Evading Equilateral Triangles without a Map

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Abstract

Consider an arrangement of equilateral non-overlapping translated triangles in the plane and two points S and T so that the segment $\overline{ST}$ is parallel to a side of each of the triangles. Assume one needs to navigate from point S to point T by evading the triangular obstacles without any previous knowledge of the location of the obstacles. The navigator becomes aware of a triangle once it is contacted along the path. The given algorithm enables the navigator to reach the target point T by a path with length less than $\sqrt{3}d(1+\frac{2}{d^2})$, where d is the length of $\overline{ST}$.

1 Introduction

In robotics, many results of the so called *bug algorithms* exist. For example J. Ng and T. Bräunl [5] compared the effectiveness of eleven such algorithms in different environments. These algorithms often contain memory of strategic points which determine a path from a starting point S and reaches a given target point T. Here algorithms address environments which contain unconstrained obstacles, thus they are stimulated mainly by the ability to reach the target point, then evaluated by how statistically effective the algorithm is for all environments. Another variation of the *bug problem* is to restrict the obstacles in the environment and create algorithms which effectively reach T by a short path. We say that an algorithm achieves a ratio c, if for each pair of points S and T, it produces a path of length $\leq c|\overline{ST}|$. A cleverly designed obstacle scene can show a lower bounds for all possible ratios. Once such bounds are known people can appreciate simple algorithms even if their ratio is not the best, but is close to such lower bounds. Consider for example the history of the shortest path problem concerning square obstacles. A lower bound is found by a packing of squares, in a brick laying design, where the shortest path of evading squares has a ratio $\frac{3}{2}$. J. Pach [2] and L. Fejes-Tóth [4] discussed different techniques of finding paths which evade squares, and both supply non heuristical methods of finding the existence of a path which achieve a ratio, equal to the lower bound, $\frac{3}{2}$. Knowing that such paths exist leaves the question of weather a simple algorithm can achieve such a path. People first created simple heuristics which produced bounds less than the trivial, evade and return technique (see [3]), then more sophisticated heuristics came closer to reaching the ratio displayed by the existing paths. The problem solved in this section was motivated by the results of A. Bezdek [1] which evades squares and moves toward a target point T. A. Bezdek uses the properties of a square to evade a square in such a way that the distance gained toward T is greater than the distance moved away from the previous direction, and the distance traveled is no more than $\frac{3}{2}$ the distance gained toward T. As in many *bug algorithms*, [1] lets the navigator travel directly toward T once the traveler has passed an obstacle. The heuristic achieves the bound which approaches that of the existing path of $\frac{3}{2}|\overline{ST}|$. The goal of this paper is to investigate similar questions in the case of triangle obstacles, yet does not use the technique discussed in [1].
We start with two remarks to explain why the bound in Theorem 1 is significant.

Observation 1 *A lower bound can be seen by challenging a heuristic in a similar manner mentioned by Papadimitriou and Yannakakis [3].*

First let $S = S_1$, then extend a line L_i, forming 60^o with $\overline{ST}$ through point S_i. Now place an equilateral triangle, T_i, which shares a side with L_i such that S_i is the midpoint of this side. The triangle that has been placed will determine a layer between L_i and L_{i+1}. The vertex, of T_i, which is not on line L_i will define L_{i+1} in the same manner as S_1. Now the path created by any heuristic must intersect L_{i+1}, thus consider the intersection point of the path and L_{i+1} to be S_{i+1} and repeat this process of placing triangles T_i. Since the size of the triangles can be determined as necessary, we can form a series of layers which connect S to T (Fig. 1).This challenge will produce a lower bound of $\frac{1+\sqrt{3}}{2}$ for the ratio of the path to $|\overline{ST}|$.

Observation 2 *A heuristic is measured by the longest path it yields around obstacles, which is known as its upper bound. The trivial heuristic of evading each triangle and returning to $\overline{ST}$ by following the boundary of the triangle obviously produces an upper bound of $2|\overline{ST}|$.*

In this paper we consider the following:
Problem: Assume a navigator wishes to traverse a plane toward a target point T from a starting point S

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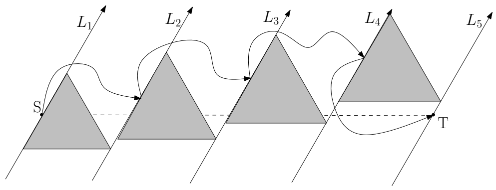

Figure 1: Layers between S and T

while avoiding equilateral triangles. The obstacles will be assumed to be non-overlapping translates with one side parallel to the segment $\overline{ST}$ and will be unknown until the navigator contacts the triangle.

Our goal is to create a heuristic which enables us to reach our target point along a path, P, that is shorter than the trivial path that is twice the length of the segment $\overline{ST}$. We will first explain, using the language of a pseudo algorithm, how we want to navigate among triangles, (Triangle Heuristic) then we will study the performance of this heuristic (Theorem 1).

2 Triangle Heuristic

When navigating through the plane toward the target point T, follow the steps below. (For a better visual understanding we included Figure 2 of a concrete path created by this heuristic.)

0. Start at S.

1. Travel along segment $\overline{ST}$ toward T. If T is reached, then go to 5. Else, next.

2. Travel along an edge of the contacted triangle toward the vertex not on the side parallel to the segment $\overline{ST}$. If you cross the line L forming a 30^o angle with $\overline{ST}$ passing through T, then go to 4. Else, next.

3. Take a 90^o turn toward $\overline{ST}$ and travel on a line E toward the segment $\overline{ST}$.

4. Travel along line E.

   a. If you reach $\overline{ST}$, repeat 1. Else,

   b. If you contact a triangle which intersects the segment $\overline{ST}$, repeat 2. Else,

   c. If you contact a triangle which does not intersect the segment $\overline{ST}$, then travel on the shortest path along the edges of the triangle back to line E and repeat 4.

5. Travel on L toward T, if a triangle is contacted, avoid the triangular obstacle in the shortest path returning to line L and repeat 5, when T is reached go to 6.

6. Stop.

Theorem 1 *If the Triangle Heuristic is followed, the ratio of the length of the path P to the length of the segment $\overline{ST}$ is less than $\sqrt{3}(1+\frac{2}{d^2}) \approx 1.732(1+\frac{2}{d^2})$, where d is the length of the segment $\overline{ST}$.*

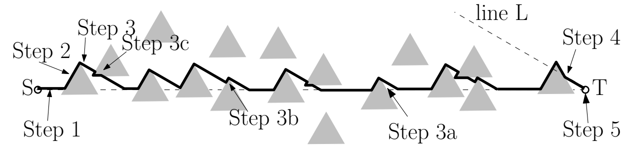

Figure 2: Path from S to T

3 Proof of Theorem 1

For orientation purposes, assume that the segment $\overline{ST}$ is horizontal and all triangles point upward, meaning the side parallel to $\overline{ST}$ in each triangle is below the third vertex.

$P \cap \overline{ST}$ is a collection of disjoint segments $\overline{a_ib_i}, i = 1, ..., k$. Let $S = a_1$ and assume the segments are labeled according to their order on $\overline{ST}$. We consider the subarcs $\phi_i \in P$, where ϕ_i is the portion of P from b_i to a_{i+1}. Through the proof of Theorem 1, we refer to ϕ as one of these subarcs. Let L_ϕ be the length of ϕ and let D_ϕ be the length of the segment $\overline{b_ia_{i+1}}$. We will show that $\frac{L_\phi}{D_\phi} < \sqrt{3}$ for each ϕ of P.

A detailed analysis of the Triangle Heuristic reveals that there are only four different types of subarcs, ϕ, to consider:

Case 1: Subarc produced by steps: 2, 3, 4a.

Case 2: Subarc produced by steps: 2, 3, 4b.

Case 3: Subarc produced by steps: 2, 3, 4c.

Case 4: Subarc produced by steps: 2, 5.

Notice that each subarc begins with step 2 by traveling to the vertex not on the side parallel to $\overline{ST}$. We will assign the variable t to the length between b_i and the before mentioned vertex. We will analyze subarcs by their progression through the triangle heuristic starting at step 2.

Case 1: Subarc produced by three consecutive steps: 2, 3, 4a.

The navigator returns to the segment $\overline{ST}$ without contacting another triangle(Figure 3),

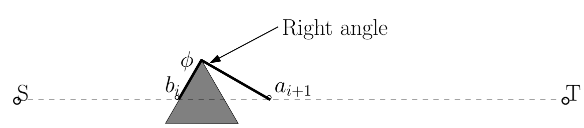

Figure 3: Case 1

Thus it is obvious that the ratio of $\frac{L_\phi}{D_\phi}$ is $\frac{1+\sqrt{3}}{2} \approx 1.366$.

Case 2: Subarc produced by three consecutive steps: 2, 3, 4b.

The navigator contacts a second triangle which intersects the segment $\overline{ST}$.

In this case we let ϕ have an end point where the path reaches this second triangle; and analyze another ϕ which begins here. Note that starting ϕ on the edge of the triangle, which intersects $\overline{ST}$, only reduces $\frac{L_\phi}{D_\phi}$ since the edge of the triangle is in a ratio of $2:1$ with respect to $\frac{L_\phi}{D_\phi}$. Thus the next ϕ can be considered as one of our 4 types of subarcs. We will distinguish two subcases depending on the position of the second triangle.

SUBCASE A: The first triangle contacted by ϕ is above the second triangle contacted.

The greatest ratio is found when the two triangles are touching (Figure 4). Here we find that $\frac{L_\phi}{D_\phi} = \frac{t+\frac{\sqrt{3}}{2}}{(\frac{t}{2}+\frac{3}{4})}$ which is maximized as t approaches 1, producing a ratio approximately equal to 1.4928, being smaller than the needed bound.

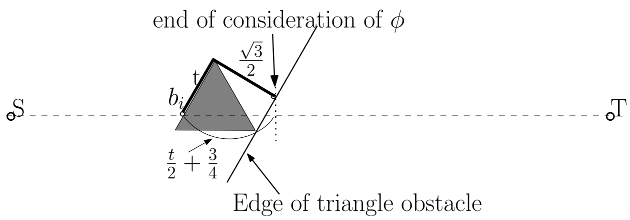

Figure 4: Maximizing Case 2-A

SUBCASE B: The first triangle contacted by ϕ is below the second triangle contacted.

Again it is obvious that the maximum ratio occurs when the two triangles are touching. It is also easy to see that the ratio increases as the second triangle slides along the first one so that its base gets closer to $\overline{ST}$. Indeed, when the second triangle moves toward the segment $\overline{ST}$ along the edge of the first triangle the part of ϕ which is not following the side of the first triangle decreases. This is significant since the portion of ϕ which is following the side of the first triangles travels with a ratio of $2:1$ with respect to $\frac{L_\phi}{D_\phi}$, whereas the second part of ϕ travels at a ratio of $\sqrt{3}:1$ with respect to $\frac{L_\phi}{D_\phi}$. Therefore as the second part of ϕ decreases, $\frac{L_\phi}{D_\phi}$ increases. Thus the position of the triangles which maximizes $\frac{\phi}{D_\phi}$ is as in Figure 5.

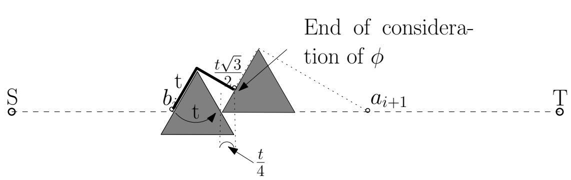

Figure 5: Maximizing Case 2-B

Here $\frac{L_\phi}{D_\phi} = \frac{t+\frac{\sqrt{3}t}{2}}{t+\frac{t}{4}} \leq \frac{4+2\sqrt{3}}{5} \approx 1.493 < \sqrt{3}$.

Case 3: Subarc produced by three consecutive steps: 2, 3, 4c.
The navigator contacts a triangle which does not intersect the segment $\overline{ST}$ and then repeats step 4.

SUBCASE A: ϕ repeats step 4 as 4a or 4c. As in the heuristic, we will avoid the triangle by following its perimeter in the shortest direction back to $\overline{ST}$. By the assumptions of the triangles, the configuration that maximizes $\frac{L_\phi}{D_\phi}$ is when the perimeter of the second triangle and ϕ have the longest part. This occurs when the second triangle is contacting the first triangle on the segment $\overline{ST}$. Therefore the configuration which maximizes $\frac{L_\phi}{D\phi}$ approaches the same configuration as in Case 2 - SUBCASE B. Since the triangle is strictly above $\overline{ST}$, ϕ ends once it has completely evaded the second triangle and arrives at the intersection of line E and $\overline{ST}$ (Figure 6).

Observe that among triangles that do not intersect segment $\overline{ST}$, at most two triangles can be in contact with any particular ϕ. In this case the "detours" caused by two triangles are shorter than the one which can be generated by one triangle.
The ratio associated with the maximizing configuration depicted in Figure 6 is $\frac{L_\phi}{D_\phi} = \frac{3+2\sqrt{3}}{2+\sqrt{3}} = \sqrt{3}$.

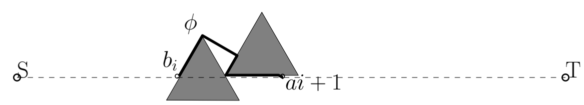

Figure 6: Maximizing Case 3 - A

SUBCASE B: ϕ repeats step 4 as 4b.
The navigator then contacts a triangle which does intersect the segment $\overline{ST}$(Figure 7).

Notice that the triangle which intersects the segment

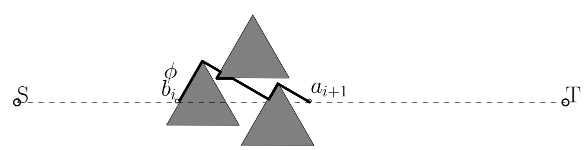

Figure 7: Case 3 - B

$\overline{ST}$ must be below the first triangle contacted as in Case 2 - SUBCASE A. Otherwise the middle triangle cannot be involved in ϕ. Therefore this case is similar to the one shown on Figure 7.

To find the configuration which maximizes $\frac{L_\phi}{D_\phi}$, just as in Case 3, assume that the triangle which does not intersect segment $\overline{ST}$ is touching the first triangle of ϕ. Also without loss of generality, assume the triangle intersecting segment $\overline{ST}$ is touching the before mentioned triangle or as in Case 3, it could be moved downward to be in contact. For now calculate $\frac{L_\phi}{D_\phi}$ as if ϕ ends when it contacts the triangle which overlaps $\overline{ST}$, as in Case 3. Finally, notice that as we lower both of these triangles, while keeping them in contact, D_ϕ stays the same and obviously, as in Case 3, the length of ϕ increases, thus the maximum configuration is as in Figure 8.

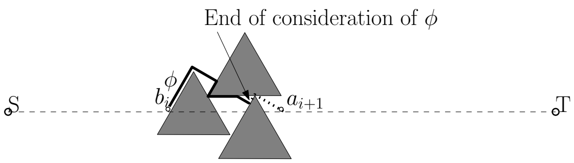

Figure 8: Maximizing Case 3 - B

For such a ϕ, $\frac{L_\phi}{D_\phi}$ depends on t. As seen above $\frac{L_\phi}{D_\phi} = \frac{4t+3+\sqrt{3}}{2t+3} = 2 + \frac{-3+\sqrt{3}}{2t+3}$ which is obviously maximized when $t = 1$, thus $\frac{\phi}{D_\phi} \leq \frac{7+\sqrt{3}}{5} \approx 1.7564$.

Currently this case can produces a ratio larger than the expected $\sqrt{3}$, but if such a ϕ exists, there are limitation on the remaining portion of ϕ. So, we will show that these two consecutive portions produce a $\frac{L_\phi}{D_\phi}$ that is smaller than $\sqrt{3}$. In this situation we will have the first ϕ maximum value as $4t + 3 + \sqrt{3}$, then we will look at the second ϕ. As the first ϕ increases it approaches the restriction that the second ϕ has a length t which approaches 0. Thus the configuration the second portion of ϕ we must navigate, no longer includes a large portion that follows the edge of a triangle having a ratio of 2 : 1 with respect to $\frac{L_\phi}{D_\phi}$. Without calculation, one can see that all other portions of ϕ are significantly smaller than $\sqrt{3}$, because all cases include the edge having a ratio of 2 : 1, but when all parts of ϕ are calculated the total $\frac{L_\phi}{D_\phi}$ is smaller than $\sqrt{3}$.

Case 4: Subarc produced by two consecutive steps: 2, 5.

We reach the line L which forms a 30^o angle with $\overline{ST}$ passing through T. Since the heuristic never allows the traveler to be more than $\frac{\sqrt{3}}{2}$ away from $\overline{ST}$, the traveler must be within 1 unit, or the length of a side of the triangles in the direction of $\overline{ST}$, from T. Notice that line L is parallel to any line E emanating from the top vertex of an obstacle, thus the traveler will produce the same results as in cases 1, 2, or 3 except when above line L. If the traveler contacts a triangle which overlaps $\overline{ST}$ the shortest distance around the obstacle and back to L is going to the top vertex as usual but then, rather than a 90^o turn, the traveler will follow the edge of the triangle back to L. Also since we are within one unit of T, there can only be one such triangle. When evading such a triangle the traveler is always moving in a ratio of 2:1, thus the last ϕ can produce at most $\frac{L_\phi}{D_\phi} = 2$ if it must travel on the edge of a triangle for its entirety and contributes the additional $\frac{2}{d^2}$ term to the bound given in Theorem 1. This maximum configuration is easily seen when a triangle is placed closely in front of the target point requiring the path P to evade this triangle by a ϕ which follows two edges of the triangle, thus $\frac{L_\phi}{D_\phi}$ approaches 2.

4 Conclusion

The lower bound given in observation 2 along with the presented upper bound leaves a gap which can be narrowed. Examination of the techniques used by A. Bezdek [1] and many *bug algorithms* have been applied to the given heuristic in an attempt to improve the upper bound. The most intuitive technique to use is to change the 90^o turn in step 3 to a turn moving in the direction of T. When this change is made to the algorithm, then case 3 produces a ϕ which has $\frac{L_\phi}{D_\phi} \geq \sqrt{3}$. Therefore, one might want to continue changing the algorithm by placing some bias in step 4c, similar to the one presented in [1], allowing the traveler to move upwards when contacting a triangle above $\overline{ST}$. This allows the traveler to move further than $\frac{\sqrt{3}}{2}$ away from $\overline{ST}$, and creating case 4 to have a situation that adds a term large enough to cancel the benefit of the alteration. Other alteration are being explored, of which non have yet to produce an upper bound less than the one proposed. Although no algorithm is known to be better than the one proposed, there is no evidence to assume that one does not exist nor does the construction of the lower bound give evidence that another construction does not produce a higher bound.

References

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