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Extreme tournaments with given primitive exponents

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Abstract

Let $e(T_n)$ be the primitive exponent of a primitive tournament T_n of order n. In this paper, we obtain the following results.

1. Let T_n be a regular or almost regular tournament of order $n \geq 7$; then $e(T_n) = 3$.
2. Let $k \in \{n, n+1, n+2\}$. We give the sufficient and necessary conditions for T_n such that $e(T_n) = k$, and obtain all T_n's such that $e(T_n) = k$.

1 Introduction

A tournament matrix of order n is a $(0, 1)$ matrix M of order n such that $M + M^t = J_n - I_n$, where J_n is the matrix of all 1's with order n, I_n the identity matrix of order

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n and M^t the transpose of M. Let $T_n = (V, E)$ be a tournament of order n. Then the adjacency matrix of T_n is a *tournament matrix* of order n. Conversely, the digraph whose adjacency matrix is a tournament matrix must be a tournament. Now let M denote the tournament matrix of order n and T_n the corresponding tournament. For $T_n = (V, E)$, the score of node $v \in V$ is the number of nodes dominated by v and is denoted by $s(v)$. If n is even and each node of T_n has score $\frac{n}{2}$ or $\frac{n-2}{2}$, then T_n is called *almost regular*. If n is odd and each node of T_n has score $\frac{n-1}{2}$, then T_n is called *regular*. The *diameter* of a strongly connected tournament T_n is the least integer d such that for every ordered pair of nodes v and u of T_n, there exists a nontrivial path of length at most d from v to u.

Let $D = (V, E)$ be a digraph. If there exists a positive integer k such that there exists a walk of length k from v to u for every ordered pair of nodes v and u of V, then D is called *primitive*, and the least such integer k is called the *primitive exponent* of D, denoted by $e(D)$. The conditions that a tournament is primitive, the bounds of primitive exponent, and the primitive exponent set, have been obtained in [1] or [2] as follows.

Theorem A *Let T_n be the tournament of order n.*

(i) T_n is primitive if and only if $n \geq 4$ and T_n is strongly connected.

(ii) If $n \geq 5$ and T_n is primitive, then $d(T_n) \leq e(T_n) \leq d(T_n) + 3$, where $d(T_n)$ denotes the diameter of T_n.

(iii) Suppose that $n \geq 6$, then the primitive exponent set of primitive tournaments of order n is $\{3, 4, \ldots, n+1, n+2\}$.

For the given primitive exponent e, it is very difficult to find all primitive tournaments T_n of order n such that $e(T_n) = e$. This problem is equivalent to finding all solutions of the Boolean matrix equation $M^e = J_n$. It is called the MS problem in [3]. In this paper, we obtain all solutions for $e = n, n+1, n+2$ and partial solutions for $e = 3$.

2 The results and proof

Lemma 1 *Let $T_n = (V, E)$ be a tournament of order $n \geq 7$ in which $V = \{v_1, v_2, \cdots, v_n\}$. If each score $s(v_i)$ $(i = 1, 2, \cdots, n)$ satisfies $\frac{n-1}{2} \leq s(v_i) \leq \frac{n}{2}$, then for every ordered pair of nodes v and u of T_n, there exists a path of length* 3 *from v to u.*

Proof. From a result of [5], T_n is strongly connected. Hence each vertex of T_n is contained in a cycle of length 3 (see [1]).

Now let v_i and v_j be two distinct vertices of T_n. We prove that there exist paths of length 3 from v_i to v_j. Let $\#S$ denote the cardinality of set S, $N(v_i) = \{u \mid \overrightarrow{v_i u} \in E, u \in V\}$ and $\tilde{N}(v_i) = \{u \mid \overrightarrow{u v_i} \in E, u \in V\}$.

Case 1 Assume $\overrightarrow{v_i v_j} \in E$. Hence we have

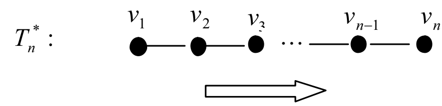


Figure 1.

$$\#(N(v_i) - v_j) \geq \frac{n-3}{2} \geq 2 (n \geq 7), \quad \#\tilde{N}(v_i) = n - 1 - s(v_i) \geq \frac{n-2}{2}.$$

If there are two distinct vertices v and u in $N(v_i) - v_j$ which dominate vertex v_j, without loss of generality, assume $\overrightarrow{vu} \in E$. Then $v_i v u v_j$ is a path of length 3 from v_i to v_j.

If there is at most one vertex of $N(v_i) - v_j$ which dominates v_j, then v_j dominates at least $\#(N(v_i) - v_j) - 1 \geq \frac{n-5}{2}$ vertices of $N(v_i) - v_j$. Thus, at most two vertices of $\tilde{N}(v_i)$ are dominated by v_j, so v_j is dominated by at least $\#\tilde{N}(v_i) - 2 \geq \frac{n-6}{2} > 0$ vertices of $\tilde{N}(v_i)$; let u be such a vertex of $\tilde{N}(v_i)$. If $N(v_i) - v_j \subseteq N(u)$, then

$$s(u) \geq \#(N(v_i) - v_j)) + 2 \geq \frac{n+1}{2} > \frac{n}{2},$$

a contradiction. Therefore u is dominated by at least one vertex of $N(v_i) - v_j$; denote such a vertex by v, so $v_i v u v_j$ is a path of length 3 from v_i to v_j.

Case 2 This case is $\overrightarrow{v_i v_j} \notin E$.

Let $N(v_i)$ replace $N(v_i) - v_j$ in Case 1. The other discussions are analogous to Case 1. We thus have completed the proof. □

Notice that $3 \leq d(T_n) \leq n - 1$ if T_n is a strongly connected tournament of order n. Hence from Lemma 1 and Theorem A, we obtain the following result.

Theorem 2 *Let T_n be a regular or almost regular tournament of order $n \geq 7$. Then T_n is primitive and $e(T_n) = 3$.*

According to the appendix of tournaments of order k $(3 \leq k \leq 6)$ in [1], we easily find that Theorem 2 does not hold for $n = 5, 6$. From Lemma 1 and this appendix, we also obtain the following result.

Corollary 3 *Suppose that T_n is a regular or almost regular tournament of order $n \geq 3$. Then $d(T_n) = 3$.*

Lemma 4 *Let T_n be a strongly connected tournament of order $n \geq 5$. Then $d(T_n) = n - 1$, if and only if $T_n \cong T_n^*$, where the sign "$\cong$" denotes isomorphism. T_n^* is a tournament of order n shown in Fig. 1, where not all arcs are included in the drawing; the sign "$\Rightarrow$" means that an arc not drawn is oriented from the left node to the right node.*

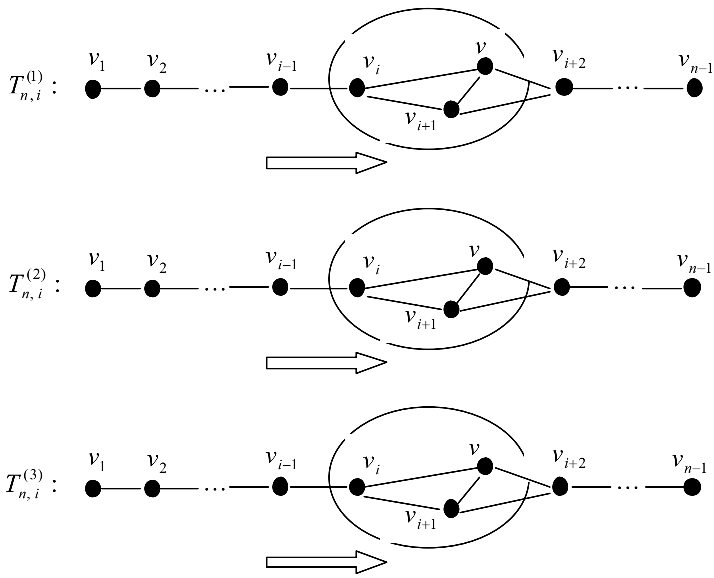


Figure 2.

Proof. Clearly, the diameter of T_n^* is equal to $n-1$. Hence the sufficiency of the lemma holds.

Now we prove the necessity of the lemma. Let $T_n = (V, E)$ be a strongly connected tournament of order n and diameter $n-1$. By the definition of diameter, there exist two distinct nodes of V, say v_1 and v_n, such that the shortest path from v_n to v_1 has length $n-1$. Let $P(v_n, v_1) \quad = v_n v_{n-1} \cdots v_1$ be such a shortest path. Clearly, all vertices of T_n are contained in the path. If there are positive integers i, j $(i+2 \le j)$ such that $\overrightarrow{v_j v_i} \in E$, then $v_n v_{n-1} \cdots v_j v_i v_{i-1} \cdots v_1$ is a path of length $n-(j-i) \le n-2$ from v_n to v_1, a contradiction to the length $n-1$ of the shortest path $P(v_n, v_1)$. Hence for arbitrary i and j with $1 \le i \le n-2$ and $i+2 \le j$, we always have $\overrightarrow{v_i v_j} \in E$. Therefor we obtain $T_n \cong T_n^*$. This completes the proof. □

It was pointed out in [1] that $e(T_n^*) = n+2$ if $n \ge 5$. The following result indicates that T_n^* is the unique tournament with order $n \ge 5$ and primitive exponent $n+2$.

Theorem 5. Let T_n be a strongly connected tournament of order $n \ge 5$. Then $e(T_n) = n+2$, if and only if $T_n \cong T_n^*$.

Proof. If $e(T_n) = n+2$, then $d(T_n) \ge n-1$ by Theorem A. Thus we have $d(T_n) = n-1$. By Lemma 4, we obtain $T_n \cong T_n^*$. If $T_n \cong T_n^*$, then $e(T_n) = n+2$ by $e(T_n^*) = n+2$ (see [1]). The proof is complete. □

Let $T_{n,i}^{(1)}$ $(1 \le i \le n-3)$, $T_{n,i}^{(2)}$ $(1 \le i \le n-2)$ and $T_{n,i}^{(3)}$ $(2 \le i \le n-3)$ be the tournaments of order n shown in Fig. 2.

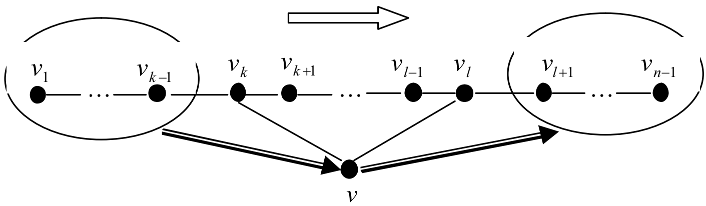


Figure 3.

Lemma 6 *Let T_n be a strongly connected tournament of order $n \geq 6$. Then $d(T_n) = n-2$ if and only if $T_n \cong T_{n,i}^{(k)}$ $(k = 1, 2, 3)$.*

Proof. It is easy to find $d(T_{n,i}^{(k)}) = n-2$ $(k = 1, 2, 3)$, so the sufficiency of the lemma holds.

Now we prove necessity. Let $T_n = (V, E)$ be a strongly connected tournament with order n and diameter $n-2$. By the definition of diameter, there exist two distinct vertices of V, say v_1 and v_{n-1}, such that the shortest path from v_{n-1} to v_1 has length $n-2$; let $P(v_{n-1}, v_1) = v_{n-1}v_{n-2}\cdots v_1$ be such a shortest path. So for arbitrary i, j $(1 \leq i \leq n-3, i+2 \leq j \leq n-1)$, we always have $\overrightarrow{v_iv_j} \in E$. Clearly, there is only one node not contained in $P(v_{n-1}, v_1)$; denote it by v. Since T_n is a strongly connected tournament, there are two distinct vertices $v_i, v_j \in V$ such that $\overrightarrow{vv_i}, \overrightarrow{v_jv} \in E$. Let

$$k = \min\{t : \overrightarrow{vv_t} \in E, v_t \in V\} \geq 1, \quad l = \max\{t : \overrightarrow{v_tv} \in E, v_t \in V\} \leq n-1.$$

Suppose that $k < l$. Then the structure of T_{n+1} is illustrated in Fig. 3, where the arcs not drawn between v and v_j $(k+1 \leq j \leq l-1)$ may be oriented arbitrarily, and the sign $W \Rightarrow Q$ means that each vertex of W dominates each of Q. If $l \geq k+3$, then

$$v_{n-1}v_{n-2}\cdots v_{l+1}v_lvv_kv_{k-1}\cdots v_1$$

is a path of length $n-(l-k) \leq n-3$ from v_{n-1} to v_1, a contradiction to the length $n-2$ of the shortest path $P(v_{n-1}, v_1)$. Hence we have $k+1 \leq l \leq k+2$. Notice that $l \leq n-1$. We have $1 \leq k \leq n-3$ if $l = k+2$, and thus we always have $T_n \cong T_{n,k}^{(1)}$ for arbitrary orientation of the arc between v and v_{k+1}; we have $1 \leq k \leq n-2$ if $l = k+1$, and thus we obtain $T_n \cong T_{n,k}^{(2)}$.

Suppose that $k > l$. According to the definitions of k and l, we have $k = l+1$, $\overrightarrow{v_iv} \in E$ and $\overrightarrow{vv_j} \in E$ for $1 \leq i \leq l$, $k \leq j \leq n$ (The corresponding drawing of tournament is obtained by only exchanging the locations of v_k and v_l in Fig. 3.) If $l = 1$ or $l = n-2$, then $d(T_n) = n-1$, a contradiction to $d(T_n) = n-2$. Therefore we have $2 \leq l \leq n-3$. So $T_n \cong T_{n,k}^{(3)}$ is obtained. The proof is completed. □

It was pointed out in [1] that $e(T_{n,n-3}^{(3)}) = n+1$ if $n \geq 6$. Indeed, we have the better results.

Theorem 7 *Let T_n be a strongly connected tournament of order $n \geq 6$. Then $e(T_n) = n+1$ if and only if $T_n \cong T_{n,i}^{(k)}$ $(k = 1, 2, 3)$.*

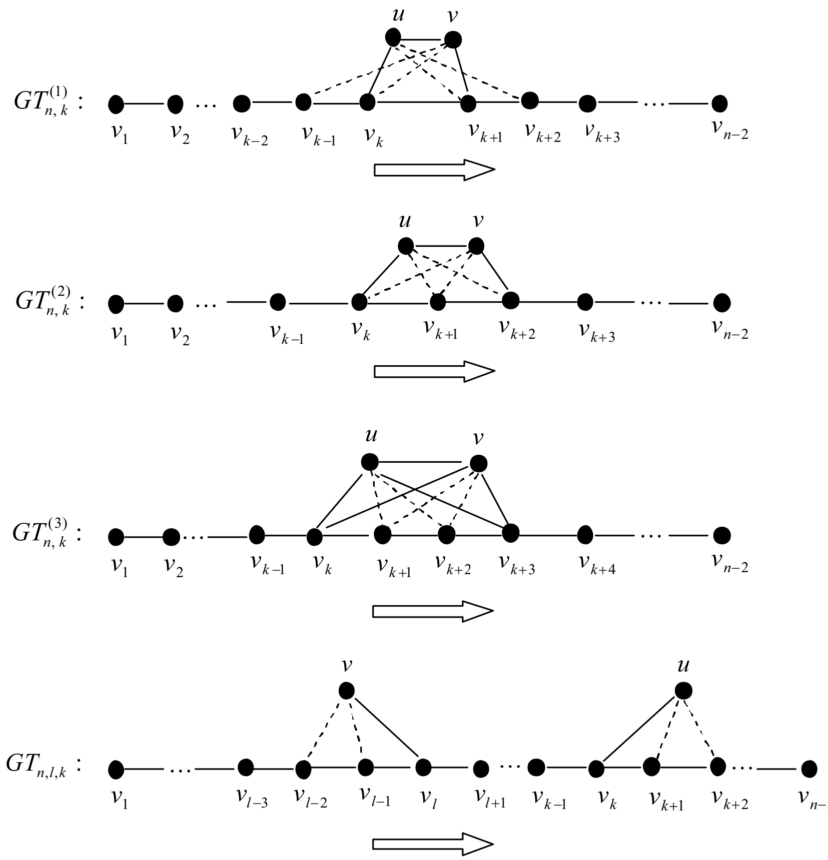


Figure 4.

Proof. For $T^{(1)}_{n,i}$, it is easy to find that there do not exist walks of lengths n, $n-1$ and $n-2$ from v_n to v_1, v_2 and v_3, respectively. Therefore we obtain $e(T^{(1)}_{n,i}) \neq n, n-1, n-2$. By Theorem A, we have $e(T^{(1)}_{n,i}) = n+1$. By the same discussion, we have $e(T^{(2)}_{n,i}) = n+1$ and $e(T^{(3)}_{n,i}) = n+1$. Thus the sufficiency of the theorem holds. If $e(T_n) = n+1$, then $d(T_n) \geq n-2$ by Theorem A; again by Lemma 4 and Theorem 5, we have $d(T_n) = n-2$; by Lemma 6, we obtain $T_n \cong T^{(k)}_{n,i}$ $(k = 1, 2, 3)$. The proof is completed. □

Let $GT^{(1)}_{n,k}, GT^{(2)}_{n,k}, GT^{(3)}_{n,k}$ and $GT_{n,l,k}$ be the tournaments of order $n \geq 7$ shown in Fig. 4, where $GT^{(1)}_{n,k}$ satisfies $2 \leq k \leq n-4$, or $k = 1$ and either $\overrightarrow{v_{k+1}u} \in E$ or $\overrightarrow{v_{k+2}u} \in E$, or $k = n-3$ and either $\overrightarrow{vv_{k-1}} \in E$ or $\overrightarrow{vv_k} \in E$; $GT^{(2)}_{n,k}$ satisfies $1 \leq k \leq n-4$; $GT^{(3)}_{n,k}$ satisfies $1 \leq k \leq n-5$; $GT_{n,l,k}$ satisfies $2 \leq l \leq k \leq n-3$. The sign "$x - - - y$" is understood to mean that the orientation of the arc between

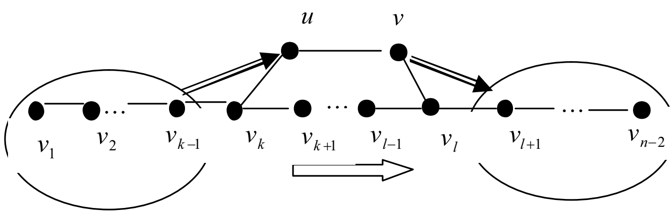


Figure 5.

x and y is arbitrary.

Lemma 8 *Let T_n be a strongly connected tournament of order $n \geq 7$. Then $d(T_n) = n-3$ if and only if $T_n \cong GT_{n,k}^{(m)}$ $(1 \leq m \leq 3)$ or $T_n \cong GT_{n,l,k}$.*

Proof. It is easy to find that $d(GT_{n,k}^{(m)}) = n-3$ $(m = 1,2,3)$ and $d(GT_{n,l,k}) = n-3$. Thus the sufficiency of the lemma holds. Now we prove the necessity of the lemma. Let $T_n = (V, E)$ be a strongly connected tournament with order $n \geq 7$ and diameter $d(T_n) = n-3$. By the definition of diameter, there exist two distinct vertices $v_1, v_{n-2} \in V$ such that the shortest path from v_{n-2} to v_1 has length $n-3$; let $P(v_{n-2}, v_1) = v_{n-2}v_{n-3}\cdots v_1$ be such a shortest path. So we always have $\overrightarrow{v_iv_j} \in E$ for i, j $(1 \leq i \leq n-3, i+2 \leq j \leq n-1)$. Clearly, there are only two vertices not contained in $P(v_{n-2}, v_1)$; denote them by v and u, without loss of generality, let $\overrightarrow{vu} \in E$. Since T_n is strongly connected, there are two vertices $v_i, v_j \in V$ such that $\overrightarrow{v_iv}, \overrightarrow{uv_j} \in E$. Let $k = \min\{t : \overrightarrow{uv_t} \in E, 1 \leq t \leq n-2\}$, and $l = \max\{t : \overrightarrow{v_tv} \in E, 1 \leq t \leq n-2\}$.

Case 1 Assume $l > k$.

According to the definitions of k and l, T_n is illustrated in Fig. 5, where all arcs between v and v_i $(1 \leq i \leq l-1)$, u and v_j $(k+1 \leq j \leq n-2)$ are not pictured. If $l \geq k+4$, then $v_{n-2}v_{n-3}\cdots v_lvuv_kv_{k-1}\cdots v_1$ is a path of length $n-(l-k) \leq n-4$ from v_{n-2} to v_1, and this is a contradiction to the length $n-3$ of the shortest path $P(v_{n-2}, v_1)$. Hence $k+1 \leq l \leq k+3$.

Suppose that $l = k+1$. If there exists a node v_i $(1 \leq i \leq k-2)$ such that $\overrightarrow{vv_i} \in E$, then $v_{n-2}v_{n-3}\cdots v_{k+1}vv_iv_{i-1}\cdots v_1$ is a path of length $n-2-(k-i) \leq n-4$ from v_{n-2} to v_1; this is a contradiction. Thus for each i $(1 \leq i \leq k-2)$, we always have $\overrightarrow{v_iv} \in E$. In the same way, we always have $\overrightarrow{uv_j} \in E$ for each j $(k+3 \leq j \leq n-2)$. If $k = 1$ and $\overrightarrow{uv_{k+1}}, \overrightarrow{uv_{k+2}} \in E$, or $k = n-3$ and $\overrightarrow{v_{k-1}v}, \overrightarrow{v_kv} \in E$, then $d(T_n) = n-2$, a contradiction. Hence we have $2 \leq k \leq n-4$, or $k = 1$ and either $\overrightarrow{v_{k+1}u} \in E$ or $\overrightarrow{v_{k+2}u} \in E$, or $k = n-3$ and either $\overrightarrow{vv_{k-1}} \in E$ or $\overrightarrow{vv_k} \in E$. Thus we obtain $T_n \cong GT_{n,k}^{(1)}$.

Suppose that $l = k+2$. By a similar discussion to the case $l = k+1$, we have $\overrightarrow{v_iv} \in E$ for each i $(1 \leq i \leq k-1)$, $\overrightarrow{uv_j} \in E$ for each j $(k+3 \leq j \leq n-2)$ and $1 \leq k \leq n-4$. Hence we obtain $T_n \cong GT_{n,k}^{(2)}$.

Suppose that $l = k+3$. By a similar discussion to the case $l = k+1$, we have $\overrightarrow{v_iv} \in E$ for each i $(1 \leq i \leq k)$, $\overrightarrow{uv_j} \in E$ for each j $(k+3 \leq j \leq n-2)$ and $1 \leq k \leq n-5$. Hence we obtain $T_n \cong GT_{n,k}^{(3)}$.

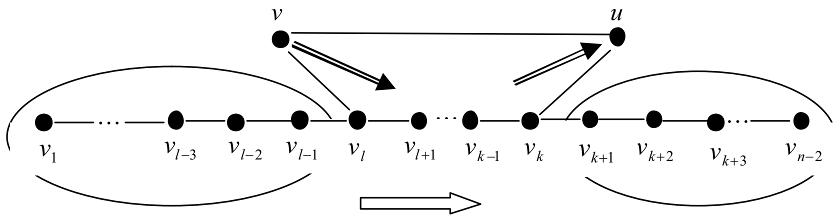


Figure 6.

Case 2 Assume $l \leq k$.

By the definitions of k and l, T_n is illustrated in Fig. 6, where all arcs between u and v_j $(k+1 \leq j \leq n-2)$, v and v_i $(1 \leq i \leq l-1)$ are not pictured. By a similar discussion to $l = k+1$ in case 1, we always have $\overrightarrow{v_i v} \in E$ for each $i(1 \leq i \leq l-3)$ and $\overrightarrow{uv_j} \in E$ for each $j(k+3 \leq j \leq n-2)$. If $l = 1$ or $k = n-2$, then $d(T_n) = n-2$, a contradiction. Thus we have $2 \leq l \leq k \leq n-3$. Thus we obtain $T_n \cong GT_{n,l,k}$. This completes the proof. □

Lemma 9 *Suppose that* $n \geq 8$ *and* $T \in \left\{GT^{(1)}_{n,k}, GT^{(2)}_{n,k}, GT^{(3)}_{n,k}, GT_{n,l,k}\right\}$. *Then* $e(T) = n$ *if and only if* T *are those tournaments of order* n *shown in Fig. 7.*

Proof. Clearly, all tournaments in Fig. 7 are strongly connected. From Theorem A, they are primitive. For the tournament $BT^{(1)}_n$, there are only paths of lengths $n-3$ or $n-2$ from v_{n-2} to v, lengths $n-4$ or $n-3$ from v_{n-2} to v_2 and lengths $n-5$ or $n-4$ from v_{n-2} to v_3. Hence there are no walks of length $n-1, n-2$ and $n-3$ from v_{n-2} to v, v_2 and v_3, respectively. Therefore we have $e(BT^{(1)}_n) \neq n-1, n-2, n-3$. Again by Theorem A, we obtain $e(BT^{(1)}_n) = n$. By a similar discussion to that above, the primitive exponents of the other tournaments in Fig. 7 are n, too. The sufficiency of the lemma holds.

Now we prove the necessity. Let x and y be two vertices of a primitive tournament G and $C(x,k)$ some cycle of length k containing x. The sign $l\exists P(x,y)$ means that there exists some path $P(x,y)$ with length l from x to y. We have the following fact.

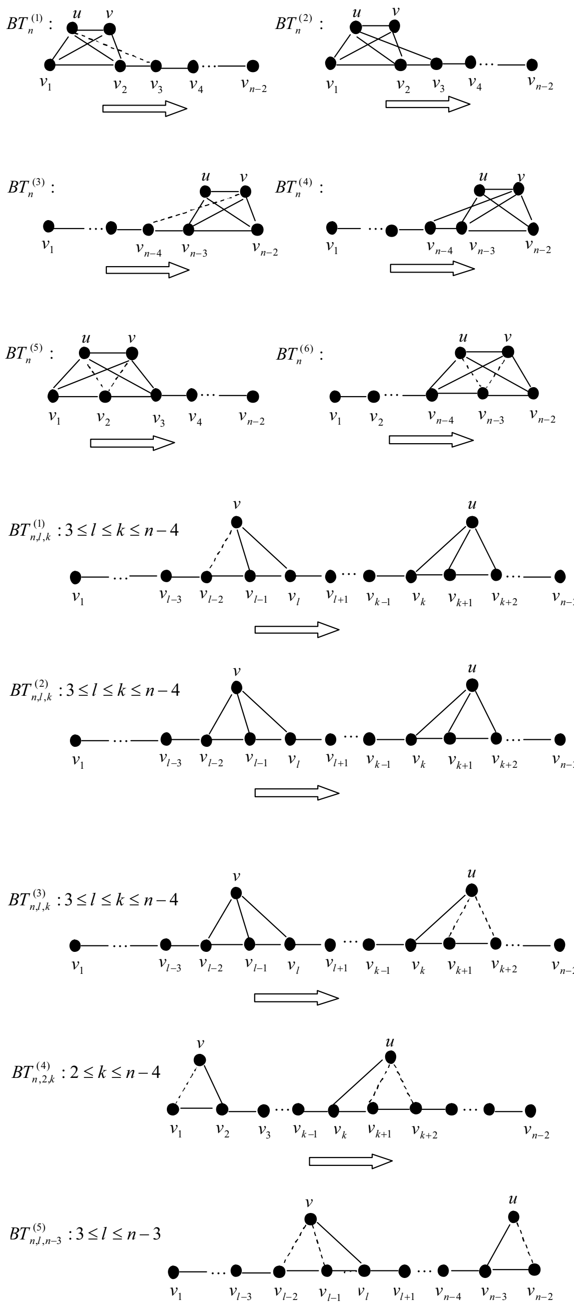
If the integer m satisfies $3 \leq m - l\exists P(x,y) \leq n$, then $P(x,y) + C(y, m - l\exists P(x,y))$ is a walk of length m from x to y. Therefore in order to prove $e(G) \leq m$, we only need prove that there exists a walk of length m from x to y for each pair of vertices x and y such that $l\exists P(x,y) \geq m-3$.

(1) Assume $T = GT^{(1)}_{n,k}$.

Clearly, T always has a path $v_{n-2}v_{n-3}\cdots v_{k+1}vuv_k\cdots v_1$ of length $n-1$ from v_{n-2} to v_1.

Assume $3 \leq k \leq n-5$. Then $l \exists P(x,y) \leq n-4$ always holds if $(x,y) \neq (v_{n-2}, v_1)$. Thus $e(T) \leq n-1$.

Assume $k = 2$. Clearly, $l \exists P(x,y) \leq n-4$ always holds if $(x,y) \neq (v_{n-2}, v_1)$, (v_{n-2}, u). If $\overrightarrow{v_4 u} \in E$, or $\overrightarrow{v_3 u} \in E$, or $\overrightarrow{v_1 v} \in E$, or $\overrightarrow{vv_2} \in E$, then from v_{n-2} to u there are the following paths with lengths $n-5, n-4, n-1$ and $n-1$, respectively.



$BT_n^{(1)}$:
u v
v_1 v_2 v_3 v_4 v_{n-2}
$BT_n^{(2)}$:
u v
v_1 v_2 v_3 v_4 v_{n-2}
$BT_n^{(3)}$:
u v
v_1 v_{n-4} v_{n-3} v_{n-2}
$BT_n^{(4)}$:
u v
v_1 v_{n-4} v_{n-3} v_{n-2}
$BT_n^{(5)}$:
u v
v_1 v_2 v_3 v_4 v_{n-2}
$BT_n^{(6)}$:
u v
v_1 v_2 v_{n-4} v_{n-3} v_{n-2}
$BT_{n,l,k}^{(1)} : 3 \le l \le k \le n-4$
v u
v_1 v_{l-3} v_{l-2} v_{l-1} v_l v_{l+1} v_{k-1} v_k v_{k+1} v_{k+2} v_{n-2}
$BT_{n,l,k}^{(2)} : 3 \le l \le k \le n-4$
v u
v_1 v_{l-3} v_{l-2} v_{l-1} v_l v_{l+1} v_{k-1} v_k v_{k+1} v_{k+2} v_{n-2}
$BT_{n,l,k}^{(3)} : 3 \le l \le k \le n-4$
v u
v_1 v_{l-3} v_{l-2} v_{l-1} v_l v_{l+1} v_{k-1} v_k v_{k+1} v_{k+2} v_{n-2}
$BT_{n,2,k}^{(4)} : 2 \le k \le n-4$
v u
v_1 v_2 v_3 v_{k-1} v_k v_{k+1} v_{k+2} v_{n-2}
$BT_{n,l,n-3}^{(5)} : 3 \le l \le n-3$
v u
v_1 v_{l-3} v_{l-2} v_{l-1} v_l v_{l+1} v_{n-4} v_{n-3} v_{n-2}

Figure 7.

$v_{n-2}v_{n-3}\cdots v_4u,\ v_{n-2}v_{n-3}\cdots v_4v_3u,\ v_{n-2}v_{n-3}\cdots v_4v_3v_2v_1vu,\ v_{n-2}v_{n-3}\cdots v_4v_3vv_2v_1u.$

Thus we assume $\overrightarrow{uv_4}, \overrightarrow{uv_3}, \overrightarrow{vv_1}, \overrightarrow{v_2v} \in E$. But $v_{n-2}v_{n-3}\cdots v_4v_3v_2vv_1u$ is a path of length $n-1$ from v_{n-2} to u. The discussion above indicates that $e(T) \leq n-1$ always holds when $k = 2$. By the similar discussion with $k = 2$, $e(T) \leq n-1$ also always holds if $k = n-4$. Hence we obtain $k \notin \{2, 3, \cdots, n-4,\}$.

Assume $k = 1$. Clearly, $l \exists P(x, y) \leq n-4$ when $(x, y) \neq (v_{n-2}, v_1), (v_{n-2}, u)$, (v_{n-2}, v), (v_{n-3}, u). Firstly, let $\overrightarrow{v_2u} \in E$; then $l\exists P(v_{n-3}, u) = n-4$. If $\overrightarrow{v_1v} \in E$, then there exist paths of length $n-1$ from v_{n-2} to u and v. So we have $e(T) \leq n-1$, a contradiction. Hence we have $\overrightarrow{vv_1} \in E$, i.e., $T \cong BT_n^{(1)}$. Secondly, let $\overrightarrow{uv_2} \in E$. Then $\overrightarrow{v_3u} \in E$ must hold and we easily find $\overrightarrow{vv_1} \in E$. Hence $T \cong BT_n^{(2)}$.

Assume $k = n-3$. By a similar discussion to the case $k = 1$, $T \cong BT_n^{(3)}$ or $T \cong BT_n^{(4)}$ hold.

(2) Assume $T = GT_{n,k}^{(2)}$.

By a similar discussion to case $3 \leq k \leq n-5$ of (1), we have $k = 1, n-4$. Firstly, suppose that $k = 1$. Obviously, $l\exists P(x, y) \leq n-4$ always holds if $(x, y) \neq (v_{n-2}, u), (v_{n-2}, v_1)$ and there always exists a path of length $n-1$ from v_{n-2} to v_1 for arbitrary orientation of the arcs among v_2 and u, v. Hence in order to make $e(T) = n$, T must not have walks of length $n-1$ from v_{n-2} to u. Notice that from v_{n-2} to u there is a path $v_{n-2}\cdots v_3v_2v_1vu$ of length $n-1$ if $\overrightarrow{v_1v} \in E$ and a path $v_{n-2}\cdots v_4v_3u$ of length $n-4$ if $\overrightarrow{v_3u} \in E$. Hence $\overrightarrow{vv_1}, \overrightarrow{uv_3} \in E$. So we have $T \cong BT_n^{(5)}$. Secondly, suppose that $k = n-4$. By a similar discussion to case $k = 1$, we have $T \cong BT_n^{(6)}$.

(3) Assume $T = GT_{n,k}^{(3)}$.

If $(x, y) \neq (v_{n-2}, v_1)$, then $l\exists P(x, y) \leq n-4$. Hence in order to make $e(T) = n$, T must not have walks of length $n-1$ from v_{n-2} to v_1. Since $v_{n-2}\cdots v_{k+3}vuv_{k+2}v_{k+1}\cdots \cdots v_1$ is a path of length $n-1$ from v_{n-2} to v_1 when $\overrightarrow{uv_{k+2}} \in E$, we must have $\overrightarrow{v_{k+2}u} \in E$. Since there is always a path of length $n-1$ from v_{n-2} to v_1 for arbitrary orientation of the arc between u and v_{k+1} when $\overrightarrow{vv_{k+2}} \in E$, we must have $\overrightarrow{v_{k+2}v} \in E$. Since $v_{n-2}\cdots v_{k+3}v_{k+2}vuv_{k+1}\cdots v_1$ is a path of length $n-1$ from v_{n-2} to v_1 when $\overrightarrow{uv_{k+1}} \in E$, we must have $\overrightarrow{v_{k+1}u} \in E$. Since $v_{n-2}\cdots v_{k+3}v_{k+2}v_{k+1}vuv_k\cdots v_1$ is a path of length $n-1$ from v_{n-2} to v_1 when $\overrightarrow{v_{k+1}v} \in E$, we must have $\overrightarrow{vv_{k+1}} \in E$. Clearly, $v_{n-2}\cdots v_{k+3}v_{k+2}vv_{k+1}uv_k\cdots v_1$ is a path of length $n-1$ from v_{n-2} to v_1, too. By the discussion above, we know that there is always a walk of length $n-1$ from v_{n-2} to v_1 for arbitrary orientation of the arc among v, u and v_{k+1}, v_{k+2}. Hence this case cannot happen.

(4) Assume $T = GT_{n,l,k}$.

Obviously, we have $T \cong BT_{n,2,k}^{(4)}$ if $l = 2$ and $T \cong BT_{n,l,n-3}^{(5)}$ if $k = n-3$. Now assume $3 \leq l \leq k \leq n-4$.

(i) Let $\overrightarrow{vv_{l-1}} \in E$. If $(x, y) \neq (v_{n-2}, v_1)$, then $l\exists P(x, y) \leq n-4$. When $\overrightarrow{v_{k+1}u} \in E$ or $\overrightarrow{uv_{k+1}}, \overrightarrow{v_{k+2}u} \in E$, there is a path of length $n-1$ from v_{n-2} to v_1. Hence $e(T) \leq n-1$, a contradiction. So we have $\overrightarrow{uv_{k+1}}, \overrightarrow{uv_{k+2}} \in E$. Therefor we obtain $T \cong BT_{n,l,k}^{(1)}$.

(ii) Let $\overrightarrow{v_{l-1}v} \in E$. By a similar discussion to (i), we have $\overrightarrow{vv_{l-2}}, \overrightarrow{uv_{k+1}}, \overrightarrow{uv_{k+2}} \in E$ or $\overrightarrow{v_{l-2}v} \in E$. So we obtain $T \cong BT^{(2)}_{n,l,k}$ or $T \cong BT^{(3)}_{n,l,k}$. We have completed this proof. □

Theorem 10 *Let T_n be a strongly connected tournament of order $n \geq 8$. Then $e(T_n) = n$, if and only if $T_n \cong BT_n^{(i)}$ $(1 \leq i \leq 6)$ or $T_n \cong BT^{(i)}_{n,l,k}$ $(1 \leq i \leq 5)$.*

Proof. If $e(T_n) = n$, then we have $d(T_n) = n - 3$ from Theorem A, Theorem 5 and Theorem 7. Hence again by Lemma 8 and Lemma 9, we obtain $T_n \cong BT_n^{(i)}$ $(1 \leq i \leq 6)$ or $T_n \cong BT^{(i)}_{n,l,k}$ $(1 \leq i \leq 5)$, i.e., the necessity of the theorem holds. The sufficiency of the theorem is obvious by Lemma 9. This completes the proof. □

Using a more careful discussion similar to Lemma 9, it is easy to obtain all T_7 with $e(T_7) = 7$.

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