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CHAINS OF STRUCTURALLY COMPLETE PREDICATE LOGICS WITH THE APPLICATION OF PRUCNAL'S SUBSTITUTION

A b s t r a c t. A logical system is structurally complete (or smooth) if structural and admissible rules are derivable in it. It is shown that if some peculiar "omitting" rules are neglected then classical predicate logic $\mathbf{L}_2$ is structurally complete. A unique structural and structurally complete extension of $\mathbf{L}_2$ is described. Next, it is shown that among negation-free intermediate predicate logics, there are chains of type $\omega^\omega + 1$ of such logics (extensions of Gödel–Dummett logic) which are hereditarily structurally complete. This is in contrast with the case of propositional logics.

Since deducibility rather than theoremhood is studied here, by logic $\mathbf{L}$ we mean a logical system $\mathbf{L} = \langle R, A\rangle$ of rules R and axioms A. Equivalently, we may consider a consequence operation Cn_L (in the sense of A. Tarski) corresponding to $\mathbf{L} = \langle R, A\rangle$. Cn_L coincides with the provability operation for $\langle R, A\rangle$; we will omit subscript in Cn_L and write Cn. The set of formulae provable in a logic $\mathbf{L}$ will be denoted by L, i.e. $\mathrm{L} = Cn(\emptyset)$. By a rule r we

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mean a set of pairs $\langle \pi, \varphi \rangle$, where π is a finite set of formulae (premises) and φ is a formula (a conclusion); we also write $r : \pi/\varphi$. By S we denote that set of all formulae. A rule r is structural if $\langle \pi, \varphi \rangle \in r$ implies $\langle h(\pi), h(\varphi) \rangle \in r$, for every substitution $h : S \to S$. We consider here only structural logics, i.e. such that $h[Cn(X)] \subseteq Cn(h[X])$, for any substitution h and $X \subseteq S$. This means that the rules in R are structural and $h(L) \subseteq L$, for all h. Let $\mathbf{L}$ be a logic. A rule r is admissible in $\mathbf{L}$, if, for every $\langle \pi, \varphi \rangle \in r$, whenever $\pi \subseteq L$, then $\varphi \in L$. A rule r is derivable in $\mathbf{L}$, if $\varphi \in Cn(\pi)$, for every $\langle \pi, \varphi \rangle \in r$. It is clear that rules derivable in $\mathbf{L}$ are admissible in $\mathbf{L}$; the converse does not hold (it holds only for complete logics).

A logic $\mathbf{L}$ is called structurally complete, $\mathbf{L} \in SCpl$, if structural and admissible rules in $\mathbf{L}$ are derivable in $\mathbf{L}$ ($\mathbf{L}$ is also called *smooth* by some authors). This notion was introduced by Pogorzelski [7]; it was shown that the classical propositional logic (with Modus Ponens as the only rule) is structurally complete. Several other propositional logics were shown to be structurally complete, e.g. modal S5 with adjunction rule, Medvedev logic (T. Prucnal, solving the problem of H. Friedman), linear logic LC of Gödel–Dummett and its extensions, (Dzik and Wroński [3]). Intuitionistic logic is not structurally complete. Rybakov in [11] presents (Kripke) semantical conditions for hereditarily structural completeness of modal and intermediate propositional logics. In case of (1-st order) predicate logics, structural completeness was considered by Pogorzelski and Prucnal [4]. It was shown that classical predicate logic $\mathbf{L}_2$ in the standard formalization i.e. based on the rules of Modus Ponens and Generalization, is not structurally complete. Addition of the rule of substitution for predicate variables to the rules of $\mathbf{L}_2$ results in structurally complete logic. The notion of substitution for predicate variables is essential here (free individual variables are preserved). It differs from those found in the literature (eg. of Church, Hilbert and Ackermann and others).

In part 1, we describe 'omitting rules', which are structural and admissible in classical logic $\mathbf{L}_2$. We show that all the structural and admissible rules in $\mathbf{L}_2$, which are not omitting, are derivable in $\mathbf{L}_2$. Then we describe a unique structurally complete extension of $\mathbf{L}_2$.

In part 2 we consider negation-free intermediate (between intuitionistic and classical) predicate logics. We provide examples of chains of type $\omega^\omega+1$ of hereditarily structurally complete predicate logics which are not finitely axiomatizable and infinitely many such logics which are Kripke incomplete. This is in contrast with the analogous results in propositional logics.

¿From now let S denotes the language of pure 1-st order logic with individual variables, predicate symbols (without equality), the connectives $\neg, \wedge, \vee, \rightarrow$ and the quantifiers $\forall$ and $\exists$. We use only the 'type ε' of substitutions for predicate variables defined in Pogorzelski and Prucnal [8]. In short, such substitution h preserves all the connectives, all signs of quantifiers and all the free individual variables, but may change indices of bounded variables. The same letter h will denote both a substitution function h from the set of atomic formula to the set S and the extension of this function to a homomorphism of S into S. We deal with the structural rules only.

1. Classical predicate logic

By classical predicate logic $\mathbf{L}_2$ we mean the standard formalization of classical predicate logic (for example as in Mendelson [5]), with axiom schemata for propositional logic plus two predicate axioms and two rules of inference: Modus Ponens and Generalization (the standard rules). L_2 denotes the set of all formulae deducible in classical predicate logic (=classical predicate tautologies).

Unlike the case of classical propositional logic, in predicate logics there are formulae $\alpha_1, \ldots, \alpha_n$ such that for no substitution h, $h(\{\alpha_1, \ldots, \alpha_n\})$ is included in L_2, and, at the same time, $\{\alpha_1, \ldots, \alpha_n\}$ is classically consistent, i.e. $Cn(\alpha_1, \ldots, \alpha_n) \neq S$ (this is due to the presence of negation connective and existential quantifier). For example, formulae $\exists_x\varphi, \exists_x\neg\varphi$ are of this type. This leads to rules that are admissible but not derivable in $\mathbf{L}_2$.

A rule r is called *omitting* if, for every $\langle\pi, \varphi\rangle \in r$ and for every substitution $h, h(\pi) \subsetneq L_2$.

Example. The rules $\exists_x\varphi, \exists_x\neg\varphi / \perp$ and $\neg(\exists_x\varphi \to \forall_x\varphi)/\psi$ are omitting.

Corollary 1. *Every omitting rule is admissible in* $\mathbf{L}_2$*. The rules from the above example are admissible but not derivable in classical predicate logic.*

Corollary 2. *Classical predicate logic in the standard (structural) formalization is not structurally complete (see [8]). The same holds for all predicate logic weaker then classical predicate logic with the standard rules.*

We will show that for classical predicate logic only omitting rules can be admissible but not derivable. Let tr be a set of formulae of the form: $\exists_x\varphi(x) \to \forall_x\varphi(x)$, $\varphi \in S$, ("quantifiers are triavial").

We define an equivalence relation of Lindenbaum–Tarski $\approx$ between formulae as follows: $\alpha \approx \beta$ iff both $\alpha \to \beta$ and $\beta \to \alpha$ are deducible in $\mathbf{L}_2$. Let $\top^n = (P_k^n(x_1, \ldots, x_n) \to P_k^n(x_1, \ldots, x_k))$ and $\perp^n = \neg(P_k^n(x_1, \ldots, x_n) \to P_k^n(x_1, \ldots, x_k)), n = 1, 2, \ldots$. Observe that the set $\{\top^n, \perp^n\}$ is closed (modulo $\approx$) with respect to $\neg, \wedge, \vee, \to, \forall, \exists$. Also $\top^n \approx \top^m$ and $\perp^n \approx \perp^m$, for all m, n. Hence only abbreviations $\top$ and $\perp$ will be used. This should not lead to a contradiction.

A substitution h is called 1-0-substitution if $h(\beta) \approx \top$ or $h(\beta) \approx \perp$ for any atomic formula β, and h is of the ε-type. Example: $h(\beta) = \beta \to \beta$ or $h(\beta) = \neg(\beta \to \beta)$ are 1-0-substitutions. It can be shown, by induction on the length of a formula α that, for every $\alpha, h(\alpha) \approx \top$ or $h(\alpha) \approx \perp$, for every 1-0- substitution h.

Lemma 1. *Let Cn be a consequence operation of* $\mathbf{L}_2$*. For any rule r let γ be a conjunction of all premises of r. Then the following conditions are equivalent:*

a) *$h(\gamma) \notin L_2$, for every substitution h, i.e. r is omitting,*

b) *$h_0(\gamma) \approx \perp$, for every 1-0-substitution h_0,*

c) *$\neg\gamma \in Cn(tr)$,*

d) *$Cn(tr \cup \{\gamma\}) = S$.*

Proof. Equivalence of a) and b) is straightforward. Now assume b) and let $\neg\gamma \notin Cn(L_2 \cup tr)$. For any formula $\varphi \in S$, let φ^0 be an open formula which is obtained by deleting all signs of quantifiers in φ, and let $X^0 = \{\varphi^0 : \varphi \in X\}$.

Let $\overline{x} = x_1, \ldots, x_n, \overline{y} = y_1, \ldots, y_n$ be arbitrary sequences of individual variables. Now observe that for every $\varphi \in S$:

if $\varphi^0 \in Cn_{MP}(L_2^0 \cup \{\varphi(\overline{x}) \to \varphi(\overline{y}) : \varphi \in S^0$, arbitrary $\overline{x}, \overline{y}\})$, then $\varphi \in Cn(L_2 \cup tr)$, where Cn_{MP} is a consequence operation with Modus Ponens rule only. In particular $\neg\gamma^0 \notin Cn_{MP}(L_2^0 \cup \{\varphi(\overline{x}) \to \varphi(\overline{y}) : \varphi \in S^0$, arbitrary $\overline{x}, \overline{y}\})$. There is a valuation $v : S^0 \to \{0, 1\}$ such that $v(L_2^0 \cup \{\varphi(\overline{x}) \to \varphi(\overline{y}) : \varphi \in S^0$, any $\overline{x}, \overline{y}\}) \subseteq \{1\}$ and $v(\neg\gamma^0) = 0$, i.e. $v(\gamma^0) = 1$. Let $h_{0v} : S \to \{\top, \bot\}$ be a substitution defined as follows:

for atomic formula $P(\overline{x}), h_{0v}(P(\overline{x})) = \top$, if $v(P(\overline{x})) = 1$ and $h_{0v}(P(\overline{x})) = \bot$ otherwise. Then, for every $\varphi \in S$, $v(\varphi^0) = 1$ iff $h_{0v}(\varphi^0) \approx \top$ iff $h_{0v}(\varphi) \approx \top$, because $\{\top, \bot\}$ is closed with respect to quantifiers. Hence $h_{0v}(\gamma) \approx \top$, which contradicts b).

Obviously c) implies d). Now, if b) is not true i.e. there is a 1-0 substitution h_0 such that $h_0(\gamma) \approx \top$ then by d): $h_0(S) \subseteq Cn(h_0(tr \cup \{\gamma\})) \subseteq \{\top\}$, which is impossible. ∎

Remark. A rule r is omitting iff negation of conjunction of its premises is valid in every model with 1-element domain.

Lemma 2. *Let h_0 be a 1-0-substitution, let γ be a sentence (a closed formula) and let us define a substitution h_γ depending on γ and h_0: $h_\gamma(P) = \gamma \wedge P$, if $h_0(P) \approx \bot$ or $h_\gamma(P) = \gamma \to P$, if $h_0(P) \approx \top$, for atomic P. Then $h_\gamma(\alpha) \approx \gamma \wedge \alpha$, if $h_0(\alpha) \approx \bot$ or $h_\gamma(\alpha) \approx \gamma \to \alpha$, if $h_0(\alpha) \approx \top$ for every formula α.*
In particular, $h_\gamma(\gamma) \approx \gamma \to \gamma$, if $h_0(\gamma) \approx \top$ and $h_\gamma(\gamma) \approx \gamma$, otherwise.

Proof. The proof (by induction on the length of a formula α) is based on classical predicate logic. ∎

The following fact is known.

Lemma 3. *Assume that Cn is a consequence operation of a logic* **L**. *Then* **L** $\in SCpl$ *if and only if the following holds*

(∗) If, for every substitution $h, h(\gamma) \in L$ implies $h(\alpha) \in L$, then $\alpha \in Cn(\gamma)$.

Theorem 1. *For every rule r which is not omitting, if r is structural and admissible in classical predicate logic, then r is derivable in it.*

Proof. Let r be a structural and admissible rule which is not omitting, let γ be universal closure of conjunction of its premises and let α be its conclusion.

We have either (a) $h(\gamma) \in \mathrm{L}_2$, for some substitution h, or (b) $h(\gamma) \notin \mathrm{L}_2$ for every substitution h. In case of (b), r is omitting, which is excluded by the assumption. In case of (a) observe that it is enough to show that the condition (∗) in Lemma 3 holds, where Cn is a consequence operation of classical predicate logic $\mathbf{L}_2$. Put in (∗) the substitution h_γ defined in Lemma 2. By Lemma 2 and 3 we have $h_\gamma(\alpha) \in \mathrm{L}_2$.

Now, since $h_\gamma(\alpha) \to (\gamma \to \alpha) \in \mathrm{L}_2$ in both cases of Lemma 2, we have $\gamma \to \alpha \in \mathrm{L}_2$, i.e. $\alpha \in Cn(\gamma)$. ∎

It is known that a logic is structurally complete iff it can not be properly extended without extending the set of its axioms. Moreover, for every logic there is a unique structurally complete extension with the same set of deducible formulae (theorems). This is also true for predicate logics.

Remark. For a given logic **L** we define an operation Σ_L, as follows:
$\alpha \in \Sigma_L(X)$ iff for every substitution $h, h(X) \subseteq L$ implies $h(\alpha) \in L$.

If Cn is a consequence operation of a (structural) logic **L**, then it can be shown that
$\mathbf{L} \in SCpl$ iff $\Sigma_L(X) = Cn(X)$, for every finite X.

Hence, in Th.1 we have proved that $Cn(X) = \Sigma_L(X)$, for every finite X, as long as $X \cup tr$ is consistent, in case of consequence operation of classical predicate logic.

Theorem 2. *A unique structurally complete extension of classical predicate logic can be obtained by adding the rules of the form:*

$$(\varrho_n) \qquad (\exists_{\overline{x}_1}\varphi_1 \wedge \exists_{\overline{x}_1}\neg\varphi_1) \vee \ldots \vee (\exists_{\overline{x}_n}\varphi_n \wedge \exists_{\overline{x}_n}\neg\varphi_n)/\ \bot,$$

for any formulae $\varphi_1, \ldots, \varphi_n$, $n \in \mathbb{N}$, *where* $\overline{x}_i$ *is here a finite sequence of individual variables* $x_{i_1}, \ldots, x_{i_m}$ *and* $\exists_{\overline{x}_i}$ *is an abbreviation of* $\exists_{x_{i_1}} \ldots \exists_{x_{i_m}}$ *(*$\perp$*= universally false sentence).*

Proof. We show that every ommitting rule r: $\gamma_1, \ldots, \gamma_n/\alpha$ is derivable by means of the rules $(\varrho_n), n \in \mathbb{N}$. Let $\gamma = \gamma_1 \wedge \ldots \wedge \gamma_n$. By Lemma 1, a) and c), $\neg\gamma \in Cn(L_2 \cup tr)$. Hence, by deduction theorem, $(\exists_{\overline{x}_1}\varphi_1 \wedge \exists_{\overline{x}_1}\neg\varphi_1) \vee \ldots \vee (\exists_{\overline{x}_n}\varphi_n \wedge \exists_{\overline{x}_n}\neg\varphi_n) \in Cn(\gamma)$, for some $\varphi_1, \ldots, \varphi_n, n \in \mathbb{N}$, and, by (ϱ_n), $\perp \in Cn(\gamma)$. Hence α is deducible from $\gamma_1, \ldots, \gamma_n$ in the logic extended with the rules (ϱ_n). ∎

Remarks

1. The rules (ϱ_n) are used only for deducing inconsistency from premises which can be consitent but are inconsistent with tr.

2. Theorem 2 describes a unique structurally complete extension of $\mathbf{L}_2$ among structural logics; in [8] another, structurally complete but non-structural extension $\mathbf{L}_2^*$ of $\mathbf{L}_2$ is obtained by adding the (non-structural) rule of substitution for predicate variables as an additional rule. Since the rules (ϱ_n) are derivable in $\mathbf{L}_2^*$, by Theorem 2 we have $\mathbf{L}_2^* \in SCpl$.

3. Theorems 1 and 2 are also true for any structural extension of classical logic.

4. There exists a proper structurally complete extension of $\mathbf{L}_2$ which is not complete.

2. Negation-free intermediate predicate logics

An intermediate (between intuitionistic and classical) predicate logic **L** has a standard set of rules. A set L of deducible formulae is closed with respect to Modus Ponens, generalization and substitution for predicate variables, INT $\subseteq$ L $\subseteq$ L_2, where INT is the set of deducible formulae in intuitionistic predicate logic.

For the rest of the paper we deal with negation–free (positive) 1-st order predicate logics i.e. such that the symbol of negation does not occur in the language. Negation-free intermediate predicate logics have been studied in several papers (see e.g. Casari, Minari [1], Minari [5]). In this case, for every set of formulae X there is a substitution h such that $h(X) \subseteq$L$_2$. Hence, there are no ommitting rules. We show that there are more various types of structurally complete predicate logics than in the case of corresponding propositional logics.

A logic **L** is called hereditarily structurally complete, $\mathbf{L} \in HSCpl$, if all its extensions (including **L**) are structurally complete. Let **LW** denote a logic which is an extension positive predicate logic of Hilbert **H** (i.e. negation-free fragment of Intuitionistic predicate logic) with the formulae:

$(\alpha \to \beta) \vee (\beta \to \alpha) = \mathbf{Lin}$, for Linearity and

$(\alpha \to \exists_x \beta) \to \exists_x(\alpha \to \beta) = \mathbf{Well}$, where α does not contain x free, for well founded relation of a Kripke frame,

i.e. $\mathbf{LW} = \mathbf{H} + \mathbf{Lin} + \mathbf{Well}$. Note, that **LW** is an extension of Gödel–Dummett logic.

T. Prucnal [9] used a particular substitution which is useful in proving structural completeness of implicational propositional calculi. We make use of Prucnal's substitution in the case of negation-free predicate logics (cf. [2]).

Lemma 4. (Prucnal's Substitution). *Let h_γ be defined by: $h_\gamma(P) = \gamma \to P$, for every atomic formula P and any fixed closed formula γ. Then $h_\gamma(\alpha) \approx \gamma \to \alpha$, for every formula α, where $\varphi \approx \psi$ iff $\varphi \to \psi$ and $\psi \to \varphi$ are deducible in* **LW**.

Proof. We prove the lemma by induction on α. If α is atomic, this is obvious. Let $\alpha = \beta \otimes \delta$, where $\otimes \in \{\wedge, \vee, \to\}$ or $\alpha = \otimes\beta$, where $\otimes \in \{\forall, \exists\}$ and assume the theorem holds for β and δ. For $\wedge, \to$ and $\forall$ the proof uses intuitionistic logic. For $\vee$ we need:

$(\gamma \to \beta \vee \delta) \to (\gamma \to \beta) \vee (\gamma \to \delta)$, which is equivalent to **Lin**. For $\exists$ we need **Well**. ∎

Theorem 3. *Every negation-free predicate logic which contains* **LW** *is hereditarily structurally complete.*

Proof. Let **L** be a logic extending **LW**. Then, combining Lemma 3 and Prucnal's substitution Lemma we have $\mathbf{L} \in SCpl$. The same proof holds for every extension of **L**. ∎

Let K be a Kripke frame, $K = \langle P, \leq, V \rangle$, where $\langle P, \leq \rangle$ is a poset (of worlds), and V is a domain function; for each world $k \in P, V(k)$ is a domain of a classical structure for predicate logic.

Let $K(\alpha)$ be a Kripke frame based on an ordinal α, with constant domain, i.e. $P = \alpha$ and, for each world $k \in \alpha, V(k) = \max(\omega, \text{card }(\alpha))$.

By $L(\alpha)$ we denote a set of negation-free formulae which are valid in every Kripke frame $K(\alpha)$ and by $\mathbf{L}(\alpha)$ a predicate logic with the set of axioms $L(\alpha)$ and with the standard rules of inference. Since $\mathbf{LW} \subseteq \mathbf{L}(\alpha)$ we have:

Corollary 3. $\mathbf{L}(\alpha)$ *is hereditarily structurally complete, for any ordinal* α.

Theorem 4. *There are descending chains of hereditarily structurally complete predicate logics over* **LW** *with the order type* $\omega^\omega + 1$. *In particular, among hereditarily structurally complete predicate logics extending* **LW***:*
a) there is a chain of type $\omega^\omega + 1$ *of logics which are not finitely axiomatizable,*
b) there is a chain of type $\omega^\omega + 1$ *of logics which are finitely axiomatizable,*
c) there are infinitely many logics which are Kripke incomplete.

Proof. Minari [5] showed that for every ordinal $\xi < \omega^\omega$, there is a negation–free formula W_ξ (defined by induction) such that $W_\xi \in L(\xi) \setminus L(\xi + 1)$. W_ξ is based on combination of instances of a schema $\forall_x((\alpha(x) \to \forall_y \alpha) \to \forall_y \alpha) \to \forall_x \alpha(x)$ and a schema

$$G_n : \ \beta_0 \vee (\beta_0 \to \beta_1) \vee \ldots \vee (\beta_n \to \beta_{n+1}).$$

By Cantor normal form theorem every ordinal $\xi < \omega^\omega$ can be uniquely presented as $\omega^n m_0 + \omega^{n-1} m_1 + \ldots + \omega^0 m_n$, hence there is a $1-1$ mapping between ordinals $\xi < \omega^\omega$ and finite sequences of natural numbers

$[n; m_0, \ldots, m_n]$. It can be shown by multiple induction, cf. [5], that $\mathrm{W}_\xi \in L(\xi)$ and that $\mathrm{W}_\xi \notin L(\xi+1)$.

Hence all the logics $\mathbf{L}\ (\xi), \xi < \omega^\omega$, are distinct and they form a strictly descending chain. It is known that for every infinite ordinal α, the logics $\mathbf{L}\ (\alpha)$ are not recursively axiomatizable, see Skvortsov [12]. This proves a).

For b) take logics $\mathbf{LW} + \mathrm{W}_\xi, \xi < \omega^\omega$.

For c) observe that logic **LW** and all its extensions which do not contain constant domain formula $D : \forall_x(\alpha \vee \beta(x)) \to (\alpha \vee \forall_x \beta(x))$ are Kripke incomplete, since every Kripke frame which validates logic LW has a constant domain. In particular, $\mathbf{LW} + G_n,\ n < \omega$, are Kripke incomplete (cf. [1]). ∎

Conjecture. There is a chain of type $\omega^\omega + 1$ of logics extending **LW**, which are Kripke incomplete.

Remark. We may compare these results with results in corresponding propositional logic.

A propositional logics corresponding to **LW** is a positive fragment of linear logic **LC** of Gödel–Dummentt. Both **LC** and positive fragment of **LC** are (hereditarily) structurally complete, cf. [3].

All extensions of **LC** form a chain of type $\omega + 1$ and all of them are hereditarily structurally complete but extensions of **LW** are much richer. Also a) and c) are in contrast with results on propositional logic: if a propositional intermediate logic **L** is hereditarily structurally complete then **L** is finitely axiomatizable and Kripke complete with respect to a class of finite frames of special type, cf. Rybakov [11].

Some limitations of structural completeness in negation–free intermediate predicate logics are presented below.

Fact. *Positive predicate logic of Hilbert* **H** *is not structurally complete. The following rule:*

$$(**) \qquad ((\forall_y \beta(y) \to \gamma) \to \exists_x \beta(x)) / \exists_x((\forall_y \beta(y) \to \gamma) \to \beta(x)),$$

where γ is a closed, is structural and admissible but not derivable in **H**.

Proof. To prove that the rule $(**)$ is admissible in **H** let us assume that the conclusion of $(**)$ is not in H. By means of Kleene slash $|$, see [4], we have $(\forall_y\beta(y) \to \gamma)|(\forall_y\beta(y) \to \gamma)$, hence, by Kleene result $\exists_x((\forall_y\beta(y) \to \gamma) \to \beta(x))$ is in H, contradiction. On the other hand the rule $(**)$ is not derivable in **H**. The formula $((\forall_y\beta(y) \to \gamma) \to \exists_x\beta(x)) \to \exists_x((\forall_y\beta(y) \to \gamma) \to \beta(x))$ is not **H**–valid. It is false in a topological model (cf. [10]) on the set of real numbers, taking for $\beta(x_n)$ open sets $B_n = (-\infty, 0) \cup (1/n, \infty)$ and for γ, a set $C = (-\infty, -1) \cup (-1, -1/2) \cup (-1/2, -1/3) \cup (-1/3, -1/4) \cup \ldots$. ■

Remarks. Both logics obtained from **LW** by deleting **Lin** or **Well** respectively are not structurally complete i.e. **H** + **Lin** $\notin$ *SCpl* and **H** + **Well** $\notin$ *SCpl*. Hence the structurally complete (positive) propositional logic **LC** has the minimal corresponding predicate logic **H** + **Lin** $\notin$ *SCpl* and a stronger corresponding predicate logic **LW** $\in$ *HSCpl* (predicate logic $\mathbf{L}_1$ corresponds to propositional logic $\mathbf{L}_0$ if propositional schemata in $\mathbf{L}_1$ and $\mathbf{L}_0$ coincide).

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