Expanding the Class of Polynomial Time Computable Well-Founded Semantics for Hybrid MKNF

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Abstract

The logic of hybrid MKNF (Minimal Knowledge and Negation as Failure) enables tight interwoven reasoning between answer set programming and open-world reasoning systems. The presence of both classical negation and negation as failure poses significant semantic challenges on issues that must be addressed before practical systems can be built. In this work, we improve well-founded reasoning for hybrid MKNF knowledge bases. Unlike traditional answer set programming, the well-founded semantics for hybrid MKNF is intractable. Prior work established a class of knowledge bases with known polynomial-time algorithms to compute their well-founded models and in this paper, we improve upon this work by expanding this class. We provide a new fixpoint operator for computing well-founded models and compare our operator to prior work.

Keywords

Hybrid MKNF Knowledge Bases, Classical Reasoning, Well-Founded Semantics, Fixpoint Operators

1. Introduction

Reasoning with ASP (Answer Set Programming) [1] is limited to the closed-world assumption. Under this assumption, we assume that unprovable facts are false. This is different from the open-world assumption which requires proof of falsity. Many description logics support powerful polynomial reasoning services [2] and hybrid MKNF can be adopted as a framework to enrich these description logics with nonmonotonic, rule-based reasoning. While the logic of hybrid MKNF [3] enables tight integration of open and closed-world reasoning, many semantic questions remain open. Well-founded semantics is one such area of interest. Polynomial-time computable well-founded semantics play a crucial role in the grounding of programs, that is, the instantiation of all variables in a program. More generally, the 3-valued semantics leveraged by well-founded semantics capture reasoning with partial information rather than an entire system which can make systems more scalable. Fixpoint operators are well-suited to express polynomial algorithms that capture well-founded semantics. However, because the well-founded semantics of hybrid MKNF knowledge bases are intractable, we do not hope to capture the entirety of the semantics [4].

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In this work, we present a well-founded operator that can compute the well-founded models for a larger class of knowledge bases than the previous state-of-the-art operators. Prior work on well-founded semantics has diverged in method with non-overlapping advancements and our work unifies these advancements and extends them further. We provide an in-depth analysis of prior operators to compare semantic differences and ultimately show that our new operator subsumes prior work.

Namely, prior operators are limited in the inferences they can make that rely on classically false information. The mixing of negation as failure and classical falsity is strong reason to use hybrid MKNF, however, current fixpoint operators do not make full use of classically false information. Our advancements are general-purpose and can improve inference power on a wide variety of problem domains.

Because a given knowledge base may not have a well-founded model, well-founded operators are sound but not necessarily complete. The fixpoint of a sound operator will always be less precise or as precise as the well-founded model if a well-founded model exists.

Section 2 details necessary preliminaries and establish notation used throughout this work. Section 3 discusses the well-founded semantics of hybrid MKNF knowledge bases. In Section 4, we present a fixpoint characterization of hybrid MKNF knowledge bases that expands the class of knowledge bases with polynomial well-founded semantics. In Section 5, we give a granular breakdown of the differences between our operators and prior operators. Finally, we summarize and conclude in Section 6.

2. Preliminaries: Hybrid MKNF Knowledge Bases

MKNF (Minimal Knowledge and Negation as Failure) is a nonmonotonic logic formulated by Lifschitz [5]. This logic flexibly unifies various nonmonotonic semantics including default and autoepistemic logic. Motik and Rosati [3] construct hybrid MKNF as a subset of the full logic of MKNF. This subset extends the MKNF characterization of stable model semantics initially introduced by Lifshitz [5] with a first-order characterization of an ontology. The resulting logic allows for joint reasoning between a set of nonmonotonic rules and an ontology. In theory, the ontology may be any external monotonic reasoning system that can be encoded as a first-order formula. Ideally, this system should support a polynomial entailment reasoning service.

While there are many hybrid frameworks that combine ASP with an external system, Motik and Rosati list several key properties of hybrid MKNF knowledge bases that make it suitable for embedding description logics [3]. (Faithfulness) The semantics of rules and embedded ontology are preserved. (Tightness) Reasoning is not limited to layering the ontology on top of the rules and vice versa. (Flexibility) A single predicate may be viewed under either the open- or closed-world assumption and (Decidability). Alberti et al. provide practical motivation to use hybrid MKNF by arguing that normative systems require it [6].

Partial model semantics have numerous advantages over their two-valued counterparts. They can serve as the basis for grounding algorithms as well as increase the scalability of a reasoning system by leaving some truth values unspecified. We adopt Knorr et al.'s [7] 3-valued semantics for hybrid MKNF; Under which there are three truth values: $\mathbf{f}$ (false), $\mathbf{u}$ (undefined), and $\mathbf{t}$ (true) that use the ordering $\mathbf{f} < \mathbf{u} < \mathbf{t}$. The min and max functions respect this ordering when applied to sets.

Hybrid MKNF relies on the standard name assumption under which every first-order interpretation in an MKNF interpretation is required to be a Herbrand interpretation with a countably infinite amount of additional constants. We use Δ to denote the set of all these constants. We use $\phi[x \to \alpha]$ to denote the formula obtained by replacing all free occurrences of variable x in ϕ with the term α.

A (3-valued) *MKNF structure* is a triple $(I, \mathcal{M}, \mathcal{N})$ where I is a (two-valued first-order) interpretation and $\mathcal{M} = \langle M, M_1 \rangle$ and $\mathcal{N} = \langle N, N_1 \rangle$ are pairs of sets of first-order interpretations such that $M \supseteq M_1$ and $N \supseteq N_1$. Using ϕ and σ to denote MKNF formulas, the evaluation of an MKNF structure is defined as follows:

$$(I, \mathcal{M}, \mathcal{N})(p(t_1, \ldots, t_n)) := \begin{cases} \mathbf{t} & \text{iff } p(t_1, \ldots, t_n) \text{ is } \boxed{\text{true}} \text{ in } I \\ \mathbf{f} & \text{iff } p(t_1, \ldots, t_n) \text{ is } \boxed{\text{false}} \text{ in } I \end{cases}$$

$$(I, \mathcal{M}, \mathcal{N})(\neg\phi) := \begin{cases} \mathbf{t} & \text{iff } (I, \mathcal{M}, \mathcal{N})(\phi) = \mathbf{f} \\ \mathbf{u} & \text{iff } (I, \mathcal{M}, \mathcal{N})(\phi) = \mathbf{u} \\ \mathbf{f} & \text{iff } (I, \mathcal{M}, \mathcal{N})(\phi) = \mathbf{t} \end{cases}$$

$$(I, \mathcal{M}, \mathcal{N})(\exists x, \phi) := max\{(I, \mathcal{M}, \mathcal{N})(\phi[x \to \alpha]) \mid \alpha \in \Delta\}$$

$$(I, \mathcal{M}, \mathcal{N})(\forall x, \phi) := min\{(I, \mathcal{M}, \mathcal{N})(\phi[x \to \alpha]) \mid \alpha \in \Delta\}$$

$$(I, \mathcal{M}, \mathcal{N})(\phi \wedge \sigma) := min((I, \mathcal{M}, \mathcal{N})(\phi), (I, \mathcal{M}, \mathcal{N})(\sigma))$$

$$(I, \mathcal{M}, \mathcal{N})(\phi \vee \sigma) := max((I, \mathcal{M}, \mathcal{N})(\phi), (I, \mathcal{M}, \mathcal{N})(\sigma))$$

$$(I, \mathcal{M}, \mathcal{N})(\phi \subset \sigma) := \begin{cases} \mathbf{t} & \textit{iff } (I, \mathcal{M}, \mathcal{N})(\phi) \geq (I, \mathcal{M}, \mathcal{N})(\sigma) \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$(I, \mathcal{M}, \mathcal{N})(\mathbf{K}\phi) := \begin{cases} \mathbf{t} & \text{iff } (J, \langle M, M_1 \rangle, \mathcal{N})(\phi) = \mathbf{t} \ \boxed{\text{for all}} \ J \in M \\ \mathbf{f} & \text{iff } (J, \langle M, M_1 \rangle, \mathcal{N})(\phi) = \mathbf{f} \ \boxed{\text{for some}} \ J \in M_1 \\ \mathbf{u} & \text{otherwise} \end{cases}$$

$$(I, \mathcal{M}, \mathcal{N})(\mathbf{not}\,\phi) := \begin{cases} \mathbf{t} & \text{iff } (J, \mathcal{M}, \langle N, N_1 \rangle)(\phi) = \mathbf{f} \ \boxed{\text{for some}} \ J \in N_1 \\ \mathbf{f} & \text{iff } (J, \mathcal{M}, \langle N, N_1 \rangle)(\phi) = \mathbf{t} \ \boxed{\text{for all}} \ J \in N \\ \mathbf{u} & \text{otherwise} \end{cases}$$

Intuitively, $\mathcal{M}$ and $\mathcal{N}$ are collections of possible worlds (two-valued first-order interpretations). The modal **K** and **not** operators check whether a condition holds for every possible world in $\mathcal{M}$ and $\mathcal{N}$ respectively. $\mathcal{M}$ and $\mathcal{N}$ are pairs of sets of possible worlds so that we can encode the third truth value, **u**. It is this separation of the evaluation of **K** and **not** that allows us to avoid transforming a program into its "reduct" as is done in stable model semantics.

The logic of MKNF as described above is very general, but we restrict our focus to hybrid MKNF Knowledge bases, a syntactic restriction that captures combining answer set programs with ontologies. An (MKNF) program $\mathcal{P}$ is a set of (MKNF) rules. A rule r is written as follows:

$$\mathbf{K}h \leftarrow \mathbf{K}p_0, \ldots, \mathbf{K}p_j, \ \mathbf{not}\, n_0, \ldots, \mathbf{not}\, n_k.$$

In the above, the atoms $h, p_0, n_0, \ldots, p_j$, and n_k are function-free first-order atoms of the form $p(t_0, \ldots, t_n)$ where p is a predicate and $t_0, \ldots, t_n$ are either constants or variables. We call an

MKNF formula ϕ *ground* if it does not contain variables. The corresponding MKNF formula $\pi(r)$ for a rule r is as follows:

$$\pi(r) \coloneqq \forall \vec{x},\ \mathbf{K}h \subset \mathbf{K}p_0 \land \dots \land \mathbf{K}p_j \land \mathbf{not}\, n_0 \land \dots \land \mathbf{not}\, n_k$$

where $\vec{x}$ is a vector of all variables appearing in the rule. We will use the following abbreviations:

$$\pi(\mathcal{P}) \coloneqq \bigwedge_{r\in\mathcal{P}} \pi(r)$$
$$head(r) = \mathbf{K}h$$
$$body^{+}(r) = \{\mathbf{K}p_0, \dots,\ \mathbf{K}p_j\}$$
$$body^{-}(r) = \{\mathbf{not}\, n_0, \dots,\ \mathbf{not}\, n_k\}$$
$$\mathbf{K}(body^{-}(r)) = \{\mathbf{K}a \mid \mathbf{not}\, a \in body^{-}(r)\}$$

An *ontology* $\mathcal{O}$ is a decidable description logic (DL) knowledge base translatable to first-order logic, we denote its translation to first-order logic with $\pi(\mathcal{O})$. We also assume that entailment reasoning with the ontology can be performed in polynomial time.

A *normal hybrid MKNF knowledge base* (or knowledge base for short) $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ contains a program and an ontology. The semantics of a knowledge base corresponds to the MKNF formula $\pi(\mathcal{K}) = \pi(\mathcal{P}) \land \mathbf{K}\pi(\mathcal{O})$. We assume, without loss of generality [7], that any given hybrid MKNF knowledge $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ base is ground, that is, $\mathcal{P}$, does not contain any variables[1] This ensures that $\mathcal{K}$ is decidable.

A *(3-valued) MKNF interpretation* (M, N) is a pair of sets of first-order interpretations where $\emptyset \subset N \subseteq M$. We say an MKNF interpretation (M, N) *satisfies* a knowledge base $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ if for each $I \in M$, $(I, \langle M, N\rangle, \langle M, N\rangle)(\pi(\mathcal{K})) = \mathbf{t}$.

Definition 2.1. *A (3-valued) MKNF interpretation (M, N) is a (3-valued) MKNF model of a normal hybrid MKNF knowledge base $\mathcal{K}$ if (M, N) satisfies $\pi(\mathcal{K})$ and for every 3-valued MKNF interpretation pair (M', N') where $M \subseteq M'$, $N \subseteq N'$, $(M, N) \neq (M', N')$, and we have some $I \in M'$ s.t. $(I, \langle M', N'\rangle, \langle M, N\rangle)(\pi(\mathcal{K})) \neq \mathbf{t}$.*

Intuitively, minimal knowledge is captured by adding more possible worlds to the MKNF interpretation used to evaluate **K**-atoms (the evaluation of **not**-atoms stays the same). The more possible worlds there are, the more likely it is for $\mathbf{K}a$ to be false.

It is often convenient to only deal with atoms that appear inside $\mathcal{P}$. We use $\mathbf{KA}(\mathcal{K})$ to denote the set of **K**-atoms that appear as either $\mathbf{K}a$ or $\mathbf{not}\, a$ in the program and we use $\mathsf{OB}_{\mathcal{O},S}$ to the objective knowledge w.r.t. to a set of **K**-atoms S.

$$\mathbf{KA}(\mathcal{K}) \coloneqq \{\mathbf{K}a \mid r \in \mathcal{P},\ \mathbf{K}a \in \Big(\{head(r)\} \cup body^{+}(r) \cup \mathbf{K}(body^{-}(r))\Big)\}$$
$$\mathsf{OB}_{\mathcal{O},S} \coloneqq \big\{\pi(\mathcal{O})\big\} \cup \big\{a \mid \mathbf{K}a \in S\big\}$$

[1] Not every knowledge base can be grounded. The prevalent class of groundable knowledge bases is the knowledge bases that are DL-safe [3].

A **K**-*interpretation* $(T, P) \in \wp(\mathbf{KA}(\mathcal{K}))^2$ is a pair of sets of **K**-atoms[2]. We say that (T, P) is consistent if $T \subseteq P$. An MKNF interpretation pair (M, N) uniquely *induces* a consistent **K**-interpretation (T, P) where for each $\mathbf{K}a \in \mathbf{KA}(\mathcal{K})$:

$$\mathbf{K}a \in \boxed{(T \cap P)} \text{ if } \forall I \in M, (I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}a) = \boxed{\mathbf{t}}$$
$$\mathbf{K}a \notin \boxed{\quad P \quad} \text{ if } \forall I \in M, (I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}a) = \boxed{\mathbf{f}}$$
$$\mathbf{K}a \in \boxed{(P \setminus T)} \text{ if } \forall I \in M, (I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}a) = \boxed{\mathbf{u}}$$

Intuitively, T contains **K**-atoms that are true and P contains **K**-atoms that are possibly true (i.e. they are not false). We can acquire the set of false **K**-atoms by taking the complement of P, e.g., $F = \mathbf{KA}(\mathcal{K}) \setminus P$ and $P = \mathbf{KA}(\mathcal{K}) \setminus F$. When we use the letters T, P, and F, we always mean sets of true, possibly true, and false **K**-atoms respectively.

We say that a **K**-interpretation (T, P) *extends* to an MKNF interpretation (M, N) if (T, P) is consistent, $\mathsf{OB}_{\mathcal{O},P}$ is consistent, and

$$(M, N) = (\{I \mid \mathsf{OB}_{\mathcal{O},T} \models I\}, \{I \mid \mathsf{OB}_{\mathcal{O},P} \models I\})$$

where I is a first-order interpretation of $\pi(\mathcal{K})$ that is satisfies $\mathsf{OB}_{\mathcal{O},T}$ or $\mathsf{OB}_{\mathcal{O},P}$ respetively. This operation extends (T, P) to the $\subseteq$-maximal MKNF interpretation that induces (T, P).

We comment on how the relation *induces* and *extends* are related.

Remark 1. *Let (M, N) be an MKNF* **model** *of an MKNF knowledge base $\mathcal{K}$. A* **K**-*interpretation (T, P) that extends to (M, N) exists, is unique and is the* **K**-*interpretation induced by (M, N).*

We say that an MKNF interpretation (M, N) *weakly induces* a **K**-interpretation (T, P) if (M, N) induces a **K**-interpretation (T^*, P^*) where $T \subseteq T^*$ and $P^* \subseteq P$. Similarly, a **K**-interpretation *weakly extends* to an MKNF interpretation (M, N) if there exists an interpretation (T^*, P^*) that extends to (M, N) such that $T \subseteq T^*$, $P^* \subseteq P$. Intuitively, we are leveraging the knowledge ordering $\mathbf{u} < \mathbf{t}$ and $\mathbf{u} < \mathbf{f}$. A **K**-interpretation is weakly induced by an MKNF interpretation if that MKNF interpretation induces a **K**-interpretation that "knows more" than the original **K**-interpretation.

There are MKNF interpretations to which no **K**-interpretation extends, however, these are of little interest; either $\mathsf{OB}_{\mathcal{O},P}$ is inconsistent or (M, N) is not maximal w.r.t. atoms that do not appear in $\mathbf{KA}(\mathcal{K})$. Similarly, there exist **K**-interpretations that extend to MKNF interpretations that do not induce them. If this is the case, then the **K**-interpretation is missing some logical consequences of the ontology and this should be corrected. We define the class of **K**-interpretations that excludes the undesirable **K**-interpretations and MKNF interpretations from focus.

Definition 2.2. *A* **K**-*interpretation (T, P) of an MKNF knowledge base $\mathcal{K}$ is* saturated *if it can be extended to an MKNF interpretation (M, N) that induces (T, P). Equivalently, a* **K**-*interpretation is saturated iff (T, P) and $\mathsf{OB}_{\mathcal{O},P}$ are consistent, $\mathsf{OB}_{\mathcal{O},P} \not\models a$ for each $\mathbf{K}a \in (\boldsymbol{KA}(\mathcal{K}) \setminus P)$ and $\mathsf{OB}_{\mathcal{O},T} \not\models a$ for each $\mathbf{K}a \in (\boldsymbol{KA}(\mathcal{K}) \setminus T)$.*

[2]We use $\wp(S)$ to denote the powerset of S, i.e., $\wp(S) = \{X \mid X \subseteq S\}$

2.1. An Example Knowledge Base

Hybrid MKNF knowledge bases are equivalent to answer set semantics when the ontology is empty. The definition of an MKNF program given is precisely the formulation of stable model semantics [1] in the logic of MKNF [5]. We give an example of a knowledge base to demonstrate the combined reasoning of an answer set program with an ontology.

Example 1. *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be the following knowledge base*

$$\pi(\mathcal{O}) = \{a \lor \neg b\}$$
$$(\textit{The definition of } \mathcal{P} \textit{ follows})$$
$$\mathbf{K}a \leftarrow \mathbf{not}\, b$$
$$\mathbf{K}b \leftarrow \mathbf{not}\, a$$

This knowledge base has two MKNF models

$$\Big(\Big\{\{a, b\}, \{a, \neg b\}, \{\neg a, \neg b\}\Big\}, \Big\{\{a, b\}\Big\}\Big)$$
$$\textit{which induces the } \mathbf{K}\textit{-interpretation } (\emptyset, \{\mathbf{K}a, \mathbf{K}b\})$$
$$(M, M) \textit{ where } M = \Big\{\{a, \neg b\}, \{a, b\}\Big\}$$
$$\textit{which induces the } \mathbf{K}\textit{-interpretation } (\{\mathbf{K}a\}, \{\mathbf{K}a\})$$

Above we have the answer set program $\mathcal{P}$ that, without $\mathcal{O}$, admits three partial stable models. The ontology $\mathcal{O}$ encodes an exclusive choice between the atoms a and b. The program allows for both a and b to be undefined. The ontology here is used as a filter to remove all interpretations where a is false but b is true.

The mixing of negation as failure and classical negation is not always straightforward. For example, let $\pi(\mathcal{O}) = \{a \lor b\}$ and $\mathcal{P} = \emptyset$. This knowledge base has just one MKNF model

$$\big(M, M\big) \textit{ where } M = \Big\{\{a, \neg b\}, \{\neg a, b\}, \{a, b\}\Big\}$$
$$\textit{which induces the } \mathbf{K}\textit{-interpretation } (\emptyset, \emptyset)$$

From the perspective of $\mathcal{P}$, which is only concerned with $\mathbf{K}$-interpretations, all atoms are false. However, the interpretation $\{\neg a, \neg b\}$ is absent from the model which ensures that $\mathcal{O}$ is still satisfied. Recall that $\pi(\mathcal{K}) = \mathbf{K}(a \lor b)$.

3. Well-Founded Semantics

3-valued semantics allow for uncertainty to be encoded using an additional truth value, undefined. Well-founded semantics are a special case of 3-valued semantics and a well-founded model does not assign a $\mathbf{K}$-atom to be true (resp. false) unless it is true (resp. false) in all 3-valued MKNF models.

We give a formal definition of well-founded semantics.

Definition 3.1. *Given a hybrid MKNF knowledge base* $\mathcal{K}$ *with an MKNF model* (M, N)*, we call* (M, N) *the* well-founded model *of* (M, N) *if for every **other** MKNF model* (M', N') *of* $\mathcal{K}$ *we have* $M' \subseteq M$ *and* $N \subseteq N'$*.*

In general, a knowledge base may have several non-comparable minimal MKNF models and a well-founded model may not exist. While the minimality condition of MKNF minimizes interpretations according to the truth ordering $\mathbf{f} < \mathbf{u} < \mathbf{t}$, a well-founded model is minimal w.r.t. the partial ordering $\mathbf{u} < \mathbf{t}$, $\mathbf{u} < \mathbf{f}$.

Crucially, a well-founded model contains information that must hold for all MKNF models of a knowledge base. This property also holds for MKNF interpretations that contain a subset of the knowledge found in the well-founded model. A polynomial algorithm will miss some well-founded models or a well-founded model may not exist [8]. However, if the algorithm is sound, then the MKNF interpretation it computes is an approximation of the well-founded model and thus it is still useful for applications such as grounding. In the following section, we present a fixpoint operator that can compute well-founded models of hybrid MKNF knowledge bases. We primarily restrict our focus to **K**-interpretations which can be extended to MKNF interpretations (See Section 2).

4. A Well-Founded Operator

We now present a fixpoint operator that captures the 3-valued semantics of hybrid MKNF knowledge bases. This operator builds directly upon prior work [9, 10].

In the following, we use $\mathsf{OB}_{\mathcal{O},P,B}$ as shorthand for $\mathsf{OB}_{\mathcal{O},P} \cup \{\neg b \mid \mathbf{K}b \in B\}$ and we use $\bot$ as an atom that is always false. The function $filter(F)$ can be any function $filter(F) : \wp(\mathbf{KA}(\mathcal{K})) \to \wp(\wp(\mathbf{KA}(\mathcal{K})))$ so long as it is $\subseteq$-order preserving, that is,

$$\forall F, F' \in \wp(\mathbf{KA}(\mathcal{K})), (F \subseteq F') \Rightarrow (filter(F) \subseteq filter(F'))$$

To limit computational complexity, it is ideal for $filter(F)$ to be polynomial-time computable and for its image is restricted to elements of polynomial size w.r.t. syntactic measure of $\mathcal{K}$. This allows our operator that embeds $filter$ to be polynomial. In the following, we use $\mathbf{lfp}\ Atmost^{(T,F)}$ to denote the least fixedpoint of the operator $Atmost^{(T,F)}$.

$$W(T, F) \coloneqq \Big(\bigcup_{k=0}^{2} add_k(T, (\mathbf{KA}(\mathcal{K}) \setminus F)), (\mathbf{KA}(\mathcal{K}) \setminus \mathbf{lfp}\ Atmost^{(T,F)}) \Big)$$

$$Atmost^{(T,F)}(P) = \Big(\bigcup_{k=0}^{2} add_k(P, T) \setminus \bigcup_{k=0}^{2} extract_k^{(T,F)}(P) \Big)$$

$$add_0(X, \_) \coloneqq \{\mathbf{K}a \in \mathbf{KA}(\mathcal{K}) \mid \mathsf{OB}_{\mathcal{O},X} \models a\}$$

$$add_1(X, Y) \coloneqq \{\mathbf{K}a \mid r \in \mathcal{P},\ \mathbf{K}a = head(r), body^+(r) \subseteq X,\ \mathbf{K}(body^-(r)) \cap Y = \emptyset\}$$

$$extract_0^{(T,F)}(P) \coloneqq \{\mathbf{K}a \in P \mid B \subseteq F, B \in filter(F), \mathsf{OB}_{\mathcal{O},P,B} \not\models \bot, \mathsf{OB}_{\mathcal{O},T,B} \models \neg a\}$$

$$extract_1^{(T,F)}(P) \coloneqq \{\mathbf{K}a \in P \mid r \in \mathcal{P}, head(r) \in F,\ \mathbf{K}(body^-(r)) \subseteq F, (body^+(r) \setminus \{\mathbf{K}a\}) \subseteq T\}$$

We use the letter T to denote a parameter that contains true atoms, P for possibly true atoms (either undefined or true) and F for false atoms. X and Y are used if the parameter could be a set of true or possibly true atoms. We give an intuitive explanation of each component of the operator.

- $add_0(X, \_)$: When X is T, everything that must be true as a consequence of the ontology is computed. When X is P, it returns everything that is possibly true as a consequence of the ontology. The second argument of the function is not used.
- $add_1(X, Y)$: Similar to $add_0(X, \_)$, except that the consequences of $\mathcal{P}$ are computed. When $(X, Y) = (T, P)$ the **not** -atoms in the body of the rules are checked against atoms that are not in P, i.e. they are false, whereas when $(X, Y) = (P, T)$, **not** -atoms are evaluated against atoms not present in T, i.e., undefined atoms.
- $extract_0^{(T,F)}(P)$: Computes atoms that are false as a consequence of the ontology. Uses atoms in F that have been previously established as false to improve inferences. Subsets of F must be checked for consistency with $\mathsf{OB}_{\mathcal{O},T}$ because atoms may not necessarily be false in conjunction. To ensure monotonicity, the consistency of $\mathsf{OB}_{\mathcal{O},P,B}$ must be checked (which implies that $\mathsf{OB}_{\mathcal{O},T}$ is consistent when (T, P) is consistent.) If $\mathsf{OB}_{\mathcal{O},P,B}$ is consistent, then we will not gain any additional inferences by checking subsets of B.
- $extract_1^{(T,F)}(P)$: Finds rules where a single atom in the body has a value of undefined and the head of the rule is false. The single atom is set to be false. This rule relies on atoms that have been previously established as false to determine that the negative bodies of rules are satisfied. It would not be correct to use the complement of P to perform this check.

It is sometimes convenient to use W as an operator that takes and returns **K**-interpretations.

$$W^+(T, P) = (T', \mathbf{KA}(\mathcal{K}) \setminus F') \text{ where } (T', F') = W(T, \mathbf{KA}(\mathcal{K}) \setminus P)$$

The operator is monotone w.r.t. the precision-ordering of **K**-interpretations, that is, the ordering on **K**-interpretations that embeds the ordering on logic values that ranks undefined beneath true and false.

Proposition 4.1. *The* $W^+(T, P)$ *operator is monotone, i.e., for each* (T_1, P_1) *and* (T_2, P_2) *such that* $T_1 \subseteq T_2$ *and* $P_2 \subseteq P_1$ *we have*

$$\begin{aligned}(T_1', P_1') &= W^+(T_1, P_1)\\ (T_2', P_2') &= W^+(T_2, P_2)\\ T_1' \subseteq T_1', &\; P_2' \subseteq P_1'\end{aligned}$$

Note that the above proposition holds for both consistent and inconsistent **K**-interpretations. We now formally claim that the operator captures a sound approximation of models.

Proposition 4.2. *Let* (M, N) *be an MKNF model of a knowledge base* $\mathcal{K}$*. For any* **K***-interpretation* (T, P) *that weakly extends to* (M, N)*,* (M, N) *weakly induces* $W^+(T, P)$*.*

If a well-founded model exists, then it weakly induces the **K**-interpretation $(\emptyset, \mathbf{KA}(\mathcal{K}))$. We can obtain a more precise approximation of this well-founded model by iteratively applying W^+ on $(\emptyset, \mathbf{KA}(\mathcal{K}))$.

We can also easily check whether the **K**-interpretation computed by W corresponds to the well-founded model.

Proposition 4.3. *Let $\mathcal{K}$ be an MKNF knowledge base with a well-founded model (M, N). Extend the least fixedpoint of W^+ to the MKNF interpretation (M', N'). We have $(M', N') = (M, N)$ iff $(M', N') \models \pi(\mathcal{K})$.*

In some cases, we can compute the well-founded model by iteratively applying the W^+ operator on $(\emptyset, \mathbf{KA}(\mathcal{K}))$. To check whether the well-founded model can be computed using W^+, we can apply the operator until a fixpoint is reached then check whether it is the well-founded model using Proposition 4.3. Even if the fixpoint is not the well-founded model, everything true or false in the fixpoint is also true and false respectively in the well-founded model due to Proposition 4.2.

The following example demonstrates basic application of the operator on a knowledge base. We demonstrate the more nuanced properties of the operator in the next section. Note that in this example, and subsequent examples, we omit **K** from atoms when describing **K**-interpretations.

Example 2. *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be the following MKNF knowledge base.*

$$\pi(\mathcal{O}) = \{(a \supset b), \neg c\}$$

(The definition of $\mathcal{P}$ follows)

$$\mathbf{K}a \leftarrow \mathbf{not}\, c$$

$$\mathbf{K}b \leftarrow \mathbf{not}\, b$$

When we iteratively apply W to $(\emptyset, \emptyset)$ we obtain the following sequence.

$Atmost$ fails to compute c as possibly true, it is added to the set of false atoms

$$W(\emptyset, \emptyset) = (\emptyset, \{c\})$$

With c false, the body of the first rule is true, so a is derived

$$W(\emptyset, \{c\}) = (\{a\}, \{c\})$$

When a is true, the ontology implies b

$$W(\{a\}, \{c\}) = (\{a, b\}, \{c\})$$

Finally, a fixpoint is reached

$$W(\{a, b\}, \{c\}) = (\{a, b\}, \{c\})$$

The operator $W(T, F)$ provides significant improvements on the prior work. In particular, our operator makes better use of false atoms and can infer more negative consequences. In the coming section, we give general examples that demonstrate inference power improvements. While it remains to be shown whether these advancements may directly benefit practical use cases, the work here improves the theoretical basis for inferences that rely on classical negation and could be built upon to support practical use cases.

5. Comparing Prior Work

The work on fixpoint operators for 3-valued hybrid MKNF is split. The well-founded operator from Ji, Liu and You [9] follows a similar presentation as our operator in Section 4. The operator that computes possibly true atoms, has access to a set of atoms that were established as false in a previous iteration. The operator defined by Liu and You [10] adheres to the conventions of approximation fixpoint theory. While it introduces improvements to the well-founded operator [9], Liu and You's operator is unable to capture false inferences to the same extent due to not having access to a well-established **K**-interpretation. As it turns out, well-founded operators, such as the one presented in Section 4, can be captured in approximation fixpoint theory, however, we save these results for a separate publication. Both of these operators improve upon the original operator defined by Knorr, Alferes, and Hitzler [7].

In this section, we describe four operators in a uniform way to highlight their semantics differences. While some restructuring of the operators was required to make meaningful comparisons, we believe the presentation below is faithful. The fixpoints of the operators are maintained. For ease of reference, we assign each operator a symbol (either γ, α, β, or δ).

- γ The alternating fixpoint operator from Knorr, Alferes, and Hitzler [7].
- α The well-founded operator from Ji, Liu and You [9].
- β The approximator defined by Liu and You [10].
- δ Our well-founded operator defined in Section 4.

In the following, we define all four operators by prefixing each row with a set of these symbols. A definition for any of the operators can be reached by deleting all rows that do not contain the symbol corresponding to the operator.

$$\langle \gamma, \alpha, \beta, \delta \rangle - W(T, F) \coloneqq \Big(\bigcup_{k=0}^{2} add_k(T, (\mathbf{KA}(\mathcal{K}) \setminus F)), (\mathbf{KA}(\mathcal{K}) \setminus \mathbf{lfp}\ Atmost^{(T,F)}) \Big)$$

$$\langle \gamma, \alpha, \beta, \delta \rangle - Atmost^{(T,F)}(P) = \Big(\bigcup_{k=0}^{2} add_k(P, T) \setminus \bigcup_{k=0}^{2} extract_k^{(T,F)}(P) \Big)$$

$$\langle \gamma, \alpha, \beta, \delta \rangle - add_0(X, \_) \coloneqq \{\mathbf{K}a \in \mathbf{KA}(\mathcal{K}) \mid \mathsf{OB}_{\mathcal{O},X} \models a\}$$

$$\langle \gamma, \alpha, \beta, \delta \rangle - add_1(X, Y) \coloneqq \{\mathbf{K}a \mid r \in \mathcal{P},\ \mathbf{K}a = head(r), body^+(r) \subseteq X,\ \mathbf{K}(body^-(r)) \cap Y = \emptyset\}$$

$$\langle \gamma \rangle - extract_0^{(T,F)}(P) \coloneqq \emptyset$$

$$\langle \alpha \rangle - extract_0^{(T,F)}(P) \coloneqq \{\mathbf{K}a \in P \mid (B = \{\mathbf{K}b\} \vee B = \emptyset), B \subseteq F,\ \mathsf{OB}_{\mathcal{O},T,B} \models \neg a\}$$

$$\langle \beta \rangle - extract_0^{(T,F)}(P) \coloneqq \{\mathbf{K}a \in P \mid \mathsf{OB}_{\mathcal{O},T} \models \neg a\}$$

$$\langle \delta \rangle - extract_0^{(T,F)}(P) \coloneqq \{\mathbf{K}a \in P \mid B \subseteq F, B \in filter(F), \mathsf{OB}_{\mathcal{O},P,B} \not\models \bot, \mathsf{OB}_{\mathcal{O},T,B} \models \neg a\}$$

$$\langle \gamma, \alpha \rangle - extract_1^{(T,F)}(P) \coloneqq \emptyset$$

$$\langle \beta \rangle - extract_1^{(T,F)}(P) \coloneqq \{\mathbf{K}a \in P \mid r \in \mathcal{P},\ \mathbf{K}b = head(r), \mathsf{OB}_{\mathcal{O},T} \models \neg b,$$

$$(body^+(r) \setminus \{\mathbf{K}a\}) \subseteq T, \mathbf{K}(body^-(r)) = \emptyset\}$$

$$\langle\delta\rangle{-}extract_1^{(T,F)}(P) := \{\mathbf{K}a \in P \mid r \in \mathcal{P}, head(r) \in F,\ \mathbf{K}(body^-(r)) \subseteq F,$$
$$(body^+(r) \setminus \{\mathbf{K}a\}) \subseteq T\}$$

In Table 1 we provide a high-level overview of the semantic difference between the operators that we will cover in the following examples.

Feature Description	**Example**	γ	α	β	δ
Ontology inferences use singleton sets of false atoms	See (3)		✓		✓
Ontology inferences with any subset of false atoms	See (3)				✓
Rule unit propagation with positive rules	See (4)			✓	✓
Rule unit propagation with all rules	See (4)				✓

Table 1
A high-level overview of operator features

Minor differences with how the operators deal with inconsistent information warrant us to limit our analysis to $\mathbf{K}$-interpretations that are consistent and cannot be used to derive new information from the ontology. For this reason, we restrict our focus to saturated $\mathbf{K}$-interpretations.

Before we compare $extract_0^{(T,F)}(X)$ for each operator, we note an ordering between them.

Lemma 5.1. *For any MKNF knowledge base $\mathcal{K}$ and saturated $\mathbf{K}$-interpretation (T, P) where $F = \boldsymbol{KA}(\mathcal{K}) \setminus P$, we have for all $X \subseteq P$*

$$\beta\text{-}extract_0^{(T,F)}(X) \subseteq \alpha\text{-}extract_0^{(T,F)}(X) \subseteq \delta\text{-}extract_0^{(T,F)}(X)$$

Because $extract_0^{(T,F)}(X)$ is computing atoms that should be false, a larger set of false atoms is more precise.

Example 3. *The difference in the treatment of negation is the core distinction between $\mathcal{P}$ and $\mathcal{O}$. In hybrid MKNF, negation as failure is looser than classical negation in the sense that $\mathsf{OB}_{\mathcal{O},T} \models \neg a$ implies $\mathsf{OB}_{\mathcal{O},T} \models$ **not** a but the inverse does not necessarily hold. We compare $extract_0^{(T,F)}$ across the α, β, and δ operators. In all three cases, we are propagating negative consequences from the ontology.*

The β-$extract_0^{(T,F)}$ operator only relies on true atoms to derive consequences with the ontology. The operator α-$extract_0^{(T,F)}$ is slightly stronger and combines individual false atoms with the ontology. When $B = \{\mathbf{K}b\}$ is a singleton set we do not need to test whether $\mathsf{OB}_{\mathcal{O},P,B}$ is consistent. When we extend (T, P) to an MKNF interpretation (M, N), there must be a first-order interpretation in (M, N) that assigns b to be false. Otherwise, $\mathsf{OB}_{\mathcal{O},P}$ would be inconsistent. When we allow B to contain multiple $\mathbf{K}$-atoms in δ-$extract_0^{(T,F)}$, we must ensure that $\mathsf{OB}_{\mathcal{O},P,B}$ is consistent. Note that the operator δ-$extract_0^{(T,F)}$ is equivalent to α-$extract_0^{(T,F)}$ when the filter function maps F to singleton subsets of F like the following.

$$filter(F) := \{\{\mathbf{K}b\} \mid \mathbf{K}b \in F\} \cup \{\emptyset\}$$

In the coming example, we assume that $filter(F)$ returns the power set of F. When we allow the set B to be larger than a singleton, i.e., we provide the ontology with a conjunction of false atoms, it is possible that the ontology does not allow these atoms to be false in conjunction. While testing $\mathsf{OB}_{\mathcal{O},T,B}$ to be consistent would be sufficient, this would result in a nonmonotonic operator.

We demonstrate these types of false inferences with a knowledge base. Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be the following

$$\pi(\mathcal{O}) = \{(a \lor b \lor c \lor d) \land$$
$$(\neg(a \land b) \land \neg(a \land c) \land \neg(a \land d) \land \neg(b \land c) \land \neg(b \land d) \land \neg(c \land d))\}$$

(*The definition of $\mathcal{P}$ follows*)

$$\mathbf{K}a \leftarrow \mathbf{not}\, b$$
$$\mathbf{K}b \leftarrow \mathbf{K}b;\ \mathbf{K}c \leftarrow \mathbf{K}c;\ \mathbf{K}d \leftarrow \mathbf{K}d$$

*Note that the last three rules simply ensure that the **K**-atoms $\mathbf{K}b$, $\mathbf{K}c$, and $\mathbf{K}d$ are present in **KA**$(\mathcal{K})$. Here, there is only one MKNF model and it induces the $\mathbf{K}$-interpretation $(\{a\}, \{a\})$. The ontology encodes that a single atom must be true while all other atoms must be false. Consider the $\mathbf{K}$-interpretation $(\{a\}, \{a, d\})$. Each operator computes the following*

$$\alpha\text{-}extract_0^{(\{a\},\{b,c\})}(\emptyset) = \beta\text{-}extract_0^{(\{a\},\{b,c\})}(\emptyset) = \emptyset$$
$$\delta\text{-}extract_0^{(\{a\},\{b,c\})}(\emptyset) = \{d\}$$

When $B = \{b, c\}$ and $T = \{a\}$, we have $\mathsf{OB}_{\mathcal{O},T,B} \models \neg d$, thus the δ operator makes a more precise inference.

Before we compare $extract_1^{(T,F)}(X)$, we note the following ordering.

Lemma 5.2. *For any MKNF knowledge base $\mathcal{K}$ and saturated $\mathbf{K}$-interpretation (T, P) where $F = \textbf{\textit{KA}}(\mathcal{K}) \setminus P$, we have for all $X \subseteq P$*

$$\beta\text{-}extract_1^{(T,F)}(X) \subseteq \delta\text{-}extract_1^{(T,F)}(X)$$

Like before, $extract_1^{(T,F)}(X)$ is computing atoms that need to be false, therefore it is better for it to return larger sets.

Example 4. *If the head of a rule must be false, then its body must also be false. If there is a single undefined atom in the positive body of such a rule preventing the body of the rule from being false, we can safely assign this atom to be false. We compare $extract_1^{(T,F)}$ between the β and the δ operators. In the $\beta\text{-}extract_1^{(T,F)}$ function, the lone undefined atom must exist in a positive rule, that is, a rule with an empty negative body. Additionally, the $\beta\text{-}extract_1^{(T,F)}$ function determines the head of a rule to be false by checking if $\mathsf{OB}_{\mathcal{O},T}$ entails it as false. As a result, false consequences computed by other parts of the operator, e.g. $extract_0^{(T,F)}$, cannot assist in determining that the head of the rule is false. Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be the following knowledge base.*

$$\pi(\mathcal{O}) = \{(a \lor b) \land (\neg a \lor \neg b)$$

(*The definition of $\mathcal{P}$ follows*)

$$\mathbf{K}x \leftarrow \textbf{not}\, y$$
$$\mathbf{K}y \leftarrow \textbf{not}\, x$$
$$\mathbf{K}a \leftarrow$$
$$\mathbf{K}b \leftarrow \mathbf{K}a, \mathbf{K}x$$

Here, $\mathcal{O}$ ensures that either a or b is true, but not both. The program $\mathcal{P}$ has a choice between $\mathbf{K}x$ or $\mathbf{K}y$. If $\mathbf{K}x$ is chosen, then both $\mathbf{K}a$ and $\mathbf{K}b$ must be true, which violates the ontology. This knowledge base has a single MKNF model that induces the $\mathbf{K}$-interpretation $(\{a, y\}, \{a, y\})$. Consider the $\mathbf{K}$-interpretation $(\{a\}, \{x, y\})$. Both the $\beta\text{-}extract_1^{(T,F)}$ and $\delta\text{-}extract_1^{(T,F)}$ functions correctly compute that x must be false. If we modify the final rule to be $\mathbf{K}b \leftarrow \mathbf{K}a, \mathbf{K}x, \textbf{not}\, b$, the MKNF models are unchanged, but the β operator fails to compute x because the rule is no longer positive.

We now formally claim that our operator in Section 4 captures both the α and β operators.

Proposition 5.1. *For any MKNF knowledge base $\mathcal{K}$ and $\mathbf{K}$-interpretation (T, P) where $F = \boldsymbol{KA}(\mathcal{K}) \setminus P$, we have*

$$\alpha\text{-}W(T, F) \sqsubseteq \delta\text{-}W(T, F)$$
$$\beta\text{-}W(T, F) \sqsubseteq \delta\text{-}W(T, F)$$

Where $(T, F) \sqsubseteq (T', F')$ iff $T \subseteq T'$ and $F \subseteq F'$.

6. Discussion

Well-founded semantics play a vital role in efficient answer set solving. The ground, solve, check paradigm is still prevalent and grounders rely on well-founded semantics. In hybrid MKNF knowledge bases, well-founded semantics are intractable and well-founded models may not exist. Additional challenges must be overcome if we are to build effective grounders for hybrid MKNF. In this work, we improved upon the well-founded semantics for hybrid MKNF knowledge bases by capturing them with a well-founded fixpoint operator. We compared our operator with prior work and demonstrated the differences to ultimately show that we expand the class of knowledge bases with known polynomial-time well-founded semantics. Because our operator performs well-founded propagation, it can also be used in conjunction with constraint propagation.

Future work could include using lookahead to achieve better unit propagation with rules. Currently, we rely on there being only a single undefined atom in the body of a rule, however, if there are two undefined atoms and assigning one to be false results in inconsistency, we may be able to safely assign the other atom as false.

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