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AUTOMATING AND COMPUTING PARACONSISTENT REASONING: CONTRACTION-FREE, RESOLUTION AND TYPE SYSTEMS

A b s t r a c t. Firstly, a contraction-free sequent system G4np for Nelson's paraconsistent 4-valued logic N4 is introduced by modifying and extending a contraction-free system G4ip for intuitionistic propositional logic. The structural rule elimination theorem for G4np can be shown by combining Dyckhoff and Negri's result for G4ip and an existing embedding result for N4. Secondly, a resolution system Rnp for N4 is introduced by modifying an intuitionistic resolution system Rip, which is originally introduced by Mints and modified by Troelstra and Schwichtenberg. The equivalence between Rnp and G4np can be shown. Thirdly, a typed λ-calculus for N4 is introduced based on Prawitz's natural deduction system for N4 via the Curry-Howard correspondence. The strong normalization theorem of this calculus can be proved by using Joachimski and Matthes' proof method for intuitionistic typed λ-calculi with premutative conversions.

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1. Introduction

1.1 Nelson's paraconsistent logic

Nelson's paraconsistent 4-valued logic N4 (or equivalently called N^{-}) [1] is a paraconsistent variant of Nelson's constructive logic N with strong negation [11], and is also known as a conservative extension of positive intuitionistic logic. The logic N4 and its versions have been studied by a number of researchers (see e.g. [5, 12, 13, 14, 24]). It is known that N4 and its versions and extensions can appropriately deal with inconsistency-tolerant reasoning [24], reasoning with negative information in logic programming [14], paraconsistent logic programming with inexact predicates [23], and non-monotonic reasoning with answer set programming [15]. For these useful applications, an efficient and simple deduction system is desired as a basis of automated reasoning and computation.

1.2 Sequent systems

A simple cut-free sequent system (called here Gn4) for N4, which is a natural extension of the positive fragment of Gentzen's system LJ for intuitionistic logic, was introduced and studied by López-Escobar [8] and later by Pearce [14] and Wansing [24]. This system is very intuitive and simple, but it is not enough to formalize a basis of automated theorem proving, because the bottom-up proof search procedure based on Gn4 is not very efficient. The first and second aims of the present paper are to give a proof-theoretic foundation for N4-based automated theorem proving in two more efficient frameworks: a contraction-free sequent system and a resolution system.

It is known that there are many cut-free sequent systems for intuitionistic propositional logic, such as LJ and its variants G1ip, G2ip, G3ip, G4ip and G5ip. [1] In particular, the contraction-free system G4ip has the useful feature that certain bottom-up proof search terminates without any loop-detection, and hence G4ip is known as a convenient basis for automated theorem proving. For this reason, some versions of G4ip have been introduced and studied by many logicians (see [2, 21] for a historical overview

[1] The names G1ip, G2ip, G3ip, G4ip and G5ip are from [21].

of G4ip). A direct simple proof of the structural rule elimination theorem (i.e. the theorem for the admissibility of cut and contraction) for G4ip was proved by Dyckhoff and Negri [3], and this result was extended by the same authors to some systems with non-logical axioms for the theories of apartness and order [4].

1.3 Natural deduction systems

As mentioned in [7], there are various natural deduction systems for N4: Prawitz's system [16], Wansing's typed λ-calculus [24], an extension of a system for the logic FDE of first-degree entailment, which was mentioned by Priest [17], an extension of Tamminga and Tanaka's system for FDE [20], a special case of Schroeder-Heister's system with general elimination rules [19], and some extensions of Negri and von Plato's systems for (positive) intuitionistic logic [10]. Some weak normalization results for some such systems were presented in [7]. The open problem of showing the strong normalization theorem of a typed λ-calculus for N4 was suggested by Wansing in Chapter 8 in [24].

The third aim of the present paper is to solve the problem of Wansing in order to obtain a natural computational interpretation for N4. This problem is solved by using the simple proof method by Joachimski and Matthes [6]. It is known that the strong normalization theorem for some typed λ-calculi with disjunction types is somewhat complex to prove because of the reduction rule of permutative conversions. Joachimski and Matthes settled such a complexity problem by obtaining a considerably simple and short proof of the strong normalization theorem for some typed λ-calculi with permutative conversions.

1.4 The contents of this paper

In Section 2, a contraction-free system G4np for (propositional) N4 is introduced by extending the positive fragment of G4ip, and the structural rule elimination theorem for G4np is shown by using an existing embedding result for N4 studied by Vorob'ev [22], Gurevich [5] and Rautenberg [18], and Dyckhoff and Negri's result for G4ip [3]. The equivalence between

G4np and Gn4 is also derived using the structural rule elimination theorem for G4np. Since the result presented is also easily adapted to the multiple succedent version like the system G4ip′ for intuitionistic logic [3], such a result is omitted here. The present result may also be extended to some systems with non-logical axioms like the systems discussed in [4].

In Section 3, a resolution system Rnp for N4 is introduced by modifying an intuitionistic resolution system Rip for intuitionistic propositional logic, and the equivalence between Rnp and an auxiliary system G5np is proved by using Troelstra and Schwichtenberg's method [21]. The system Rip was introduced by Troelstra and Schwichtenberg, and as mentioned in [21], it is regarded as a modification of Mints' original system RIp [9]. The system G5np presented here is regarded as a modified extension of Troelstra and Schwichtenberg's system G5ip for intuitionistic propositional logic.

In Section 4, a typed λ-calculus for N4 is introduced based on Prawitz's natural deduction system for N4 via the Curry-Howard correspondence, and Joachimski and Matthes' method of proving strong normalization is simply adapted to this calculus.

1.5 Preliminaries

Prior to the detailed discussion, the language and basic notations used in this paper are introduced below. The usual propositional language with the strong negation connective $\sim$ and without falsum and truth constants is used in this paper. Greek lower-case letters $\alpha, \beta, \gamma, ...$ are used to denote formulas. Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) *multisets* of formulas. A *sequent* is an expression of the form $\Gamma \Rightarrow \gamma$. If a sequent S is provable in a system L, then such a fact is denoted as $L \vdash S$.

2. Contraction-free system

2.1 G4np

Definition 2.1. [G4np] Let p be an arbitrary propositional variable. The initial sequents of G4np are of the form:

$$p, \Gamma \Rightarrow p \qquad \sim\!p, \Gamma \Rightarrow \sim\!p.$$

The inference rules of G4np are of the form:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma}\ (\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee \beta, \Gamma \Rightarrow \gamma}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right1}) \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right2})$$

$$\frac{p, \beta, \Gamma \Rightarrow \gamma}{p, p\to\beta, \Gamma \Rightarrow \gamma}\ (\to\text{left0}) \quad \frac{\sim\!p, \beta, \Gamma \Rightarrow \gamma}{\sim\!p, \sim\!p\to\beta, \Gamma \Rightarrow \gamma}\ (\sim\to\text{left0})$$

$$\frac{\alpha_1\to(\alpha_2\to\beta), \Gamma \Rightarrow \gamma}{(\alpha_1 \wedge \alpha_2)\to\beta, \Gamma \Rightarrow \gamma}\ (\wedge\to\text{left}) \quad \frac{\alpha_1\to\beta, \alpha_2\to\beta, \Gamma \Rightarrow \gamma}{(\alpha_1 \vee \alpha_2)\to\beta, \Gamma \Rightarrow \gamma}\ (\vee\to\text{left})$$

$$\frac{\alpha_1, \alpha_2\to\beta, \Gamma \Rightarrow \alpha_2 \quad \beta, \Gamma \Rightarrow \gamma}{(\alpha_1\to\alpha_2)\to\beta, \Gamma \Rightarrow \gamma}\ (\to\to\text{left}) \quad \frac{\sim\!\alpha_1\to\beta, \sim\!\alpha_2\to\beta, \Gamma \Rightarrow \gamma}{\sim\!(\alpha_1 \wedge \alpha_2)\to\beta, \Gamma \Rightarrow \gamma}\ (\sim\wedge\to\text{left})$$

$$\frac{\sim\!\alpha_1\to(\sim\!\alpha_2\to\beta), \Gamma \Rightarrow \gamma}{\sim\!(\alpha_1 \vee \alpha_2)\to\beta, \Gamma \Rightarrow \gamma}\ (\sim\vee\to\text{left}) \quad \frac{\alpha_1\to(\sim\!\alpha_2\to\beta), \Gamma \Rightarrow \gamma}{\sim\!(\alpha_1\to\alpha_2)\to\beta, \Gamma \Rightarrow \gamma}\ (\sim\to\to\text{left})$$

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\to\beta}\ (\to\text{right}) \quad \frac{\alpha, \Gamma \Rightarrow \gamma}{\sim\sim\!\alpha, \Gamma \Rightarrow \gamma}\ (\sim\text{left}) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim\sim\!\alpha}\ (\sim\text{right})$$

$$\frac{\sim\!\alpha, \Gamma \Rightarrow \gamma \quad \sim\!\beta, \Gamma \Rightarrow \gamma}{\sim\!(\alpha \wedge \beta), \Gamma \Rightarrow \gamma}\ (\sim\wedge\text{left})$$

$$\frac{\Gamma \Rightarrow \sim\!\alpha}{\Gamma \Rightarrow \sim\!(\alpha \wedge \beta)}\ (\sim\wedge\text{right1}) \quad \frac{\Gamma \Rightarrow \sim\!\beta}{\Gamma \Rightarrow \sim\!(\alpha \wedge \beta)}\ (\sim\wedge\text{right2})$$

$$\frac{\sim\!\alpha, \sim\!\beta, \Gamma \Rightarrow \gamma}{\sim\!(\alpha \vee \beta), \Gamma \Rightarrow \gamma}\ (\sim\vee\text{left}) \quad \frac{\Gamma \Rightarrow \sim\!\alpha \quad \Gamma \Rightarrow \sim\!\beta}{\Gamma \Rightarrow \sim\!(\alpha \vee \beta)}\ (\sim\vee\text{right})$$

$$\frac{\alpha, \sim\!\beta, \Gamma \Rightarrow \gamma}{\sim\!(\alpha\to\beta), \Gamma \Rightarrow \gamma}\ (\sim\to\text{left}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \sim\!\beta}{\Gamma \Rightarrow \sim\!(\alpha\to\beta)}\ (\sim\to\text{right}).$$

The $\sim$-free part of G4np is called here G4ip$^{\perp}$, which is the $\perp$-free fragment of G4ip. [2]

Roughly speaking, the rules ($\to$left0), ($\sim\to$left0), ($\wedge\to$left), ($\vee\to$left), ($\to\to$left), ($\sim\wedge\to$left), ($\sim\vee\to$left) and ($\sim\to\to$left) can be regarded as some divided versions of the rule:

$$\frac{\alpha\to\beta, \Gamma \Rightarrow \alpha \quad \beta, \Gamma \Rightarrow \gamma}{\alpha\to\beta, \Gamma \Rightarrow \gamma},$$

[2] The name G4ip is from [21]. For a historical overview of G4ip, see [2, 21].

where α is divided into p, $\sim p$, $\alpha_1 \wedge \alpha_2$, $\alpha_1 \vee \alpha_2$, $\alpha_1 \rightarrow \alpha_2$, $\sim(\alpha_1 \wedge \alpha_2)$, $\sim(\alpha_1 \vee \alpha_2)$ and $\sim(\alpha_1 \rightarrow \alpha_2)$. In the rule just displayed above, the principal formula $\alpha \rightarrow \beta$ appears twice, i.e. in one of the upper sequents and in the lower sequent. Such occurences of $\alpha \rightarrow \beta$ derive some inefficient proof search procedures with *loops*. Since G4np is loop-free, it is regarded as efficent.

2.2 Admissibility of structural rules

Definition 2.2. [Structural rule] The following strucutral rules are called *cut, contraction* and *weakening*, respectively:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma} \text{(cut)} \quad \frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \text{(co)} \quad \frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \text{(we)}.$$

In order to prove the structural rule elimination theorem for G4np, we give an embedding f of G4np into G4ip$^{\perp}$, which was studied in [5, 18, 22].

Definition 2.3. We fix a set Φ of propositional variables, used as a component of the language using $\sim$, and define the set $\Phi' := \{p' \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}^{\sim}$ is defined by using Φ, $\rightarrow$, $\wedge$, $\vee$ and $\sim$. The language $\mathcal{L}$ is obtained from $\mathcal{L}^{\sim}$ by adding Φ' and by deleting $\sim$.

A mapping f from $\mathcal{L}^{\sim}$ to $\mathcal{L}$ is defined as follows.

1. $f(p) := p$ and $f(\sim p) := p' \in \Phi'$ for any $p \in \Phi$,
2. $f(\alpha \circ \beta) := f(\alpha) \circ f(\beta)$ where $\circ \in \{\rightarrow, \wedge, \vee\}$,
3. $f(\sim\sim\alpha) := f(\alpha)$,
4. $f(\sim(\alpha \rightarrow \beta)) := f(\alpha) \wedge f(\sim\beta)$,
5. $f(\sim(\alpha \wedge \beta)) := f(\sim\alpha) \vee f(\sim\beta)$,
6. $f(\sim(\alpha \vee \beta)) := f(\sim\alpha) \wedge f(\sim\beta)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

We can easily prove:

Lemma 2.4 (Key lemma)**.** *Let* Γ *be a multiset of formulas in* $\mathcal{L}^{\sim}$*,* γ *be a formula in* $\mathcal{L}^{\sim}$*,* R *be the set* {(cut), (co), (we)}*, and* f *be the mapping defined in Definition 2.3.*

(1) *If* G4np + R $\vdash \Gamma \Rightarrow \gamma$*, then* G4ip$^\perp$ + R $\vdash f(\Gamma) \Rightarrow f(\gamma)$.

(2) *If* G4ip$^\perp$ $\vdash f(\Gamma) \Rightarrow f(\gamma)$*, then* G4np $\vdash \Gamma \Rightarrow \gamma$.

Using Lemma 2.4, we can prove:

Theorem 2.5 (Structural rule elimination theorem for G4np)**.** *The rules* (cut)*,* (co) *and* (we) *are admissible in* G4np*.*

Proof. Suppose G4np + {(cut), (co), (we)} $\vdash \Gamma \Rightarrow \gamma$. Then, we have G4ip$^\perp$ + {(cut), (co), (we)} $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by Lemma 2.4 (1), and hence G4ip$^\perp \vdash f(\Gamma) \Rightarrow f(\gamma)$ by the structural rule elimination theorem for G4ip$^\perp$, which was directly proved by Dyckhoff and Negri [3]. [3] By Lemma 2.4 (2), we obtain the required fact: G4np $\vdash \Gamma \Rightarrow \gamma$. □

2.3 Equivalence between systems

We have introduced a strucutral rule free system G4np, but we have not yet shown that G4np is a system for N4. We then show the equivalence between G4np and the cut-free system called here Gn4 which was introduced and discussed in [8, 14, 24].

Definition 2.6. [Gn4 [8, 14, 24]] The sequent system Gn4 for N4 is obtained from G4np by deleting the initial sequents and the rules ($\rightarrow$left0), ($\sim\rightarrow$left0), ($\wedge\rightarrow$left), ($\vee\rightarrow$left), ($\rightarrow\rightarrow$left), ($\sim\wedge\rightarrow$left), ($\sim\vee\rightarrow$left) and ($\sim\rightarrow\rightarrow$left), and by adding the initial sequents of the form $\alpha \Rightarrow \alpha$, the rules (cut), (co), (we) and the rule of the form:

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Gamma \Rightarrow \gamma}{\alpha\rightarrow\beta, \Gamma \Rightarrow \gamma}\ (\rightarrow\text{left}).$$

Then we can prove:

Theorem 2.7 (Equivalence between Gn4 and G4np)**.** *For any sequent* S*,* S *is provable in* G4np *if and only if it is provable in* Gn4*.*

Proof. ($\Longrightarrow$): We can straightforwardly prove that if a sequent S is provable in G4np then it is provable in Gn4. This is proved by induction

[3]Strictly speaking, it is shown in [3] that (cut), (co) and (we) are admissible in G4ip. By this fact and the conservativity w.r.t. $\perp$, the same fact holds for G4ip$^\perp$.

on the proof P of S in G4np. We distinguish the cases according to the last inference in P. We only illustrate the case that the last inference in P is of the form:

$$\dfrac{\alpha_1, \alpha_2{\rightarrow}\beta, \Gamma \Rightarrow \alpha_2 \quad \beta, \Gamma \Rightarrow \gamma}{(\alpha_1{\rightarrow}\alpha_2){\rightarrow}\beta, \Gamma \Rightarrow \gamma}\ ({\rightarrow}{\rightarrow}\mathrm{left}).$$

By the hypothesis of induction, both $\alpha_1, \alpha_2{\rightarrow}\beta, \Gamma \Rightarrow \alpha_2$ and $\beta, \Gamma \Rightarrow \gamma$ are provable in Gn4. Then, we have the required proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha_2 \Rightarrow \alpha_2}{\alpha_1, \alpha_2 \Rightarrow \alpha_2}\,(\mathrm{we})}{\alpha_2 \Rightarrow \alpha_1{\rightarrow}\alpha_2} \quad \dfrac{\beta \Rightarrow \beta}{\beta, \alpha_2 \Rightarrow \beta}\,(\mathrm{we})}{(\alpha_1{\rightarrow}\alpha_2){\rightarrow}\beta, \alpha_2 \Rightarrow \beta}}{(\alpha_1{\rightarrow}\alpha_2){\rightarrow}\beta \Rightarrow \alpha_2{\rightarrow}\beta} \quad \dfrac{\alpha_1, \alpha_2{\rightarrow}\beta, \Gamma \Rightarrow \alpha_2}{\alpha_2{\rightarrow}\beta, \Gamma \Rightarrow \alpha_1{\rightarrow}\alpha_2}}{(\alpha_1{\rightarrow}\alpha_2){\rightarrow}\beta, \Gamma \Rightarrow \alpha_1{\rightarrow}\alpha_2}\,(\mathrm{cut}) \quad \dfrac{\beta, \Gamma \Rightarrow \gamma}{\beta, (\alpha_1{\rightarrow}\alpha_2){\rightarrow}\beta, \Gamma \Rightarrow \gamma}\,(\mathrm{we})}{\dfrac{(\alpha_1{\rightarrow}\alpha_2){\rightarrow}\beta, (\alpha_1{\rightarrow}\alpha_2){\rightarrow}\beta, \Gamma \Rightarrow \gamma}{(\alpha_1{\rightarrow}\alpha_2){\rightarrow}\beta, \Gamma \Rightarrow \gamma}\,(\mathrm{co}).}$$

($\Longleftarrow$): We prove that if a sequent S is provable in Gn4 then it is provable in G4np. This is proved by induction on the proof P of S in Gn4. We distinguish the cases according to the last inference of P. We show some cases. The cases that the last inference of P is (cut), (co) or (we) can be shown by Theorem 2.5. The case that the last inference of P is (→left) can be proved using the fact that (→left) is admissible in G4np. This fact can be shown in the same way as the proof of Theorem 2.5: we use the result by Dyckhoff and Negri [3] [4] that (→left) is admissible in G4ip, and prove a similar lemma as Lemma 2.4 with respect to (→left) using the embedding f of Definition 2.3. □

3. Resolution system

3.1 Rnp

For the multiset with multiplicity one which is obtained from a multiset Γ, we write (Γ), i.e. the multiset (Γ) contains the formulas of Γ with multiplicity one. For example, if Γ is the multiset $\{\alpha, \alpha, \beta\}$, then (Γ) is the multiset $\{\alpha, \beta\}$.

[4] Lemma 4.1 in page 1503 in [3].

Definition 3.1. [Intuitionistic clause] A formula is called a *literal* if it is an atomic formula or a negated atomic formula. A sequent is called an *intuitionistic clause* if it is one of the following forms:

$$(P \rightarrow Q) \Rightarrow R, \qquad P \Rightarrow (Q \vee R), \qquad P_1, ..., P_n \Rightarrow Q$$

where $P, Q, R, P_1, ..., P_n$ represent literals, and n can be 0.

Definition 3.2. [Rnp] Let P, Q, R, S be literals and all the sequents of Rnp be intuitionistic clauses.

The axioms of Rnp are of the form:

$$P, \Delta \Rightarrow P.$$

The inference rules of Rnp are of the form:

$$\frac{\Gamma \Rightarrow P \quad P, \Delta \Rightarrow Q}{(\Gamma, \Delta) \Rightarrow Q} \text{ (resol)}$$

$$\frac{P \Rightarrow Q \vee R \quad \Gamma \Rightarrow P \quad Q, \Delta \Rightarrow S \quad R, \Sigma \Rightarrow S}{(\Gamma, \Delta, \Sigma) \Rightarrow S} \text{ ($\vee$resol)}$$

$$\frac{(P \rightarrow Q) \Rightarrow R \quad [P], \Delta \Rightarrow Q}{\Delta \Rightarrow R} \text{ ($\rightarrow$resol)}$$

where $[P]$ represents P or the empty multiset.

It is remarked that Rnp is a modification of the system Rip (for intuitionistic logic) introduced in [21]. In Rip, the formulas P, Q, R, S, which are used as literals in Rnp, are atomic formulas. As mentioned in [21], Rip was based on Mints' framework in [9]. It is also remarked that the axioms in Rip are of the forms $P \Rightarrow P$ and $\bot \Rightarrow P$, but the axioms in Rnp are of the form $P, \Delta \Rightarrow P$.

3.2 G5np

Definition 3.3. [G5np] The initial sequents of G5np are of the form:

$$\alpha, \Gamma \Rightarrow \alpha.$$

The inference rules of G5np are of the form:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{(\Gamma, \Delta) \Rightarrow \gamma} \text{ (cut}*\text{)}$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{(\alpha \wedge \beta, \Gamma) \Rightarrow \gamma}\ (\wedge\text{left1}*) \quad \frac{\beta, \Gamma \Rightarrow \gamma}{(\alpha \wedge \beta, \Gamma) \Rightarrow \gamma}\ (\wedge\text{left2}*)$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{(\alpha \wedge \beta, \Gamma) \Rightarrow \gamma}\ (\wedge\text{left3}*) \quad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{(\Gamma, \Delta) \Rightarrow \alpha \wedge \beta}\ (\wedge\text{right}*)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Delta \Rightarrow \gamma}{(\alpha \vee \beta, \Gamma, \Delta) \Rightarrow \gamma}\ (\vee\text{left}*) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right1}*) \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right2}*)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{(\alpha \to \beta, \Gamma, \Delta) \Rightarrow \gamma}\ (\to\text{left}*) \quad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\to\text{right1}*) \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\to\text{right2}*)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{(\sim\sim\alpha, \Gamma) \Rightarrow \gamma}\ (\sim\text{left}*) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim\sim\alpha}\ (\sim\text{right}*)$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \gamma \quad \sim\beta, \Delta \Rightarrow \gamma}{(\sim(\alpha \wedge \beta), \Gamma, \Delta) \Rightarrow \gamma}\ (\sim\wedge\text{left}*)$$

$$\frac{\Gamma \Rightarrow \sim\alpha}{\Gamma \Rightarrow \sim(\alpha \wedge \beta)}\ (\sim\wedge\text{right1}*) \quad \frac{\Gamma \Rightarrow \sim\beta}{\Gamma \Rightarrow \sim(\alpha \wedge \beta)}\ (\sim\wedge\text{right2}*)$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \gamma}{(\sim(\alpha \vee \beta), \Gamma) \Rightarrow \gamma}\ (\sim\vee\text{left1}*) \quad \frac{\sim\beta, \Gamma \Rightarrow \gamma}{(\sim(\alpha \vee \beta), \Gamma) \Rightarrow \gamma}\ (\sim\vee\text{left2}*)$$

$$\frac{\sim\alpha, \sim\beta, \Gamma \Rightarrow \gamma}{(\sim(\alpha \vee \beta), \Gamma) \Rightarrow \gamma}\ (\sim\vee\text{left3}*) \quad \frac{\Gamma \Rightarrow \sim\alpha \quad \Delta \Rightarrow \sim\beta}{(\Gamma, \Delta) \Rightarrow \sim(\alpha \vee \beta)}\ (\sim\vee\text{right}*)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{(\sim(\alpha \to \beta), \Gamma) \Rightarrow \gamma}\ (\sim\to\text{left1}*) \quad \frac{\sim\beta, \Gamma \Rightarrow \gamma}{(\sim(\alpha \to \beta), \Gamma) \Rightarrow \gamma}\ (\sim\to\text{left2}*)$$

$$\frac{\alpha, \sim\beta, \Gamma \Rightarrow \gamma}{(\sim(\alpha \to \beta), \Gamma) \Rightarrow \gamma}\ (\sim\to\text{left3}*) \quad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \sim\beta}{(\Gamma, \Delta) \Rightarrow \sim(\alpha \to \beta)}\ (\sim\to\text{right}*).$$

It is remarked that G5np is a modified extension of the system G5ip which is introduced by Troelstra and Schwichtenberg [21] in order to give the equivalence between Rip and G5ip. It is also remarked that the initial sequents of G5np are different from G5ip: G5ip has the initial sequents of the forms $\alpha \Rightarrow \alpha$ and $\bot \Rightarrow \alpha$, but G5np has the initial sequents of the form $\alpha, \Gamma \Rightarrow \alpha$.

Proposition 3.4 (Equivalence between G5np and G4np)**.** (1) *If* G5np $\vdash \Gamma \Rightarrow \gamma$, *then* G4np $\vdash \Gamma \Rightarrow \gamma$. (2) *If* G4np $\vdash \Gamma \Rightarrow \gamma$, *then* G5np $\vdash \Gamma' \Rightarrow \gamma$ *for some* $\Gamma' \subseteq \Gamma$.

3.3 Equivalence between systems

An expression l_α, called the label of a formula α, means a literal which corresponds to a formula α. For a multiset $\Gamma \equiv \{\gamma_1, ..., \gamma_n\}$ of formulas, l_Γ means the multiset $\{l_{\gamma_1}, ..., l_{\gamma_n}\}$. For any atomic formula p, l_p and $l_{\sim p}$ are defined by p and $\sim p$, respectively. For any compound formula γ, the interpretation of l_γ is considered below.

Definition 3.5. For any formulas α and β, we define

$$
\begin{aligned}
C_{\alpha\wedge\beta} &= \{\ l_{\alpha\wedge\beta} \Rightarrow l_\alpha; \quad l_{\alpha\wedge\beta} \Rightarrow l_\beta; \quad l_\alpha, l_\beta \Rightarrow l_{\alpha\wedge\beta}\} \\
C_{\alpha\vee\beta} &= \{\ l_{\alpha\vee\beta} \Rightarrow l_\alpha \vee l_\beta; \quad l_\alpha \Rightarrow l_{\alpha\vee\beta}; \quad l_\beta \Rightarrow l_{\alpha\vee\beta}\} \\
C_{\alpha\to\beta} &= \{\ l_{\alpha\to\beta}, l_\alpha \Rightarrow l_\beta; \quad (l_\alpha \to l_\beta) \Rightarrow l_{\alpha\to\beta}\} \\
C_{\sim\sim\alpha} &= \{\ l_{\sim\sim\alpha} \Rightarrow l_\alpha; \quad l_\alpha \Rightarrow l_{\sim\sim\alpha}\} \\
C_{\sim(\alpha\wedge\beta)} &= \{\ l_{\sim(\alpha\wedge\beta)} \Rightarrow l_{\sim\alpha} \vee l_{\sim\beta}; \quad l_{\sim\alpha} \Rightarrow l_{\sim(\alpha\wedge\beta)}; \\
&\qquad\qquad l_{\sim\beta} \Rightarrow l_{\sim(\alpha\wedge\beta)}\} \\
C_{\sim(\alpha\vee\beta)} &= \{\ l_{\sim(\alpha\vee\beta)} \Rightarrow l_{\sim\alpha}; \quad l_{\sim(\alpha\vee\beta)} \Rightarrow l_{\sim\beta}; \\
&\qquad\qquad l_{\sim\alpha}, l_{\sim\beta} \Rightarrow l_{\sim(\alpha\vee\beta)}\} \\
C_{\sim(\alpha\to\beta)} &= \{\ l_{\sim(\alpha\to\beta)} \Rightarrow l_\alpha; \quad l_{\sim(\alpha\to\beta)} \Rightarrow l_{\sim\beta}; \\
&\qquad\qquad l_\alpha, l_{\sim\beta} \Rightarrow l_{\sim(\alpha\to\beta)}\}
\end{aligned}
$$

An expression $Nsub(\alpha)$ represents the set of all non-literal subformulas and non-literal negated-subformulas of a formula α. Let γ be an arbitrary non-literal formula. Then, we define

$$Cl(\gamma) = \bigcup\{C_\delta \mid \delta \in Nsub(\gamma)\}.$$

Let Γ be a set $\{\gamma_1, ..., \gamma_n\}$ $(n \geq 2)$ of non-literal formulas. Then, we define

$$Cl(\Gamma) = Cl(\gamma_1) \cup \cdots \cup Cl(\gamma_n).$$

It is remarked that $Cl(\gamma)$ is a set of intuitionistic clauses. Suppose that an expression $\alpha \leftrightarrow \beta$ means $\vdash \alpha \Rightarrow \beta$ and $\vdash \beta \Rightarrow \alpha$. Assuming $Cl(\gamma)$ is intended to address the intrepretation of the label expression l_γ as $l_p \leftrightarrow p$, $l_{\sim p} \leftrightarrow {\sim}p$, $l_{\alpha\wedge\beta} \leftrightarrow (l_\alpha \wedge l_\beta)$, $l_{\alpha\vee\beta} \leftrightarrow (l_\alpha \vee l_\beta)$, $l_{\alpha\to\beta} \leftrightarrow (l_\alpha \to l_\beta)$, $l_{\sim\sim\alpha} \leftrightarrow l_\alpha$, $l_{\sim(\alpha\wedge\beta)} \leftrightarrow (l_{\sim\alpha} \vee l_{\sim\beta})$, $l_{\sim(\alpha\vee\beta)} \leftrightarrow (l_{\sim\alpha} \wedge l_{\sim\beta})$ and $l_{\sim(\alpha\to\beta)} \leftrightarrow (l_\alpha \wedge l_{\sim\beta})$. Such an interpretation is regarded as the label version of the embedding [5, 18, 22] discussed in the previous section.

Lemma 3.6. *If* G5np + $Cl(\gamma) \vdash\ \Rightarrow l_\gamma$, *then* G5np $\vdash\ \Rightarrow \gamma$.

Proof. Suppose G5np + $Cl(\gamma) \vdash\ \Rightarrow l_\gamma$. Let P be a proof of $\Rightarrow l_\gamma$ in G5np + $Cl(\gamma)$. If we substitute α for all the labels l_α everywhere in P, then l_γ becomes γ and all the sequents in $Cl(\gamma)$ appearing in P become G5np-provable sequents. Therefore G5np $\vdash\ \Rightarrow \gamma$. $\square$

Lemma 3.7. *If* G5np $\vdash \Gamma \Rightarrow \gamma$, *then* Rnp + $Cl(\Gamma, \gamma) \vdash l_\Gamma \Rightarrow l_\gamma$.

Proof. By induction on a proof P of $\Gamma \Rightarrow \gamma$ in G5np. We distinguish the cases according to the last inference of P. We show some cases.

Case ($\sim\vee$left3$*$): The last inference of P is of the form:

$$\frac{\sim\alpha, \sim\beta, \Sigma \Rightarrow \gamma}{(\sim(\alpha \vee \beta), \Sigma) \Rightarrow \gamma}\ (\sim \vee \text{left3}*).$$

By the hypothesis of induction, we have

$$\text{Rnp} + Cl(\sim\alpha, \sim\beta, \Sigma, \gamma) \vdash l_{\sim\alpha}, l_{\sim\beta}, l_{\Sigma} \Rightarrow l_{\gamma}.$$

We then obtain:

$$\frac{l_{\sim(\alpha\vee\beta)} \Rightarrow l_{\sim\beta} \quad \dfrac{l_{\sim(\alpha\vee\beta)} \Rightarrow l_{\sim\alpha} \quad l_{\sim\alpha}, l_{\sim\beta}, l_{\Sigma} \Rightarrow l_{\gamma}}{(l_{\sim(\alpha\vee\beta)}, l_{\sim\beta}, l_{\Sigma}) \Rightarrow l_{\gamma}}\ (\text{resol})}{(l_{\sim(\alpha\vee\beta)}, l_{\Sigma}) \Rightarrow l_{\gamma}}\ (\text{resol}).$$

Since $\sim\alpha, \sim\beta \in Nsub(\sim(\alpha \vee \beta))$, we have Rnp $+$ $Cl(\sim(\alpha \vee \beta), \Sigma, \gamma) \vdash l_{\sim(\alpha\vee\beta)}, l_{\Sigma} \Rightarrow l_{\gamma}$.

Case ($\sim\vee$right$*$): The last inference of P is of the form:

$$\frac{\Sigma \Rightarrow \sim\alpha \quad \Delta \Rightarrow \sim\beta}{(\Sigma, \Delta) \Rightarrow \sim(\alpha \vee \beta)}\ (\sim \vee \text{right}*).$$

By the hypothesis of induction, we have Rnp $+$ $Cl(\Sigma, \sim\alpha) \vdash l_{\Sigma} \Rightarrow l_{\sim\alpha}$ and Rnp $+$ $Cl(\Delta, \sim\beta) \vdash l_{\Delta} \Rightarrow l_{\sim\beta}$. We then obtain:

$$\frac{l_{\Delta} \Rightarrow l_{\sim\beta} \quad \dfrac{l_{\Sigma} \Rightarrow l_{\sim\alpha} \quad l_{\sim\alpha}, l_{\sim\beta} \Rightarrow l_{\sim(\alpha\vee\beta)}}{(l_{\Sigma}, l_{\sim\beta}) \Rightarrow l_{\sim(\alpha\vee\beta)}}\ (\text{resol})}{(l_{\Sigma}, l_{\Delta}) \Rightarrow l_{\sim(\alpha\vee\beta)}}\ (\text{resol}).$$

Case ($\sim\wedge$left$*$): The last inference of P is of the form:

$$\frac{\sim\alpha, \Sigma \Rightarrow \gamma \quad \sim\beta, \Delta \Rightarrow \gamma}{(\sim(\alpha \wedge \beta), \Sigma, \Delta) \Rightarrow \gamma}\ (\sim \wedge \text{left}*).$$

By the hypothesis of induction, we have Rnp $+$ $Cl(\sim\alpha, \Sigma, \gamma) \vdash l_{\Sigma}, l_{\sim\alpha} \Rightarrow l_{\gamma}$ and Rnp $+$ $Cl(\sim\beta, \Delta, \gamma) \vdash l_{\Delta}, l_{\sim\beta} \Rightarrow l_{\gamma}$. We then obtain:

$$\frac{l_{\sim(\alpha\wedge\beta)} \Rightarrow l_{\sim\alpha} \vee l_{\sim\beta} \quad l_{\sim(\alpha\wedge\beta)} \Rightarrow l_{\sim(\alpha\wedge\beta)} \quad l_{\Sigma}, l_{\sim\alpha} \Rightarrow l_{\gamma} \quad l_{\Delta}, l_{\sim\beta} \Rightarrow l_{\gamma}}{(l_{\sim(\alpha\wedge\beta)}, l_{\Sigma}, l_{\Delta}) \Rightarrow l_{\gamma}}\ (\vee\text{resol}).$$

Case ($\rightarrow$right1$*$): The last inference of P is of the form:

$$\frac{\alpha, \Delta \Rightarrow \beta}{\Delta \Rightarrow \alpha \rightarrow \beta}\ (\rightarrow\text{right1}*).$$

By the hypothesis of induction, we have Rnp + $Cl(\alpha, \Delta, \beta) \vdash l_\alpha, l_\Delta \Rightarrow l_\beta$. We then obtain:

$$\frac{l_\alpha \rightarrow l_\beta \Rightarrow l_{\alpha\rightarrow\beta} \quad l_\alpha, l_\Delta \Rightarrow l_\beta}{l_\Delta \Rightarrow l_{\alpha\rightarrow\beta}}\ (\rightarrow\text{resol}).$$

Case ($\rightarrow$left$*$): The last inference of P is of the form:

$$\frac{\Sigma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{(\alpha\rightarrow\beta, \Sigma, \Delta) \Rightarrow \gamma}\ (\rightarrow\text{left}*).$$

By the hypothesis of induction, we have Rnp + $Cl(\Sigma, \alpha) \vdash l_\Sigma \Rightarrow l_\alpha$ and Rnp + $Cl(\beta, \Delta, \gamma) \vdash l_\beta, l_\Delta \Rightarrow l_\gamma$. We then obtain:

$$\frac{l_\Sigma \Rightarrow l_\alpha \quad \dfrac{l_{\alpha\rightarrow\beta}, l_\alpha \Rightarrow l_\beta \quad l_\beta, l_\Delta \Rightarrow l_\gamma}{(l_{\alpha\rightarrow\beta}, l_\alpha, l_\Delta) \Rightarrow l_\gamma}\ (\text{resol})}{(l_{\alpha\rightarrow\beta}, l_\Sigma, l_\Delta) \Rightarrow l_\gamma}\ (\text{resol}).$$

□

Lemma 3.8. *For any intuitionistic clause* $\Gamma \Rightarrow P$*, if* Rnp $\vdash \Gamma \Rightarrow P$*, then* G5np $\vdash \Gamma \Rightarrow P$.

Proof. By induction on a proof of $\Gamma \Rightarrow P$ in Rnp. □

Theorem 3.9 (Equivalence between Rnp and G5np)**.** G5np $\vdash\ \Rightarrow \gamma$ *if and only if* Rnp + $Cl(\gamma) \vdash\ \Rightarrow l_\gamma$.

Proof. ($\Longrightarrow$): By Lemma 3.7.
($\Longleftarrow$): Suppose Rnp + $Cl(\gamma) \vdash\ \Rightarrow l_\gamma$. By Lemma 3.8 with an obvious generalization, we have G5np + $Cl(\gamma) \vdash\ \Rightarrow l_\gamma$. We thus obtain G5np $\vdash$ $\Rightarrow \gamma$ by Lemma 3.6. □

4. Type system

Prawitz's natural deduction system for N4, which was presented in [16] [5], is obtained from the usual natural deduction system for positive intuitionistic logic by adding the following inference rules with respect to the strong negation connective $\sim$:

$$\frac{\alpha}{\sim\sim\alpha} \qquad \frac{\sim\sim\alpha}{\alpha} \qquad \frac{\alpha \quad \sim\beta}{\sim(\alpha\to\beta)} \qquad \frac{\sim(\alpha\to\beta)}{\alpha} \qquad \frac{\sim(\alpha\to\beta)}{\sim\beta}$$

$$\frac{\sim\alpha}{\sim(\alpha\wedge\beta)} \qquad \frac{\sim\beta}{\sim(\alpha\wedge\beta)} \qquad \frac{\sim(\alpha\wedge\beta) \quad \begin{matrix}[\sim\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\sim\beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma}$$

$$\frac{\sim\alpha \quad \sim\beta}{\sim(\alpha\vee\beta)} \qquad \frac{\sim(\alpha\vee\beta)}{\sim\alpha} \qquad \frac{\sim(\alpha\vee\beta)}{\sim\beta}$$

which correspond to the Hilbert-style axiom schemes $\sim\sim\alpha \leftrightarrow \alpha$, $\sim(\alpha\to\beta) \leftrightarrow \alpha\wedge\sim\beta$, $\sim(\alpha\wedge\beta) \leftrightarrow \sim\alpha\vee\sim\beta$ and $\sim(\alpha\vee\beta) \leftrightarrow \sim\alpha\wedge\sim\beta$.

In this section, a new typed λ-calculus for N4 is introduced based on Prawitz's system via the Curry-Howard correspondence. The calculus presented is constructed using two new constructors neg and neg^{-1} in order to interprete the natural deduction rules that correspond to $\sim\sim\alpha \leftrightarrow \alpha$. For the other rules in the natural deduction, we do not have to introduce any new constructors, since $\sim(\alpha\to\beta)$ and $\sim(\alpha\vee\beta)$ are interpreted as the conjunction of α (or $\sim\alpha$) and $\sim\beta$, and also $\sim(\alpha\wedge\beta)$ is interpreted as the disjunction of $\sim\alpha$ and $\sim\beta$. The resulting system is thus a natural extension of the usual typed λ-calculus for positive intuitionistic logic, but has the non-standard consequence of non-unique typedness of terms.

Definition 4.1. Let always $i \in \{0,1\}$. *Terms* $r, s, t, ...$ are inductively defined using variables $x, y, x_0, x_1, ...$ by the following grammar:

$$r ::= x \mid rr \mid \lambda yr \mid \mathrm{inj}_i\, r \mid r(x_0.r, x_1.r) \mid \langle r, r\rangle \mid r\, \mathrm{proj}_i \mid \mathrm{neg}\, r \mid r\, \mathrm{neg}^{-1}.$$

The variables y, x_0 and x_1 get bound in λyr and $r(x_0.r, x_1.r)$, respectively.

[5] Strictly speaking, Prawitz introduced a natural deduction system for Nelson's constructive logic N with strong negation [11]. Since N4 is a sublogic of N, such a natural deduction system can also be adapted for N4.

An expression $r\vec{s}$ with a possibly empty vector $\vec{s}$ is used for $rs_1 \cdots s_n \equiv (\cdots(rs_1)\cdots s_n)$. The *substitution* of a term s for a variable x is defined as usual and denoted as $r_x[s]$ or $r[s/x]$. It is assumed that α-equal terms are equal.

Definition 4.2. *Types* α, β, γ ... are inductively defined by the following grammar from basic types $b, c, d, ...$:

$$\alpha ::= b \mid \alpha{\to}\alpha \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid {\sim}\alpha.$$

Definition 4.3. For an assignment $x : \alpha$ of a type α to a variable x, the following rules give the typable terms and their types:

$$\frac{r : \alpha{\to}\beta \quad s : \alpha}{rs : \beta}\ ({\to}E) \qquad \frac{x : \alpha \quad r : \beta}{\lambda xr : \alpha{\to}\beta}\ ({\to}I)$$

$$\frac{r : \alpha_i}{\mathrm{inj}_i\ r : \alpha_0 \vee \alpha_1}\ (\vee I) \qquad \frac{r : \alpha_0 \vee \alpha_1 \quad s_0 : \beta \quad s_1 : \beta}{r(x_0^{\alpha_0}.s_0, x_1^{\alpha_1}.s_1) : \beta}\ (\vee E)$$

$$\frac{r : \alpha_0 \quad s : \alpha_1}{\langle r, s\rangle : \alpha_0 \wedge \alpha_1}\ (\wedge I) \qquad \frac{r : \alpha_0 \wedge \alpha_1}{r\ \mathrm{proj}_i : \alpha_i}\ (\wedge E)$$

$$\frac{r : \alpha}{\mathrm{neg}\ r : {\sim}{\sim}\alpha}\ ({\sim}I) \qquad \frac{r : {\sim}{\sim}\alpha}{r\ \mathrm{neg}^{-1} : \alpha}\ ({\sim}E)$$

$$\frac{r : \alpha \quad s : {\sim}\beta}{\langle r, s\rangle : {\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}I) \quad \frac{r : {\sim}(\alpha{\to}\beta)}{r\ \mathrm{proj}_0 : \alpha}\ ({\sim}{\to}E1) \quad \frac{r : {\sim}(\alpha{\to}\beta)}{r\ \mathrm{proj}_1 : {\sim}\beta}\ ({\sim}{\to}E2)$$

$$\frac{r : {\sim}\alpha_i}{\mathrm{inj}_i\ r : {\sim}(\alpha_0 \wedge \alpha_1)}\ ({\sim}\wedge I) \qquad \frac{r : {\sim}(\alpha_0 \wedge \alpha_1) \quad s_0 : \beta \quad s_1 : \beta}{r(x_0^{{\sim}\alpha_0}.s_0, x_1^{{\sim}\alpha_1}.s_1) : \beta}\ ({\sim}\wedge E)$$

$$\frac{r : {\sim}\alpha_0 \quad s : {\sim}\alpha_1}{\langle r, s\rangle : {\sim}(\alpha_0 \vee \alpha_1)}\ ({\sim}\vee I) \qquad \frac{r : {\sim}(\alpha_0 \vee \alpha_1)}{r\ \mathrm{proj}_i : {\sim}\alpha_i}\ ({\sim}\vee E).$$

The superscripts of variables in the rules $(\vee E)$ and $({\sim}\wedge E)$ denote the type assignments for those variables. An expression $r : \alpha$ is also denoted as r^{α}.

It is remarked that the typing rules derive the non-standard consequence of non-unique typedness of terms. For example, $\langle r^{\alpha}, s^{{\sim}\beta}\rangle$ has two types ${\sim}(\alpha{\to}\beta)$ and $\alpha \wedge {\sim}\beta$, a fact which corresponds to the axiom scheme ${\sim}(\alpha{\to}\beta) \leftrightarrow \alpha \wedge {\sim}\beta$.

It is also remarked that neg and neg^{-1} are very similar to $\langle , \rangle$ and proj_i, respectively, and hence in the strong normalization proof presented, the cases for neg and neg^{-1} can be treated in the same way as that for $\langle , \rangle$ and proj_i.

In the following discussion, we treat only on typable terms.

Definition 4.4. *Eliminations* R, S, T, ... are defined by the following grammar: [6]

$$R ::= r \mid (x_0.r, x_1.r) \mid \text{proj}_i \mid \text{neg}^{-1}.$$

An expression $(x_0.s_0, x_1.s_1)_x[t]$ is used for the capture-free substitution $(x_0.s_0[t/x],\ x_1.s_1[t/x])$. A similar vector expression $\vec{R}$ is also adopted for eliminations. A square bracket notion [...] is used for optional syntax elements.

Definition 4.5. The set TE of typed terms is inductively defined by the following grammar:

$$\text{TE} \ni r ::=$$
$$x \mid x\vec{r}[R] \mid \lambda xr \mid \text{inj}_i\, r \mid \langle r, r\rangle \mid \text{neg}\, r \mid (\lambda xr)r\vec{R} \mid x\vec{r}(x_0.r, x_1.r)R\vec{R} \mid$$
$$(\text{inj}_i r)(x_0.r, x_1.r)\vec{R} \mid \langle r, r\rangle\ \text{proj}_i\ \vec{R} \mid (\text{neg}\ r)\ \text{neg}^{-1}\ \vec{R}.$$

The set NF of typed terms in normal forms is inductively defined by the following grammar:

$$\text{NF} \ni r ::= x \mid x\vec{r}[(x_0.r, x_1.r)] \mid \lambda xr \mid \text{inj}_i\ r \mid \langle r, r\rangle \mid \text{neg}\ r.$$

An expression $FV(S)$ denotes the set of all free variables in S.

Definition 4.6. The reduction relation $\rhd$ on TE is defined by

$(\to\beta)$: $(\lambda xr)s \ \rhd\ r_x[s]$,

$(\vee\beta)$: $(\text{inj}_i\ r)(x_0.s_0, x_1.s_1) \ \rhd\ s_i[r/x_i]$,

$(\wedge\beta)$: $\langle r_0, r_1\rangle\ \text{proj}_i \ \rhd\ r_i$,

$(\sim\beta)$: $(\text{neg}\ r)\ \text{neg}^{-1} \ \rhd\ r$,

(Perm): $r(x_0.s_0, x_1.s_1)S \ \rhd\ r(x_0.s_0S, x_1.s_1S)\ \ x_i \notin FV(S)$,

(Comp): if $r \ \rhd\ r'$, then $sr \ \rhd\ sr'$, $rS \ \rhd\ r'S$, $\lambda xr \ \rhd\ \lambda xr'$, $\text{inj}_i\ r \rhd \text{inj}_i\ r'$, $s(x.r, y.t) \rhd s(x.r', y.t)$, $s(y.t, x.r) \rhd s(y.t, x.r')$, $\langle r, s\rangle \ \rhd\ \langle r', s\rangle$, $\langle s, r\rangle \ \rhd\ \langle s, r'\rangle$, $\text{neg}\ r \ \rhd\ \text{neg}\ r'$.

Definition 4.7. An expression $r \Downarrow$ (r is strongly normalizable) means that there is no infinite reduction sequence starting with r. The set WF is defined by $\{r \mid r \Downarrow\}$.

[6] Using this notion, the permutative conversions can uniformly be presented as (Perm) in the definition of reductions.

Definition 4.8. The set SN is inductively defined by the following rules where $(x.r, y.t) \in \mathrm{SN}$ denotes an abbreviation for $r, t \in \mathrm{SN}$.

$$\frac{}{x \in \mathrm{SN}}\ (Var_0) \qquad \frac{\vec{r}[R] \in \mathrm{SN}}{x\vec{r}[R] \in \mathrm{SN}}\ (Var) \qquad \frac{x\vec{r}(x_0.s_0S, x_1.s_1S)\vec{R} \in \mathrm{SN}}{x\vec{r}(x_0.s_0, x_1.s_1)S\vec{R} \in \mathrm{SN}}\ (Perm^{-1})$$

$$\frac{r \in \mathrm{SN}}{\lambda xr \in \mathrm{SN}}\ (\lambda) \qquad \frac{r \in \mathrm{SN}}{\mathrm{inj}_i\ r \in \mathrm{SN}}\ (\mathrm{Inj}) \qquad \frac{r_0, r_1 \in \mathrm{SN}}{\langle r_0, r_1\rangle \in \mathrm{SN}}\ (Pair) \qquad \frac{r \in \mathrm{SN}}{\mathrm{neg}\ r \in \mathrm{SN}}\ (Neg)$$

$$\frac{r_x[s]\vec{S} \in \mathrm{SN} \quad s \in \mathrm{SN}}{(\lambda xr)s\vec{S} \in \mathrm{SN}}\ (\beta{\to}) \qquad \frac{r_0\vec{S}, r_1\vec{S} \in \mathrm{SN}}{\langle r_0, r_1\rangle\ \mathrm{proj}_i\ \vec{S} \in \mathrm{SN}}\ (\beta\wedge)$$

$$\frac{s_i[r/x_i]\vec{S} \in \mathrm{SN} \quad s_{1-i}\vec{S} \in \mathrm{SN} \quad r \in \mathrm{SN}}{(\mathrm{inj}_i\ r)(x_0.s_0, x_1.s_1)\vec{S} \in \mathrm{SN}}\ (\beta\vee) \qquad \frac{r\vec{S} \in \mathrm{SN}}{(\mathrm{neg}\ r)\ \mathrm{neg}^{-1}\ \vec{S} \in \mathrm{SN}}\ (\beta{\sim}).$$

The rule $(Perm^{-1})$ has the proviso $x_i \notin FV(S)$.

In the following, we give only a sketch of the strong normalization proof, since the proof is almost the same as the one of [6].

Lemma 4.9. SN = WF.

Lemma 4.10. *If* $r(x_0.s_0S, x_1.s_1S)\vec{R} \in \mathrm{SN}$*, then* $r(x_0.s_0, x_1.s_1)S\vec{R} \in \mathrm{SN}$.

Using Lemma 4.10, we can prove the following lemma.

Lemma 4.11. *For any type* α*, for any* $r \in \mathrm{SN}$*,*

(1) *if* $s^{\alpha} \in \mathrm{SN}$*, then* $rs \in \mathrm{SN}$*;*

(2) *if* $r : \alpha$ *where* $\alpha \in \{\alpha_0 \vee \alpha_1,\ {\sim}(\alpha_0 \wedge \alpha_1)\}$ *and* $s_0^{\delta_0}, s_1^{\delta_1} \in \mathrm{SN}$ *where* $\langle\delta_0, \delta_1\rangle \in \{\langle\alpha_0, \alpha_1\rangle,\ \langle{\sim}\alpha_0, {\sim}\alpha_1\rangle\}$*, then* $r(x_0.s_0, x_1.s_1) \in \mathrm{SN}$*;*

(3) *if* $s^{\alpha} \in \mathrm{SN}$*, then* $r_x[s] \in \mathrm{SN}$.

Proof. By simultaneous induction on α, side induction on $r \in \mathrm{SN}$. We distinguish the cases according to $r \in \mathrm{SN}$. We always first prove (1) (2) in parallel, and later infer (3), possibly with the help of (2). □

It is remarked that by Definition 4.8, the following conditions hold:

(4) if $\alpha \in \{\beta_0 \wedge \beta_1,\ {\sim}(\beta_0{\to}\beta_1),\ {\sim}(\beta_0 \vee \beta_1)\}$ and $r^{\delta_0}, s^{\delta_1} \in \mathrm{SN}$ where $\langle\delta_0, \delta_1\rangle \in \{\langle\beta_0, \beta_1\rangle,\ \langle\beta_0, {\sim}\beta_1\rangle,\ \langle{\sim}\beta_0, {\sim}\beta_1\rangle\}$, then $\langle r, s\rangle^{\alpha} \in \mathrm{SN}$, and

(5) if $r^{\alpha} \in \mathrm{SN}$, then $(\mathrm{neg}\ r) \in \mathrm{SN}$.

Theorem 4.12 (Strong normalization). *All typed terms are strongly normalizable.*

Proof. For any term r, we can show $r \in \mathrm{SN}$ by induction on the structure of r. In order to show this, Lemma 4.11 is used, e.g. the case $r \equiv st$ is shown by using Lemma 4.11 (1). By Lemma 4.9, we obtain $r \in \mathrm{WF}$, i.e. $r \Downarrow$. □

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