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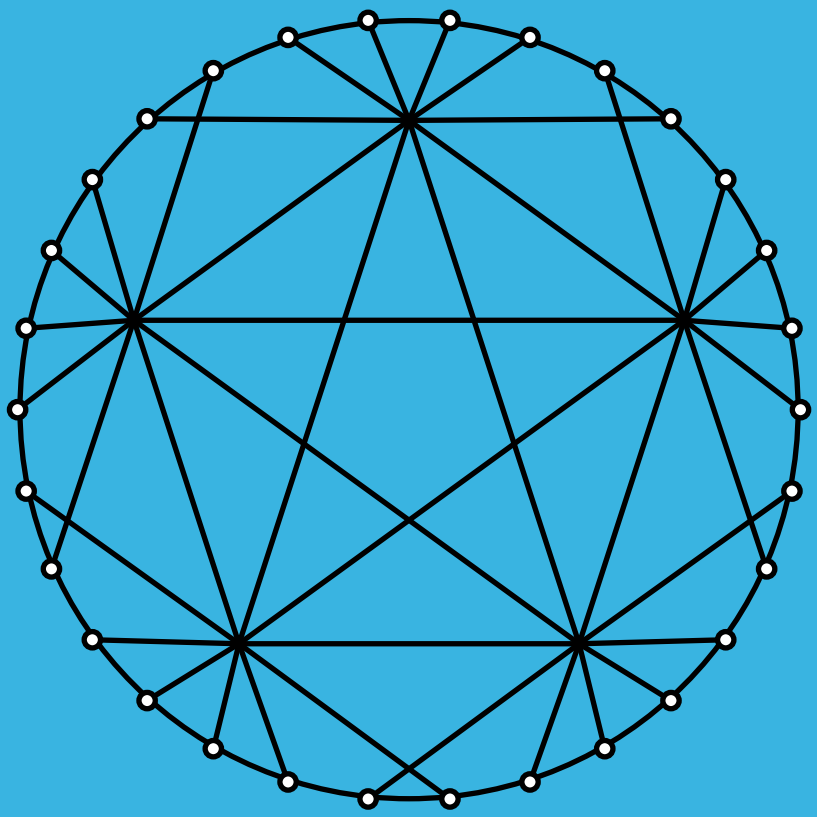

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Even orientations and pfaffian graphs

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Abstract. We give a characterization of pfaffian graphs in terms of even orientations, extending the characterization of near bipartite non-pfaffian graphs by Fischer and Little [4]. Our graph theoretical characterization is equivalent to the one proved by Little in [6] (cf. [8]) using linear algebra arguments.

1 Introduction

All graphs considered are finite and simple (without loops or multiple edges) unless otherwise stated. Most of our terminology is standard and can be found in many textbooks such as [2] and [9].

Let F be a 1-factor of a graph G. Then a cycle C is said to be *F-alternating* if $|E(C)| = 2|E(F) \cap E(C)|$. In particular, each F-alternating cycle has an even number of edges. An F-alternating cycle C in an orientation $\vec{G}$ of G is *evenly (oddly) oriented* if for either choice of direction of traversal around C, the number of edges of C directed in the direction of traversal is even (odd). Since C is even, this is clearly independent of the initial choice of direction around C.

Let $\vec{G}$ be an orientation of G and F be a 1-factor of G. If every F-alternating cycle is evenly oriented then $\vec{G}$ is said to be an *even F-orientation of G*.

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On the other hand if every F-alternating cycle is oddly oriented then $\vec{G}$ is said to be an *odd F-orientation of G*.

An even subdivison of G is any graph G^* which can be obtained from G by replacing edges (u, v) of G by paths $P(u, v)$ of odd length, such that $V(P(u, v)) \cap V(G) = \{u, v\}$.

An F-orientation $\vec{G}$ of a graph G is *pfaffian* if it is odd. It turns out that if $\vec{G}$ is a pfaffian F-orientation then $\vec{G}$ is a pfaffian $\widetilde{F}$-orientation for all 1-factors $\widetilde{F}$ of G (cf.[9, Theorem 8.3.2 (3)]). In this case we simply say that G is *pfaffian*.

It is well known that every planar graph is pfaffian and that the smallest non-pfaffian graph is the complete bipartite graph $K_{3,3}$ (cf.[3]). The Petersen graph is a further example of a non-pfaffian graph (see [1, Section 3] for details).

The literature on pfaffian graph is extensive and the results often profound (see [16] for a complete survey). In particular, the problem of characterizing pfaffian bipartite graphs was posed by Pólya [14]. Little [7] obtained the first such characterization in terms of a family of forbidden subgraphs. Unfortunately, his characterization does not give rise to a polynomial algorithm for determining whether a given bipartite graph is pfaffian, or for calculating the permanent of its adjacency matrix when it is. Such a characterization was subsequently obtained independently by McCuaig [11, 12], and Robertson, Seymour and Thomas [15]. As a special case their result gives a polynomial algorithm, and hence a good characterization, for determining when a balanced bipartite graph G with adjacency matrix A is *det-extremal*, i.e. it has $|det(A)| = per(A)$. For a structural characterization of det-extremal cubic bipartite graphs the reader may also refer to [17], [10], [12] and [5]. In this context the *pfaffian* of the skew adjacency matrix is defined and the sign of a 1-factor F (denoted by $sgn(F)$) is the sign of the term corresponding to F in such a pfaffian (cfr. [9, Chapter 8]). In this paper, the sign of a 1-factor of a graph will be used in the proof of Lemma 4.1.

The problem of characterizing pfaffian general graphs seems much harder. Nevertheless, some very interesting connections in terms of *bricks* and *near bipartite graphs* have been found (cf. e.g. [4], [9], [13], [16], [18]).

A graph G is said to be 1-*extendable* if each edge of G is contained in at least one 1-factor of G. A subgraph J of a graph G is *central* if $G - V(J)$ has a 1-factor.

A 1-extendable non-bipartite graph G is said to be *near bipartite* if there exist edges e_1 and e_2 such that $G\backslash\{e_1, e_2\}$ is 1-extendable and bipartite.

The pfaffian property which holds for odd F-orientations does not hold for even F-orientations. Indeed, the Wagner graph W (cf. Section 2) is pfaffian, so there is an odd orientation which works for all 1-factors. On the other hand, it has an even F_1-orientation and no even F_2-orientation where F_1 and F_2 are chosen 1-factors of W (cf. Lemma 2.4).

A graph G is said to be *simply reducible to a graph* H_0 if G has an odd length cycle C such that H_0 can be obtained from G by contracting C, i.e. by removing an edge from the graph while simultaneously merging the two vertices that it previously joined and disregarding loops or multiple edges. More generally G is said to be *reducible to a graph* H if for some fixed integer k there exist graphs $G_0, G_1, \ldots, G_k$ such that $G_0 = G$, $G_k = H$ and for i, $1 \leq i \leq k$, G_{i-1} is simply reducible to G_i.

Fischer and Little [4] proved the following characterization of near bipartite non-pfaffian graphs:

Theorem 1.1 (Fischer and Little [4])**.** *A near bipartite graph G is non-pfaffian if and only if G contains a central subgraph J which is reducible to an even subdivision of $K_{3,3}$, the cubeplex Γ_1 or the twinplex Γ_2 (cf. Fig. 1 in Section 3)*

In [13] this result was restated in terms of matching minors.

In this context, recently we have examined the structure of 1-extendable graphs G which have no even F-orientation [1], where F is a fixed 1-factor of G. We have given in [1] a characterization in the case of graphs of connectivity at least four and of k-regular graphs, $k \geq 3$. Part of this characterization is stated in Theorem 2.6.

In this note, as a consequence of the cited characterization of graphs with no even F-orientations, we characterize non-pfaffian graphs in terms of even orientations (cf. Theorem 4.3), extending the characterization of near bipartite non-pfaffian graphs by Fischer and Little [4] cited in Theorem 1.1.

Note that Theorem 4.3 gives a graph theoretical proof of an equivalent formulation that is stated in Little and Rendl [8] and proved using linear algebra arguments in [6].

2 Preliminaries

In this section we introduce some definitions and notation useful to state and then prove our main Theorem 4.3.

Definition 2.1. (Zero-sum sets) *Let G be a graph with a 1-factor F. Suppose that $\mathcal{A} := \{C_1, \ldots, C_k\}$ is a set of F-alternating cycles such that each edge of G is contained in an even number of elements of $\mathcal{A}$. Then $\mathcal{A}$ is said to be a* zero-sum F-set.

If k is even or odd we say that the zero-sum F-set is respectively an even F-set *or an* odd F-set.

The following Lemma will be particularly useful in the proof of Corollary 2.3, Proposition 4.2 and the main Theorem 4.3.

Lemma 2.2. *Let G be a graph with a 1-factor F and an odd zero-sum F-set $\mathcal{C} := \{C_1, \ldots, C_k\}$. Suppose that $C_1, \ldots, C_{k_1}$ are oddly F-oriented and $C_{k_1+1}, \ldots, C_k$ are evenly F-oriented in an orientation $\vec{G}$ of G. Let $k_2 := k - k_1$. Then, if k_1 is odd or k_2 is odd, G cannot have respectively an even F-orientation or an odd F-orientation.*

Proof. Firstly suppose that k_1 is odd and that G has an even F-orientation. Then there exists a set S of edges such that $|E(C_i) \cap S| \equiv 1 \pmod 2$, $i = 1, \ldots, k_1$ and $|E(C_j) \cap S| \equiv 0 \pmod 2$, $j = k_1 + 1, \ldots, k$. This follows since to change $\vec{G}$ into an even F-orientation we must reverse an odd number of orientations in the oddly oriented F-cycles and an even number of orientations in the evenly oriented F-cycles. Set $S := \{e_1, \ldots, e_l\}$ and write

$$a_{i,j} := \begin{cases} 1 & \text{if } e_i \in E(C_j) \quad (j = 1, \ldots, k) \\ 0 & \text{otherwise} \end{cases}$$

Then, since $\mathcal{C}$ is a zero-sum F-set

$$\sum_{j=1}^{k} a_{i,j} \equiv 0 \pmod 2, \quad i = 1, \ldots, l \tag{1}$$

and, from the definition of S,

$$\sum_{i=1}^{l} a_{i,j}\, i \equiv 1 \pmod 2, \quad j = 1, \ldots, k_1 \tag{2}$$

$$\sum_{i=1}^{l} a_{i,j} \equiv 0 \pmod 2, \quad j = k_1 + 1, \ldots, k \tag{3}$$

Since k is odd, (1), (2) and (3) give a contradiction. Note that the same contradiction holds if $k_2 = 0$. Hence, if k_1 is odd G cannot have an even F-orientation. Similarly, (reversing the roles of (2) and (3)) if k_2 is odd then G cannot have an odd F-orientation. □

Corollary 2.3. *Let G be a graph with a 1-factor F and an odd F-set. Then G cannot have both an odd F-orientation and an even F-orientation.*

Proof. In the notation of Lemma 2.2, since k is odd either k_1 is odd or k_2 is odd. Then the result follows directly from Lemma 2.2. □

The *Wagner graph* W is the cubic graph having vertex set $V(W) = \{1, \ldots, 8\}$ and edge set $E(W)$ consisting of the edges of the cycle $C = (1, \ldots, 8)$ and the chords $\{(1,5), (2,6), (3,7), (4,8)\}$.

Let C_1 and C_2 be cycles of G such that both include the pair of distinct independent edges $e = (u_1, u_2)$ and $f = (v_1, v_2)$. We say that e and f are *skew relative to C_1 and C_2* if the sequence (u_1, u_2, v_1, v_2) occurs as a subsequence in exactly one of these cycles. Equivalently, we may write, without loss of generality, $C_1 := (u_1, u_2, \ldots, v_1, v_2, \ldots)$ and $C_2 := (u_1, u_2, \ldots, v_2, v_1, \ldots)$ i.e. if the cycles C_1 and C_2 are regarded as directed cycles, the orientation of the pair of edges e and f occur differently.

Lemma 2.4. *[1] Let $F_1 := \{(1,5), (2,6), (3,7), (4,8)\}$ and $F_2 := \{(1,2), (3,4), (5,6), (7,8)\}$ be 1-factors of the Wagner graph W. Set $e := (1,8)$ and $f := (4,5)$. Then the Wagner graph W satisfies the following:*

(i) *W is 1-extendable*

(ii) *$W - \{e, f\}$ is bipartite and 1-extendable (i.e. W is near bipartite).*

(iii) *W has an even F_1-orientation and an odd F_1-orientation.*

(iv) *W is pfaffian.*

(v) *W has no even F_2-orientation.*

(vi) *There exist no pair of F_1-alternating cycles relative to which e and f are skew.*

(vii) *The edges e and f are skew relative to the F_2-alternating cycles $C_1 = (1, \ldots, 8)$ and $C_2 = (1, 2, 6, 5, 4, 3, 7, 8)$.*

Definition 2.5. (Generalized Wagner graphs $\mathcal{W}$) *A graph G is said to be a* generalized Wagner graph *if*

(i) *G is 1-extendable;*

(ii) *G has a subset $R := \{e, f\}$ of edges such that $G - R$ is 1-extendable and bipartite (i.e. G is near bipartite);*

(iii) *$G - R$ has a 1-factor F and F-alternating cycles C_1 and C_2 relative to which e and f are skew.*

The set of such graphs is denoted by $\mathcal{W}$, and a 1-factor F of G satisfying (iii) *is said to be a W-factor of G.*

Note that, if we say that $G \in \mathcal{W}$, we will assume the notation of Definition 2.5, i.e. that F is a $\mathcal{W}$-factor of G and R, C_1 and C_2 are as described in Definition 2.5(ii) and (iii), respectively.

Recently the authors proved in [1] the following result:

Theorem 2.6 (Abreu et al. [1])**.** *Let G be a 1-extendable graph containing a 1-factor F such that G has no even F-orientation. Then G contains an F-central subgraph G_0 such that $G_0 \in \mathcal{W}$ and F^* is a $\mathcal{W}$-factor of G_0.*

Note that in a companion paper [1], we complete this characterization in the case of regular graphs, graphs of connectivity at least four and of k-regular graphs for $k \geq 3$. Moreover, note that if $G_0 \in \mathcal{W}$ then G_0 is near bipartite. Furthermore F^* is the 1-factor of G_0 induced by F in the obvious way.

3 Bad graphs

In this section we introduce the definition of *bad* graphs and we study their relation with even and odd F-orientations. The results contained in this section will be fundamental in proving our main Theorem 4.3

Definition 3.1. (Bad Graph) *A graph G is said to be* bad *if G contains a 1-factor F such that:*

(i) *G has a zero-sum F-set $\mathcal{A}$;*

(ii) *G has an orientation in which exactly an odd number of elements of $\mathcal{A}$ are evenly F-oriented (the other elements of $\mathcal{A}$ being oddly F-oriented).*

This definition is equivalent to the one of *intractable set of alternating circuits* given by Little and Rendl in [8]. We will prove, in Theorem 4.3 that a graph is bad if and only if it is non-pfaffian, which corresponds to the equivalent result proved by Little in [6], using linear algebra arguments.

Definition 3.2. (Simply Bad Graph) *Let G be a graph. G is said to be* simply bad *if G contains a 1-factor F such that:*

(i) *G has an odd F-set $\mathcal{A}$;*

(ii) *G has an F-orientation in which each element of $\mathcal{A}$ is evenly F-oriented.*

Remark 3.3. *By definition, a simply bad graph is also bad. Definitions of bad and simply bad are, in fact, equivalent (this follows from Proposition 4.2 and Theorem 4.3).*

The next two lemmas will be used in the proof of Proposition 4.2.

Lemma 3.4. *The graphs cubeplex Γ_1, twinplex Γ_2 and $K_{3,3}$ are simply bad.*

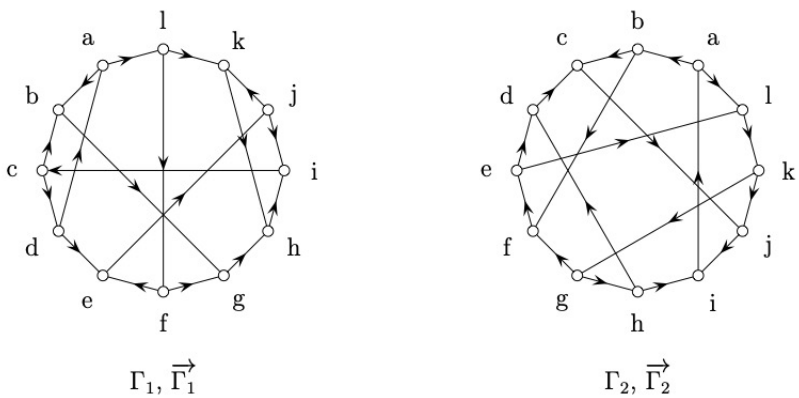

Figure 1: The graphs Γ_1, Γ_2 and their orientations

Proof. (i) Γ_1 *is simply bad:*

Let $F_1 := \{(a,d),(b,g),(i,c),(j,e),(h,k),(f,l)\}$. Let $\mathcal{A}$ be the set of F_1-alternating cycles defined by:

C_1:=$(a,d,c,i,j,e,f,l,k,h,g,b,a)$ C_2:=$(a,d,e,j,k,h,i,c,b,g,f,l,a)$
C_3:=(b,g,f,l,k,h,i,c,b) C_4:=(a,d,c,i,j,e,f,l,a)
C_5:=(a,d,e,j,k,h,g,b,a)

Thus, $\mathcal{A}$ is an odd F_1-set in which each element of $\mathcal{A}$ is evenly F_1-oriented. Hence, Γ_1 is simply bad.

(ii) Γ_2 *is simply bad:*

Note that Γ_2 may be obtained from the Petersen graph by subdividing two fixed edges at a maximum distance apart and then joining the vertices of degree 2 by an edge. Let

$$F_2 := \{(a,b),(c,d),(e,f),(g,h),(i,j),(k,l)\}.$$

Let $\mathcal{A}$ be the set of F_2-alternating cycles defined by:

$$\begin{array}{ll} C_1 := (a,b,f,e,l,k,g,h,d,c,j,i,a) & C_2 := (h,g,f,e,l,k,j,i,h) \\ C_3 := (a,b,f,e,d,c,j,i,a) & C_4 := (a,b,c,d,h,g,k,l,a) \\ C_5 := (a,b,c,d,e,f,g,h,i,j,k,l,a) & \end{array}$$

Thus, $\mathcal{A}$ is an odd F_2-set in which each element of $\mathcal{A}$ is evenly F_2-oriented. Hence, Γ_2 is simply bad.

(iii) $K_{3,3}$ *is simply bad:*

Finally, it is easily shown that $K_{3,3}$ is simply bad (see Figure 2).

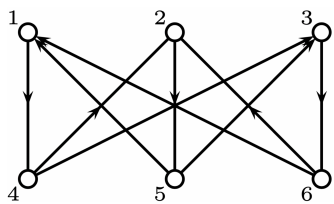

Figure 2: Orientation of the graph $K_{3,3}$

Using the notation of (i) and (ii), set $F_3 := \{(1,4),(2,5),(3,6)\}$ and $\mathcal{A} := \{C_i | i = 1,2,\ldots,5\}$ where

$$\begin{array}{ll} C_1 := (1,4,2,5,3,6,1) & C_2 := (1,4,3,6,2,5,1) \\ C_3 := (1,4,2,5,1) & C_4 := (1,4,3,6,1) \\ C_5 := (2,5,3,6,2) & \end{array}$$

The proof follows immediately. $\square$

In the following lemmas we examine the relations between even subdivision, reducibility and simply bad graphs.

Lemma 3.5. *An even subdivision H of a simply bad graph G is also simply bad.*

Proof. Let F be a 1-factor of G, let $\mathcal{A}$ be an odd F-set and $\overrightarrow{G}$ an orientation of G in which all elements of $\mathcal{A}$ are evenly oriented. Let F^* be the 1-factor of H naturally induced by F, in which from each path P_e in H which replaced an edge $e \in E(G)$, alternating edges are chosen into F^* according to e belonging to F or not. Similarly, $\mathcal{A}$ induces a set of cycles $\mathcal{A}^*$ in H, in which each edge of a cycle of $\mathcal{A}$ which had been replaced by a path in H, is replaced by that same path in the corresponding cycle in $\mathcal{A}^*$. Finally, $\overrightarrow{G}$ induces an orientation $\overrightarrow{G^*}$ in H in which every path P_e of H which replaced an edge e from G, has all edges oriented in correspondence to the orientation of $e \in E(\overrightarrow{G})$. Since H is an even subdivision, by definition $\mathcal{A}^*$ turns out to be an odd F^*-set and $\overrightarrow{G^*}$ turns out to be an orientation in which every cycle of $\mathcal{A}^*$ is evenly F^*-oriented, so H is simply bad. $\square$

Definition 3.6. *Let $\vec{G}$ be an orientation of G. We define a $(0,1)$-function $\omega := \omega_{\vec{G}}$ on the set of paths and cycles of G as follows:*

(i) For any path $P := P(u,v) = (u_0, \ldots, u_n)$

$$\omega(P) \;:=\; |\{i \,:\, [u_i, u_{i+1}] \in E(\vec{G}), 0 \leq i \leq n-1\}|\,(mod\ 2)\,.$$

Note that $\omega(P(u,v)) \equiv \omega(P(v,u)) + n\,(mod\ 2)$.

(ii) For any cycle $C = (u_1, \ldots, u_n, u_1)$

$$\omega(C) \;:=\; |\{i \,:\, [u_i, u_{i+1}] \in E(\vec{G}), 0 \leq i \leq n-1\}|\,(mod\ 2)\,;$$

where the suffixes are integers taken modulo n.

We say that ω is the orientation function associated with $\vec{G}$.

The following lemmas will be necessary in the proof of Proposition 4.2.

Lemma 3.7. *Suppose that G is a graph which is simply reducible to a graph H. Then if H is simply bad, G is simply bad.*

Proof. Suppose that H is obtained from G by contracting the cycle D to a vertex u, where for some integer k ($k \geq 1$) $D := (u_1, u_2, \ldots, u_{2k+1})$.

Suppose that H is simply bad. Let F be a 1-factor of H such that H contains an odd F-set $\mathcal{A}$ and H has an F-orientation $\overrightarrow{H}$ in which each element of $\mathcal{A}$ is evenly oriented. Let ω be the associated orientation function. Suppose that $e_i := (u, v_i)$, $i = 1, 2, \ldots, 2k+1$ are a subset of the edges incident to u such that $e_i^* := (u_i, v_i)$ are edges in G (we will assume that such edges exist and this makes no difference to the argument). We may assume that $e_1 \in F$. Set $F_1 := \{(u_{2i}, u_{2i+1}) |\ i = 1, 2, \ldots, k)\}$ and $F_2 := F_1 \cup \{F \setminus \{e_1\} \cup \{e_1^*\}\}$. Thus, F_2 is a 1-factor of G. Now define an F_2-orientation $\overrightarrow{G}$ of G with orientation function ω_2, as follows:

(i) $\omega_2(a, b) := \omega(a, b)$ for each $(a, b) \in E(H \setminus \{u\})$;

(ii) $\omega_2(u_i, v_i) := \omega(u, v_i)$ for $i = 1, 2, \ldots, 2k+1$;

(iii) $\omega_2(u_i, u_{i+1}) := 1$ for $i = 1, 2, \ldots, 2k+1$ (indices taken modulo $2k+1$).

Let C_j be a typical F-alternating cycle of H containing e_1 and e_j. Then there is a natural one to one correspondence with F_2-alternating cycles C_j^* in G. Thus set C_{2i}^* to be the F_2-alternating cycle in G obtained from C_{2i} on replacing the path (v_1, u, v_{2i}) by $(v_1, u_1, u_{2k+1}, u_{2k}, \ldots, u_{2i}, v_{2i})$. Similarly set C_{2i+1}^* to be the F_2-alternating cycle obtained from C_{2i+1} by replacing (v_1, u, v_{2i+1}) by $(v_1, u_1, u_2, \ldots, u_{2i+1}, v_{2i+1})$. By definition $w(C_j^*) = w(C_j) = 0$.

Let $\mathcal{A}^*$ be the set of F_2-alternating cycles which is obtained form $\mathcal{A}$ by replacing each C_j by C_j^*. Thus each element of $\mathcal{A}^*$ is evenly F_2-oriented in $\overrightarrow{G}$. Furthermore, consider the modulo 2 sums of the cycles in $\mathcal{A}^*$. Thus this is an Eulerian graph contained in D (since $\mathcal{A}$ is an odd F-set) and hence is a union of even cycles contained in D. Hence since D is an odd length cycle, this Eulerian graph is empty and $\mathcal{A}^*$ is and odd F_2-set. Hence, G is simply bad. □

Lemma 3.8. *If G contains a simply bad central subgraph J, then G is simply bad.*

Proof. Let J be as in the statement. Since J is simply bad, J has a 1-factor F such that J contains and odd F-set $\mathcal{A}$ and J has an F-orientation in which each element of $\mathcal{A}$ is evenly oriented. Now set $F_2 := F \cup F_1$ where F_1 is a 1-factor of $G - V(J)$. Thus G contains $\mathcal{A}$ and $\mathcal{A}$ is and odd F_2-set and in G, $\mathcal{A}$ has the induced F_2-orientation in which each element of $\mathcal{A}$ is evenly F_2-oriented. □

Lemma 3.9. *If G is reducible to H and H is simply bad then G is simply bad.*

Proof. It is an immediate consequence of Lemmas 3.7 and 3.8. □

4 Equivalence between bad and non-pfaffian graphs

In this section we prove our main characterization Theorem 4.3. Firstly we need the following lemma which relates pfaffian graphs to even F-orientations, and an accessory characterization of non-pfaffian graphs in terms of simply bad graphs (c.f. Proposition 4.2).

Lemma 4.1. *Let G be a non-pfaffian graph containing a 1-factor F. Suppose that G has an even F-orientation. Then G is simply bad.*

Proof. We use the proofs of Lemma 8.3.1 and Theorem 8.3.8 contained in [9].

Let G be a non-pfaffian graph with a 1-factor F such that G has an even F-orientation $\overrightarrow{G}$.

By Theorem 8.3.7(4) in [9] there is a set of 1-factors $F_1, F_2, \ldots, F_r$ $(r > 0)$ of G such that

$$\sum_{j=1}^{r} F_j \equiv 0 \pmod 2 \tag{4}$$

(i.e. each edge belongs to an even number of these 1-factors)

and

$$\sum_{j=1}^{r} \ell(F_j) \equiv 1 \pmod 2, \tag{5}$$

where for each F_j, $\ell(F_j)$ satisfies $sgn(F_j) = (-1)^{\ell(F_j)}$ and $\ell(F_j) \in \{0, 1\}$.

Let $\mathcal{A}$ be the family of all F-alternating cycles formed from $F\Delta F_j$ for $j = 1, 2, \ldots, r$ (where Δ stands for the symmetric difference). Also let k_j denote the number of F-alternating cycles formed from $F\Delta F_j$. We may assume that the vertices of G are labelled so that $sgn(F) = 1$. Hence, as in Lemma 8.3.1.:

$$sgn(F)sgn(F_j) = sgn(F_j) = (-1)^{k_j} \tag{6}$$

and thus, as in the proof of Lemma 8.3.8, $\ell(F_j) \equiv k_j \pmod 2$. Hence, from (5),

$$|\mathcal{A}| = \sum_{j=1}^{r} k_j = \sum_{j=1}^{r} \ell(F_j) \equiv 1 \pmod 2. \tag{7}$$

Furthermore consider the sum of the cycles in $\mathcal{A}$ modulo 2. If $e \notin F$ then, from (4), e is contained in an even number of F_j $(j = 1, 2, \ldots, r)$. Thus the modulo 2 sum of the cycles in $\mathcal{A}$ is a subset of F. But since the modulo 2 sum of cycles must be an Eulerian graph, it follows that the modulo 2 sum of cycles in $\mathcal{A}$ is zero. Hence $\mathcal{A}$ is a simply bad F-set and G is simply bad. □

We give a characterization of non-pfaffian graphs in terms of simply bad graphs and then use it to prove our main result which characterizes non-pfaffian graphs in terms of bad graphs.

Proposition 4.2. *Let G be a graph. Then G is simply bad if and only if it is non-pfaffian.*

Proof. Let G be a simply bad graph. From the definition of *simply bad graph*, it follows that $k_2 = k$ in Lemma 2.2, and k is odd so G has no odd F-orientation. Hence G is non-pfaffian.

Now suppose that G is non-pfaffian. There are two cases to consider: (i) G has an even F-orientation where F is a 1-factor of G; (ii) G has no even F-orientation, for all 1-factors F.

Case (i). Let G be a graph with an even F-orientation where F is a 1-factor of G. Then, G is simply bad by Lemma 4.1.

Case (ii). Suppose that for all 1-factors F, G has no even F-orientation. Then, from Theorem 2.6 and subsequent note, for each 1-factor F, the graph G contains an F-central subgraph G_0 which is near bipartite and non-pfaffian. Hence, from Theorem 1.1, G_0 contains a central subgraph J which is reducible to an even subdivision of $K_{3,3}$, Γ_1 or Γ_2.

By Lemma 3.4, $K_{3,3}$, Γ_1 and Γ_2 are simply bad, and so is any even subdivision by Lemma 3.5. Thus, applying Lemma 3.9, the subgraph J is simply bad. Hence, applying Lemma 3.8 twice, both G_0 and G are simply bad. □

Theorem 4.3. *Let G be a graph. Then G is bad if and only if it is non-pfaffian.*

Proof. Let G be a bad graph. From the definition of *bad graph*, it follows that $k_2 = k$ in Lemma 2.2, and k is odd so G has no odd F-orientation. Hence G is non-pfaffian.

Let G be a non-pfaffian graph. By Proposition 4.2, G is simply bad and by Remark 3.3, G is bad. □

As mentioned earlier, this result was equivalently stated in Little and Rendl [8] and proved using linear algebra arguments in [6]. Here, we have given a graph theoretical proof and extended Fischer and Little's result [4] on near-bipartite graphs in terms of bad graphs.

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