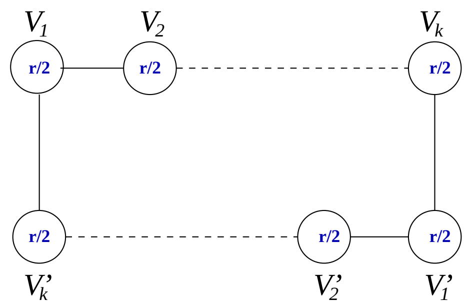

Figure 4: A graph $G \in \mathbb{B}(r,k)$ where r is even and $k > 3$.

a $2k$-cycle $v_1v_2\cdots v_kv_1'v_2'\cdots v_k'$. Replace each vertex v_i (respectively, v_i') of C by an independent set V_i (respectively, V_i') of cardinality $\frac{r}{2}$. Then, for any two adjacent vertices of C, we add all possible edges between two corresponding independent sets. Thus, we obtain a connected r-regular bipartite graph G of order kr. See Fig. 4. It is clear that G has pair length k. $\square$

Note on constructions in Lemma 3.6 and beyond: The construction in Lemma 3.6 is a graph blow-up. The constructions in subsequent statements are modified blow-ups of graphs. For instance, see the blow-up lemma at `http://mathworld.wolfram.com/Blow-UpLemma.html` .

Lemma 3.7 *Let $k > 3$ and $r > 1$ be integers. Assume that r is odd and k is even.*
(i) If $k \equiv 0(mod\ 4)$, then $\min_{G\in\mathbb{B}(r,k)}\{|V(G)|\} = kr$.
(ii) If $k \equiv 2(mod\ 4)$, then $\min_{G\in\mathbb{B}(r,k)}\{|V(G)|\} = kr + 2$.

Proof. By Proposition 3.5, $\min_{G\in\mathbb{B}(r,k)}\{|V(G)|\} \geq kr$. By Corollary 2.5, $|V(G)| = 2n$ where n is even since k is even. Hence, $|V(G)| = 2n \equiv 0(\text{mod } 4)$.

Case 1. $k \equiv 0(\text{mod } 4)$. Then $kr \equiv 0(\text{ mod } 4)$. We show that $\min_{G\in\mathbb{B}(r,k)}\{|V(G)|\} \geq kr$ is a sharp bound by constructing a desired bipartite graph of order kr. See Fig. 5. Start with a $2k$-cycle $C = v_1v_2\cdots v_kv_1'v_2'\cdots v_k'$ where $k = 4m$ for some positive integer m. Replace each vertex v_i (respectively, v_i') of C by an independent set V_i (respectively, V_i') such that

$$|V_i| = |V_i'| = \begin{cases} \frac{r-1}{2}, & \text{if } i = 4j-3, 4j \text{ for } 1 \leq j \leq m, \\ \frac{r+1}{2}, & \text{otherwise.} \end{cases}$$

Then, for any two adjacent vertices of C, we add all possible edges between two corresponding independent sets. Thus, we obtain a connected r-regular bipartite graph G of order kr and pair length k.

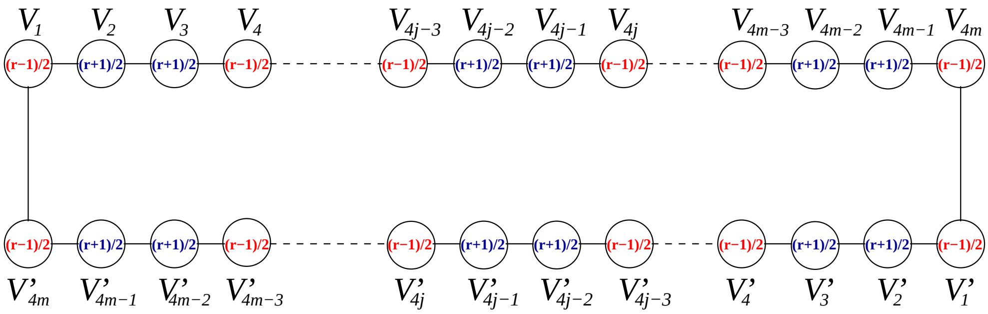

Figure 5: A graph G in $\mathbb{B}(r,k)$ where r is odd and $k = 4m$.

Case 2. $k \equiv 2 \pmod 4$. Then $kr \equiv 2 \pmod 4$ since r is odd. It follows that $\min\limits_{G\in\mathbb{B}(r,k)} \{|V(G)|\} \geq kr + 2$ since $|V(G)| \equiv 0 \pmod 4$. We show that this bound is sharp by constructing a desired bipartite graph of order $kr+2$. See Fig. 6. Start with

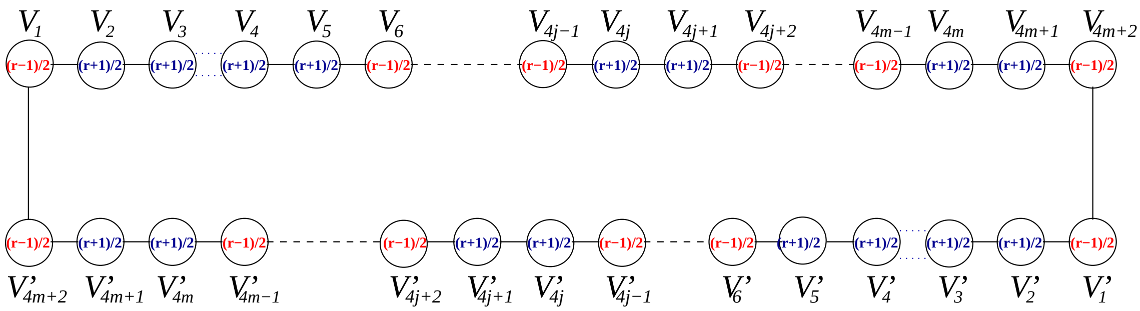

Figure 6: A graph G in $\mathbb{B}(r,k)$ where r is odd and $k = 4m + 2$.
The dotted double lines between V_3 and V_4 (respectively, V'_3 and V'_4) denote all possible edges between the two vertex subsets but one perfect matching.

a $2k$-cycle $C = v_1v_2\cdots v_kv'_1v'_2\cdots v'_k$ where $k = 4m + 2$ for some positive integer m. Replace each vertex v_i (respectively, v'_i) of C by an independent set V_i (respectively, V'_i) such that

$$|V_i| = |V'_i| = \begin{cases} \frac{r-1}{2}, & \text{if } i = 1, 6 \text{ or } i = 4j-1, 4j+2 \text{ for } 2 \leq j \leq m, \\ \frac{r+1}{2}, & \text{otherwise.} \end{cases}$$

Then for any two adjacent vertices of C, add all possible edges between two corresponding independent sets. Finally, remove a perfect matching between V_3 and V_4 (respectively, V'_3 and V'_4). Then we obtain a connected r-regular bipartite graph G of order $kr + 2$ and pair length k. □

3.3 On the remaining case

When both $k \geq 5$ and $r \geq 3$ are odd numbers, it becomes much more challenging to determine $\min\limits_{G\in\mathbb{B}(r,k)} \{|V(G)|\}$. We have provided a lower bound $\min\limits_{G\in\mathbb{B}(r,k)} \{|V(G)|\} \geq$

$kr+1$ in Proposition 3.5. Here we give an upper bound as follows.

Proposition 3.8 For odd numbers $k \geq 5$ and $r \geq 3$, $\min_{G \in \mathbb{B}(r,k)} \{|V(G)|\} \leq kr + (k-4)(r-2)$.

Proof. A desired graph attaining the bound can be constructed as follows.

Start with a $2k$-cycle $C = v_1 v_2 \cdots v_k v_1' v_2' \cdots v_k'$. Replace each vertex v_i (respectively, v_i') of C by an independent set I_i (respectively, I_i') such that

$$|I_i| = |I_i'| = \begin{cases} \frac{r+1}{2}, & \text{if } i = 2, 3, \\ \frac{r-1}{2}, & \text{otherwise.} \end{cases}$$

Moreover, if $|I_i| = \frac{r+1}{2}$, then $I_i = \{x_i\} \cup J_i$ where x_i is a single vertex and J_i is an independent set of size $\frac{r-1}{2}$, and if $|I_i'| = \frac{r+1}{2}$, then $I_i' = \{x_i'\} \cup J_i'$ where x_i' is a single vertex and J_i' is an independent set of size $\frac{r-1}{2}$.

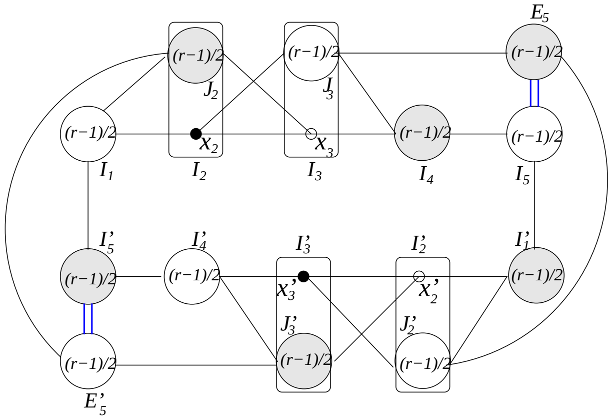

Figure 7: A graph G in $\mathbb{B}(r,5)$ where $r \geq 3$ is odd.
The double lines between I_5 and E_5 (respectively, I_5' and E_5') denote a perfect matching between the two vertex subsets.

Then for any two adjacent vertices of C, add all possible edges between two corresponding independent sets. Next, remove all edges between J_2 and J_3 (respectively, J_2' and J_3'). Finally, for $5 \leq i \leq k$, add an independent set E_i (respectively, E_i') and a perfect matching between E_i and I_i (respectively, E_i' and I_i'). Add all possible edges joining E_i and E_{i+1} and those joining E_i' and E_{i+1}' for $i = 5, \ldots, k-1$. Add all possible edges between E_5 and J_3 (respectively, E_5' and J_3'), and all possible edges between E_k and J_2' (respectively, E_k' and J_2). Then we obtain a connected r-regular bipartite graph G that has order $kr + (k-4)(r-2)$ and pair length k. □

Note. By Lemma 3.4, we can see that if $k = 3$, the upper bound given in Proposition 3.8 equals exactly the desired minimum order of a graph in $\mathbb{B}(r,3)$ for any odd number $r > 1$. We conjecture that the same also holds for the special case when $r = 3$.

Conjecture:

For any odd number $k > 1$, the minimum order of a connected cubic bipartite graph of pair length k is $4k - 4$.

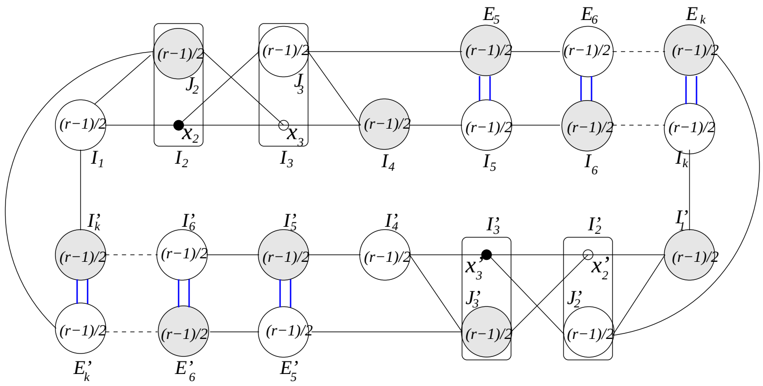

Figure 8: A graph G in $\mathbb{B}(r,k)$ where both $r \geq 3$ and $k > 5$ are odd. The double lines between I_i and E_i (respectively, I'_i and E'_i) denote a perfect matching between the two vertex subsets.

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