A Theorem on the Manipulability of Redundant Serial Kinematic Chains

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***Abstract*—The concept of 'manipulability' is particularly important to characterize the ability of a serial kinematic chain – artificial manipulator or natural limb – to move quickly its end-effector in any direction of its operational space in response to given joint velocities. Yoshikawa's manipulability definition has shown its benefit for robotics. According to him the robot manipulability can be measured, in any joint configuration $\boldsymbol{q}$, by $\sqrt{\det[\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})]}$ where $\boldsymbol{J}(\boldsymbol{q})$ denotes the robot Jacobian matrix. From this specific manipulability criterion an original theorem is derived which helps to express in closed-form the manipulability of a redundant serial kinematic chain in resolving it into manipulability factors associated to non-redundant subrobots of which the redundant original robot consists. The proposed approach is suitable for robotics but also for the analysis of naturally redundant biomechanical kinematic systems as illustrated in the study of a redundant 4R kinematic model defined as being the regional structure of a 7R anthropomorphic robot-arm.**

***Index Terms*—Manipulability, Redundant manipulators, Biomechanical kinematic systems.**

I. Introduction

The development of advanced industrial robot-arms as humanoid robots leads robot designers to consider more and more spatial redundant robot limbs kinematic structures i.e. with more d.o.f than necessitated by the task. For example, the recent Mitsubishi PA-10 has 7 d.o.f. as all upper limbs of actual humanoid robots; some of them have even 9 d.o.f. to mimic the shoulder complex mobility and generating human-like gesture. This d.o.f. number defines the dimension of the robot joint space we will note 'n'. The end-effector of the robot limb performs its task in a m-dimensional operational space. In redundant case – i.e. $n > m$ – the robot Jacobian $m\times n$ matrix $\boldsymbol{J}(\boldsymbol{q})$ – $\boldsymbol{q}$ denotes the robot joint vector – expressing the robot end-effector velocities in operational space as a function of the joint velocities is not a square matrix. A problem occurs in consequence to determine the inverse relationship necessary to the robot control in its operational space. The Moore-Penrose inverse $\boldsymbol{J}^{+}(\boldsymbol{q})$ gives an elegant solution to this problem in the form :

$$\boldsymbol{J}^{+}(\boldsymbol{q}) = \boldsymbol{J}^T(\boldsymbol{q})[\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})]^{-1} \tag{1}$$

so long as the matrix $\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})^{\mathrm{T}}$ is not singular.

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Consequently $\sqrt{\det[\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})]}$ can be regarded as the distance of any redundant configuration from singular ones. Yoshikawa [1], [2] has proposed to call manipulability this criterion denoted by $\omega(\boldsymbol{q})$.

A zero-manipulability configuration – i.e. a singular configuration – expresses the impossibility for the robot end-effector to be moved in any direction of its working area, whereas a non-zero manipulability expresses the possibility for the robot end-effector to be moved in any direction of its working area with task velocities more higher for given joint velocities as the manipulability criterion is high.

Manipulability analysis can be very useful to specify the kinematic possibilities of a given robot, particularly in comparison with simplified non-redundant versions it comes from. However, it is well known in redundant arms robotics that the matrix product $\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q})$ is often so complex that its inverse can be determined only by numerical computation. In particular the look for a closed-form expression of $\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q})$ even helped by symbolic computation software, can be difficult to obtain for $n > 3$. Nevertheless a closed-form expression written in a simple way would be particularly interesting to found the robot manipulability analysis. In order to help its search for we propose in the framework of this paper an original resolution theorem of the manipulability criterion. This manipulability theorem presented in section 2 is the consequence of a fundamental resolution formula of $\det(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}})$; it is followed in section 3 by its application to a manipulability analysis of the a 4R redundant kinematic model of the regional joints of a 7R anthropomorphism robot upper limb.

II. A Theorem on Redundant Robot Limb Manipulability Resolution

The proposed theorem is based on an important property of theory of determinants which, according to us, has been less used in robotics and particularly in the framework of robot manipulability theory : the Cauchy-Binet formula. This formula can be expressed as follows (Cauchy, 1815 [3]): let us consider two $n\times m$ matrices $\boldsymbol{A} = [\boldsymbol{A}_1\,\boldsymbol{A}_2\,.....\,\boldsymbol{A}_m]$ and $\boldsymbol{B} = [\boldsymbol{B}_1\,\boldsymbol{B}_2\,.....\,\boldsymbol{B}_m]$ with $\boldsymbol{A}_i$, $\boldsymbol{B}_i$ vectors of the n-dimensional real space (naturally m<n) and let us define $\boldsymbol{A}_{i_1 \ldots i_m}$ and $\boldsymbol{B}_{i_1 \ldots i_m}$ as the $m\times m$ matrices formed by rows $i_1,\ldots,i_m$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively; we get :

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$$\det(\boldsymbol{A}^T\boldsymbol{B}) = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \det(\boldsymbol{A}_{i_1 \ldots i_m}) \det(\boldsymbol{B}_{i_1 \ldots i_m}) \tag{2}$$

where the summation is over the $\binom{n}{m}$ m-combinations of $\{1, 2, ..., n\}$. Besides the original proof given by Cauchy [3], similar proofs can be found in classic algebra books [4], [5]. If $\boldsymbol{A}$=$\boldsymbol{B}$, the Cauchy-Binet formula comes to :

$$\det(\boldsymbol{A}^T\boldsymbol{A}) = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \det{}^2(\boldsymbol{A}_{i_1 \ldots i_m}) \tag{3}$$

which is also called Lagrange's identity [5]. This identity founds the generalization of the Pythagorean theorem [6]-[7] : in the n-dimensional real vector space the square of the volume of a m-dimensional parallelepiped can be determined as the sum of the squares of the volumes of its projections on the $\binom{n}{m}$ m-dimensional coordinates planes.

In the case of a redundant robot, the robot Jacobian matrix $\boldsymbol{J}$ to be considered is a $m \times n$ matrix with $m < n$. In consequence $\boldsymbol{J}^T$ is a $n \times m$ matrix and the Cauchy-Binet formula can be applied. If we note $\boldsymbol{J} = [\boldsymbol{J}_1 \boldsymbol{J}_2 \ldots\ldots \boldsymbol{J}_n]$ where $\boldsymbol{J}_{i, 1 \le i \le n}$ designates the i^{th} column of $\boldsymbol{J}$, the rows $i_1, \ldots, i_m$ of $\boldsymbol{J}^T$ are also the $i_1, \ldots, i_m$ columns of $\boldsymbol{J}$. We get :

$$\det(\boldsymbol{J}\boldsymbol{J}^T) = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \det{}^2(\boldsymbol{J}_{i_1}, \boldsymbol{J}_{i_2}, \ldots, \boldsymbol{J}_{i_m}) \tag{4}$$

This relationship resolves the computation of $\det(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}})$ into the sum of the m-minors of $\boldsymbol{J}$. Furthermore, the interesting following corollary can indeed be deduced :

$$\det(\boldsymbol{J}\boldsymbol{J}^T) = 0 \iff \forall (i_1, i_2, \ldots, i_m) \text{ m-combination of } \{1, 2, \ldots, \mathrm{n}\}, \quad \det(\boldsymbol{J}_{i_1}, \boldsymbol{J}_{i_2}, \ldots, \boldsymbol{J}_{i_m}) = 0 \tag{5}$$

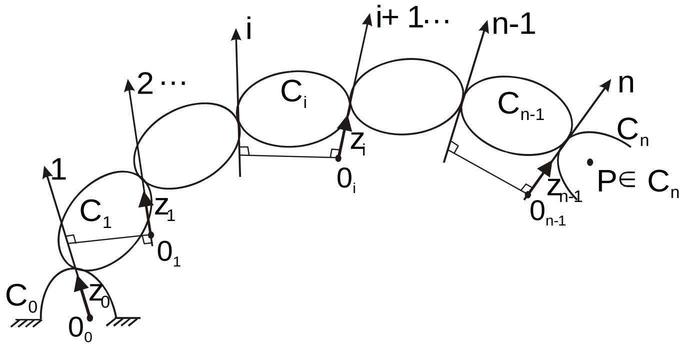
The robot in consequence becomes singular if and only if all m-minors of $\boldsymbol{J}$ are equal to zero.

We are now going to adapt equ.(4) formula to the manipulability notion in introducing an original concept of subrobot of a redundant robot.

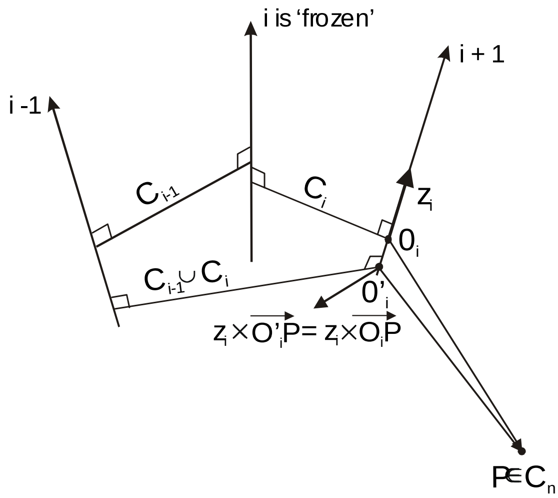
A. Notion of subrobot of a redundant robot

Let us consider a n-d.o.f. robot limb whose axes are numbered from 1 to n. Let us now consider a m-combination of $\{1, 2...., n\}$ whose elements are numbered from the smallest to the biggest $1 \le i_1 < i_2 < \ldots. < i_m \le n$ and let us define the kinematic structure derived from the n-d.o.f. robot whose $i_1, i_2, \ldots, i_m$ axes are supposed to be mobile and the other ones 'frozen'. We propose to call m-order subrobot such kinematic structure derived from the original robot. When the original robot is composed of n mobile link $C_{i, 1 \le i \le n}$ defined by the common perpendicular to axis i and $i+1$ – expected the end-effector C_n – the m-order subrobot is composed in consequence of m mobile links resulting of the combination of proximal links jointed by frozen axis. A given n-d.o.f. has consequently $\binom{n}{m}$ m-order subrobots.

Furthermore, it is well known that a given robot limb kinematic structure can be entirely characterized by its Jacobian matrix which constitutes its differential model. An important result in industrial robotics gives a general expression of the robot Jacobian matrix in associating a frame R_i to each robot link as illustrated in Fig. 1.a.



(a)



(b)

Fig.1. Kinematic analysis for determining the Jacobian matrix components of a subrobot derived from a serial n-d.o.f. robot limb, (a) Classical kinematic scheme of a serial n-d.o.f. robot limb by associating to each link C_i a frame R_i placed on the common perpendicular to axis i and i+1: according to R.Paul's notation [8] – preferred in this paper to J.Craig's notation [9] – the frame R_i is placed on the axis i+1 and the last frame whose origin is noted $\boldsymbol{P}$ is attached to the end-effector C_n , (b) Effect of 'freezing' the axis i mobility : the link C_i is now rigidly fixed to the link C_{i-1} (see text).

The robot Jacobian matrix associated to the point $\boldsymbol{P}$ of the end-effector can be written in a reference frame R_j as follows:

$$ {}^{j}\boldsymbol{J}=\begin{bmatrix} {}^{j}\boldsymbol{z}_0\times{}^{j}\overrightarrow{O_0P} & {}^{j}\boldsymbol{z}_1\times{}^{j}\overrightarrow{O_1P} & \cdots & {}^{j}\boldsymbol{z}_{n-1}\times{}^{j}\overrightarrow{O_{n-1}P} \\ {}^{j}\boldsymbol{z}_0 & {}^{j}\boldsymbol{z}_1 & \cdots & {}^{j}\boldsymbol{z}_{n-1} \end{bmatrix} \tag{6} $$

Let us consider the m-order sub-robot defined by the combination $(i_1, i_2,..., i_m)$. The same relation (6) can be applied to any considered subrobot. Let us assume, for example, that the axis i is frozen. As illustrated in Fig.1.b, the frame R_i associated to link i joining axis i to axis i+1 must now be considered as the frame associated to 'link (i-1) + link i' joining axis (i-1*)* to axis (i+1) : the new ${}^{j}\boldsymbol{z}_i$ axis is the same but the new origin – denoted ***O'***$_i$ in Fig. 1.b – has shifted. Concerning the robot Jacobian matrix the ^{i}th column disappears and the ${}^{i+1}$th column is now composed of the same ${}^{j}\boldsymbol{z}_i$ component and a new ${}^{j}\boldsymbol{z}_i\times{}^{j}\overrightarrow{O'_iP}$ component but this last one is equal to the original one since $\overrightarrow{O'_iO_i}$ is collinear to $\boldsymbol{z}_i$ as illustrated in Fig. 1.b. Let us note $\boldsymbol{J}_{\text{subrobot } i_1i_2...i_m}$ the Jacobian matrix of the considered subrobot. It results that its determinant can be directly derived from the knowledge of the corresponding $\boldsymbol{J}_{i_1}, \boldsymbol{J}_{i_2}, ..., \boldsymbol{J}_{i_m}$ column vectors of the $\boldsymbol{J}$ matrix as follows (since the robot Jacobian determinant is independent from any frame choice no reference frame is mentioned) :

$$ \det\boldsymbol{J}_{\text{subrobot } i_1, i_2 ... i_m} = \det(\boldsymbol{J}_{i_1}, \boldsymbol{J}_{i_2}, ..., \boldsymbol{J}_{i_m}) \tag{7} $$

In other terms, the determinant of any considered subrobot of the original robot is the corresponding minor of the robot Jacobian matrix.

Let us now interpret the notion of subrobot in the case of a task performed by a n-d.o.f. robot in a m-dimensional operational space – i.e. a task needing only m of the n available d.o.f. of the considered robot. In this case, $\binom{n}{m}$ m-order subrobots can be defined. We think that the existence of these subrobots characterize in some way the robot redundancy. To perform its m-dimensional task, the redundant robot can use everyone of these sub-robots. Without trying to demonstrate it rigorously we think that the optimal character of Moore-Penrose pseudo-inverse physically corresponds to choose the 'best' subrobot minimizing at any time of the redundancy control the velocity norm in the robot operational space. The resulting non-cyclicity property would come from this 'blind' choice of a non-redundant inverse kinematic solution in the form of a subrobot choice independently of the closed character of the performed operational path. Inversely, cyclic methods as Seraji's configuration control method [10] imposes in some way a given subrobot or a fixed subrobots combination.

Let us go back now to our manipulability problem. Thanks to this subrobot notion, we are now able to derive from fundamental relationship (4) an original theorem on redundant robot manipulability.

B. Resolution theorem of redundant robot limbs manipulability

Let us note by ω_{robot} the manipulability of the considered redundant robot limb and by $\omega_{\text{subrobot } i_1 i_2 ... i_m}$ the manipulability of the sub-robot $(i_1, i_2, ..., i_m)$. Since each of the considered subrobots is non-redundant, we get :

$$ \omega_{\text{subrobot } i_1 i_2 ... i_m} = \left|\det\boldsymbol{J}_{\text{subrobot } i_1 i_2 ... i_m}\right| = \left|\det(\boldsymbol{J}_{i_1}, \boldsymbol{J}_{i_2}, ..., \boldsymbol{J}_{i_m})\right| \tag{8} $$

The following relationship results in consequence :

$$ \omega^2_{\text{robot}} = \sum_{1\le i_1 < i_2 < ... < i_m \le n} \omega^2_{\text{subrobot } i_1 i_2 ... i_m} = \sum \omega^2_{\text{m-subrobot}} \tag{9} $$

In other terms, the manipulability squared of a redundant robot limb in a m-dimensional operational space is equal to the sum of manipulability squared of all m-order subrobots. The equ. (5) corollary can now be explained as follows : a redundant robot is singular in the performance of a m-dimensional task if and only if all its m-order subrobots are singular.

Beyond the meaning that this theorem opens about the passage from non-redundancy to redundancy in robotics it offers also a great practical advantage : it allows to substitute to the complex manipulability computation of a redundant robot-limb a sum of factors much more simple to express in closed form because they correspond to manipulability criteria of non-redundant kinematic structures. In a one hand, singularity analysis is indeed helped through the look for singular configuration of each concerned subrobot; in a second hand, the look for maximum manipulability configuration is helped as follows. From

$$ \mathrm{d}\omega^2_{\text{robot}} = 2\omega_{\text{robot}}\mathrm{d}\omega_{\text{robot}} \tag{10} $$

we get indeed in all non-singular configuration the equivalence :

$$ \mathrm{d}\omega^2_{\text{robot}} = 0 \iff \mathrm{d}\omega_{\text{robot}} = 0 \tag{11} $$

The maximum manipulability configuration according to a given joint variable q_i can be in consequence determined from the equivalence :

$$ \frac{\partial\omega^2_{\text{robot}}}{\partial q_i} = 0 \iff \sum_{\text{m-order subrobots}} \frac{\partial\omega^2_{\text{subrobot } i_1 i_2 ... i_m}}{\partial q_i} = 0 \tag{12} $$

The look for the robot maximum configuration can so be led as follows : a joint variable is chosen and the value of this variable maximizing the manipulability criterion can be determined from relation (12). The corresponding optimal value is integrated in general manipulability expression and the process is repeated until all optimal joint values have been obtained. In a similar way, the optimization of the studied redundant structure can be made in order to determine the specific robot geometric parameter ratios maximizing the robot manipulability. We illustrate now this approach in the case of a 4R redundant regional structure of a anthropomorphic upper limb model.

III. Application to Manipulability Analysis of Anthropomorphic Robot Upper Limb

The 4R kinematic model of the upper limb regional structure illustrated in Fig. 2 is considered : the shoulder is reduced to a 3 d.o.f. spherical joint and the elbow is a revolute joint; the arm length is noted A and B the 'forearm + hand' length. From a previous study about the kinematic modeling of a 7R anthropomorphic robot upper-limb [11] whose the 4R considered structure is the "regional structure", the following expression of the robot Jacobian in frame R_4 is as follows :

$$ {}^4\boldsymbol{J} = \begin{bmatrix} C_2S_3(AC_4+B) & C_3(AC_4+B) & 0 & B \\ AC_2C_3-BS_2S_4+BC_2C_3C_4 & -S_3(A+BC_4) & BS_4 & 0 \\ AC_2S_3S_4 & AC_3S_4 & 0 & 0 \end{bmatrix} \quad (13) $$

where θ_1 $(0 \le \theta_1 \le \pi)$ designates the shoulder abduction-adduction, θ_2 $(-\pi/2 \le \theta_2 \le \pi)$ the shoulder flexion-extension, θ_3 $(-\pi/2 \le \theta_3 \le \pi/2)$ the internal/external arm rotation and θ_4 $(0 \le \theta_4 \le \pi)$ the elbow flexion-extension. If the concerned task consists in positioning the point $\boldsymbol{P}$, the operational space is a 3-dimensional space when the joint space is a 4-dimensional space.

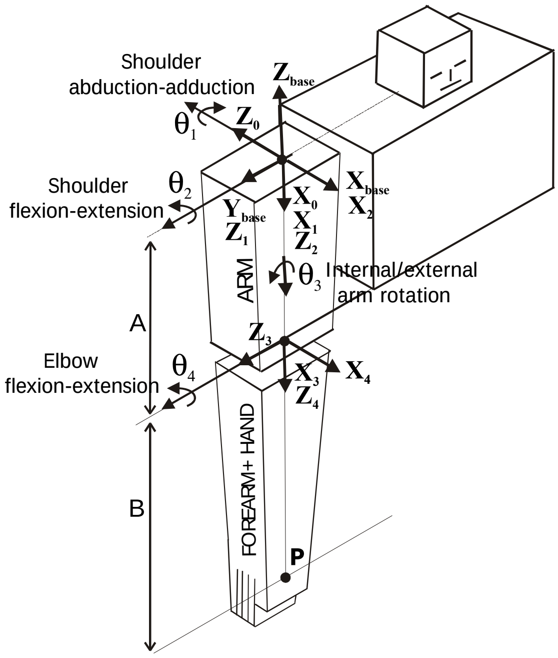


Fig.2. Kinematic model of the considered anthropomorphic redundant 4R structure (the frame notations R_{base} to R_4 are similar to the ones used in our study about the kinematic modeling of a 7R anthropomorphic upper-limb robot [11]).

The manipulability of this redundant robot will be denoted by ω_{4R} and we will note ω_{123} , ω_{124} , ω_{134} and ω_{234} the corresponding 3-order subrobots manipulabilities. We get :

$$ \begin{cases} \omega_{123} = 0 \\ \omega_{124} = ABS_4\left|C_2(A+BC_4) - BS_2C_3S_4\right| \\ \omega_{134} = AB^2S_4^2\left|C_2S_3\right| \\ \omega_{234} = AB^2S_4^2\left|C_3\right| \end{cases} \quad (14) $$

and :

$$ \omega_{4R}^2(\theta_2, \theta_3, \theta_4) = \omega_{124}^2 + \omega_{134}^2 + \omega_{234}^2 \quad (15) $$

A. Singularity analysis

The singularities of our redundant 4R kinematic structure correspond to the joint configurations making zero the 4 subrobot manipulabilities. The first subrobot '123' is always singular since the subset of mobiles axes 1, 2 and 3 is not able to position the point $\boldsymbol{P}$ in a volumic area. The joint conditions making zero each of three other subrobot manipulabilities can be gathered in a table as the following one where each minor is resolved into elementary factors (Table I).

$\lvert minor \rvert$	**Factors**		
ω_{124}	S_4	$C_2(A+B\ C_4) - BS_2C_3S_4$	
ω_{134}	S_4	S_3	C_2
ω_{234}	S_4	C_3	

Table I. Factors making zero each subrobot manipulability

A simple factor analysis leads to highlight two singular configurations :

- S_4=0 , corresponding to the well-known *elbow singularity* occurring when the elbow is in full flexion or extension ;
- C_2=0 and C_3=0 , corresponding to a *shoulder-type singularity* : arm and forearm are placed in the horizontal plane ($\boldsymbol{O}_{base}$, $\boldsymbol{X}_{base}$, $\boldsymbol{Y}_{base}$).

This result is in accordance with Kreutz-Delgado *et alia* classic work on the kinematic analysis of 7R robots [12] but our approach leads to a rigorous proof when the mentioned paper deduces the 7R robot singularities from a qualitative study of the robot Jacobian columns.

B. Manipulability analysis

We pursue our study by a more global analysis of the robot manipulability in order to determine and quantify the contribution of the redundancy. Let us start from the subrobot '124' : it corresponds to the regional structure of the fundamental non-redundant anthropomorphic 6R robot. If we put θ_3 = 0 in ω_{124} expression, we get indeed the corresponding manipulability expression of this 3R regional structure we will denote ω_{3R} :

$$ \omega_{3R}(\theta_2, \theta_4) = ABS_4\left|AC_2 + BC_{24}\right| \quad (16) $$

The two conditions making zero the 3R model manipulability corresponds to the two well-known singularities [13] :

- *elbow singularity*, occurring when S_4 = 0 i.e. when the elbow is in full flexion or extension ;
- *shoulder singularity*, occurring when (AC_2 + BC_{24}) = 0 i.e. when the end point $\boldsymbol{P}$ is located on the $\boldsymbol{Y}_{base}$ axis as indicated by the corresponding robot direct kinematic model :

$$\begin{cases} P_{3R,\,X} = S_1(AC_2+BC_{24}) \\ P_{3R,\,Y} = AS_2+BS_{24} \\ P_{3R,\,Z} = -C_1(AC_2+BC_{24}) \end{cases} \quad (17)$$

Let us try now to determine the robot maximum manipulability. We start optimizing the θ_2 angle :

$$\frac{\partial \omega_{3R}^2(\theta_2,\theta_4)}{\partial \theta_2} = -2A^2B^2S_4^2(AC_2+BC_{24})(AS_2+BS_{24}) \quad (18)$$

leading to the optimal relationship between θ_2 et θ_4 angles : $AS_2+BS_{24}=0$.
(19)
If we reintegrate this relation inside the manipulability expression, we can pursue the optimization process according to the angle θ_4 . From

$$\omega_{3R}^{\theta_2\mathrm{opt}}(\theta_4) = ABS_4\sqrt{A^2+B^2+2ABC_4} \quad (20)$$

we get :

$$\frac{\mathrm{d}[\omega_{3R}^{\theta_2\mathrm{opt}}(\theta_4)]^2}{\mathrm{d}\theta_4} = 2A^2B^2S_4[3ABC_4^2+(A^2+B^2)C_4-AB] \quad (21)$$

Solving the second order equation in C_4 leads to the following maximum manipulability :

$$\omega_{3R}^{opt} = ABS_4\sqrt{A^2+B^2+2ABC_4} \quad \text{with} \quad (22)$$

$$C_4 = [-(A^2+B^2)+\sqrt{(A^2+B^2)^2+12A^2B^2}]/6AB$$

corresponding to the optimal configuration :

$$\theta_4 = \mathrm{atan2}(\sqrt{1-C_4^2},\, C_4) \;\text{ and }\; \theta_2 = \mathrm{atan}(-BS_4/(A+BC_4)) \quad (23)$$

It can be asked finally if a privileged A/B ratio can be selected. Let us consider the λ-ratio between 0 and 1 defined as follows :

$$\lambda = A/(A+B) \quad (24)$$

The new manipulability expression function of λ is then :

$$\omega_{3R}^{opt}(\lambda) = (A+B)^3S_4\lambda(1-\lambda)\sqrt{\lambda^2+(1-\lambda)^2+2\lambda(1-\lambda)C_4} \quad (25)$$

A simple graphical simulation leads to determine the optimal λ-ratio : $\lambda = 0.5$, which corresponds to a ‘forearm + hand’ length equal to the arm length. Fig. 3 illustrates the manipulability variation in function of both the shoulder flexion-extension angle and the elbow flexion-extension angle (the $A+B$ sum is normalized to 1 with numerical values for A and B equal to 0.5). The zero-manipulability sides correspond to the bound *elbow singularity* when the middle ‘singular valley’ bottom corresponds to the internal *shoulder singularity*. The tops of the ‘hills’ corresponds to the manipulability maxima. Let us compare this surface with the corresponding one derived from the redundant 4R kinematic manipulability illustrated in Fig. 4.a in the case θ_3 = 0. The benefit of the redundancy clearly appears : the 3R ‘singularity valley’ is now markedly raised. The same manipulability simulation performed with θ_3 = +/– π/2 is given in Fig. 4.b : an internal singularity re-appears corresponding to the double condition previously highlighted : θ_2 = +/– π/2 and θ_3 = +/– π/2. These two graphs emphasize in which way the adding of the internal-external arm rotation (θ_3 angle) improves the kinematic behaviour of the non-redundant arm : thanks to it, and if we consider a human-like θ_3 joint range limited to [– 90°, + 90°] the internal *shoulder singularity* is now removed at bounds of the robot workspace.

It can also be observed that, as in the non-redundant 3R case, points of maximum manipulability exist in the redundant case and, in consequence, an optimum A/B ratio optimizing the manipulability in those points can be determined. Let us try to do it in comparison with the non-redundant case. Let us privilege the case θ_3 = 0, which appears like a middle and partic-

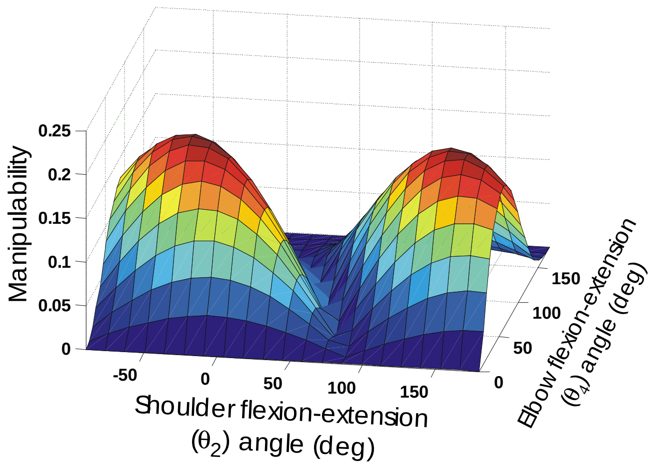
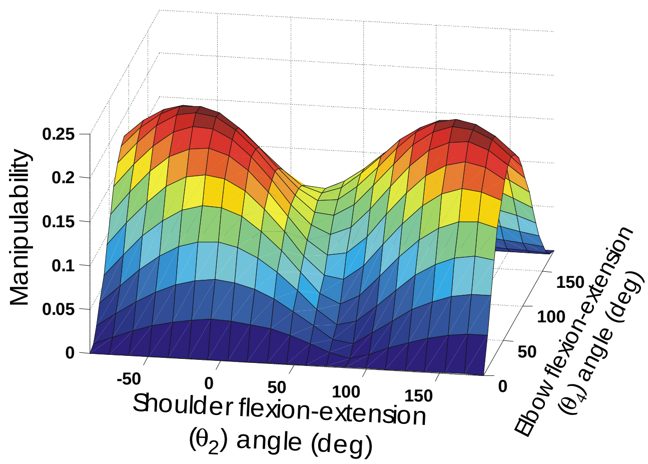
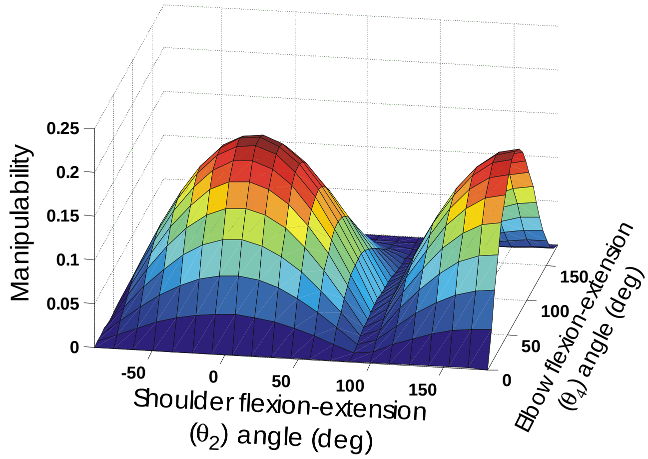


Fig.3. Manipulability of the fundamental non-redundant 3R anthropomorphic kinematic structure.



(a)



(b)

Fig.4. Manipulability of the considered redundant 4R kinematic structure at constant θ_3 joint angle (A = B =0.5), (a) θ_3 = 0 case, (b) θ_3 = +/– pi/2 case.

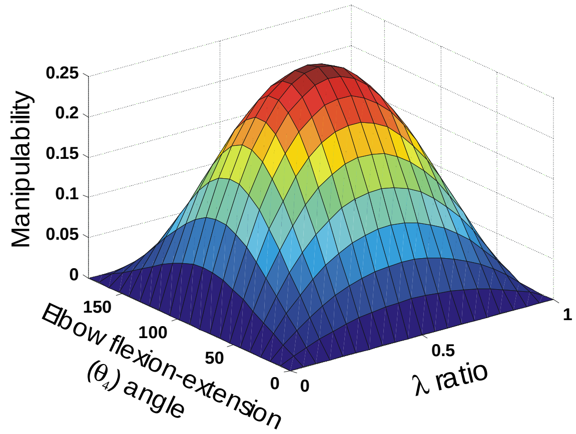
ularly favourable value of this joint and let us look for corresponding optimal θ_2 and θ_4 values. We get first :

$$\frac{\partial\omega_{4R}^2}{\partial\theta_2}(\theta_2,\theta_3{=}0,\theta_4)=\frac{\partial\omega_{3R}^2}{\partial\theta_2}(\theta_2,\theta_4) \tag{26}$$

and, in consequence, the same relationship (19) occurs between θ_2 and θ_4 to maximize the manipulability function of θ_2 . It results the final expression of the manipulability as a function of
both θ_4 and the λ-ratio :

$$\omega_{4R,\theta_3=0}^{\theta_{2opt}}(\theta_4,\lambda)=(A+B)^3S_4\lambda(1-\lambda)\sqrt{\lambda^2+(1-\lambda)^2+2\lambda(1-\lambda)C_4+(1-\lambda)^2S_4^2} \tag{27}$$

Fig. 5.a illustrates the variation of this criterion in function of θ_4 and λ : it appears clearly that in the elbow flexion-extension physiological-like range [0, +180°] and since λ belongs to [0,1] only one optimal solution in θ_4 and λ exists which maximizes the robot manipulability. Because it also appears difficult to solve in closed-form the search for the optimum θ_4 configuration, we propose to do it numerically, from the graphical variation of the manipulability with θ_4 angle at constant λ – ratio, as illustrated in Fig. 5.b.



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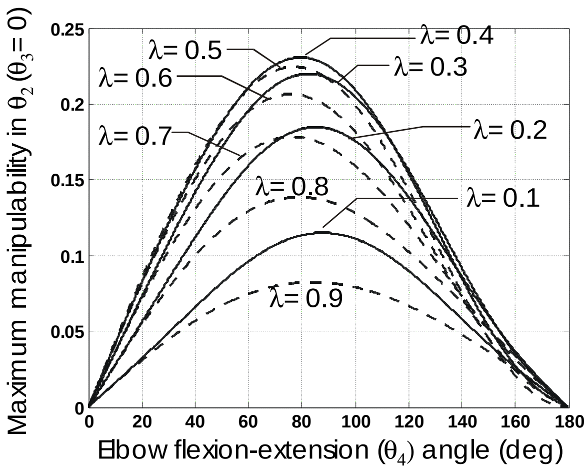


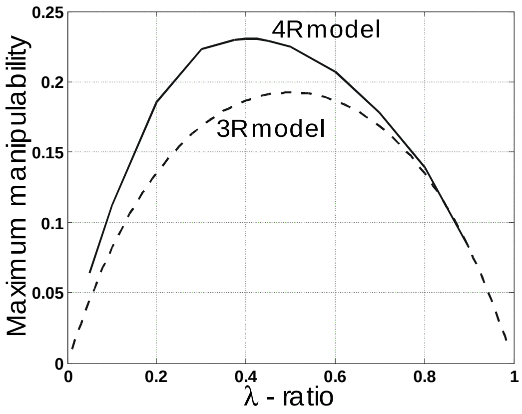
(b)

Fig.5. Maximum manipulability in θ_2 of the considered 4R redundant kinematic structure (for θ_3 = 0), (a) Manipulability as a function of the elbow flexion-extension angle θ_4 and the λ -ratio, (b) Corresponding constant curves λ -ratio.

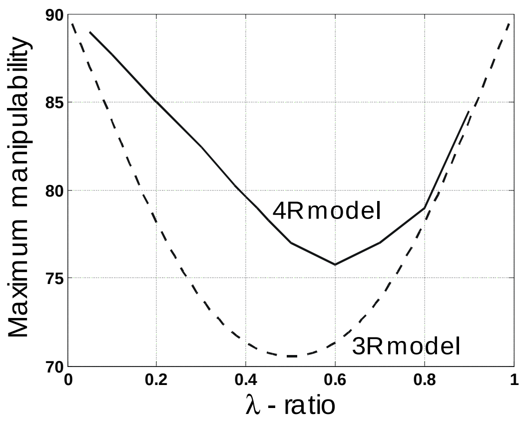
For each value of the λ – ratio it is possible to determine the corresponding θ_4-angle optimum value, from knowledge of which the maximum manipulability can be deduced, as illustrated in Fig. 6. First it appears (Fig. 6.a) that the optimum λ value maximizing the robot manipulability is now about 0.4 instead of 0.5 in the case of the non-redundant robot. It is interesting to note that this ratio is relatively well in accordance with human biometric data. If we measure A like the distance between the glenohumeral joint center and the elbow pivot, and B like the distance between the elbow pivot and the hand center of mass, we get from New Orleans's *Naval Biodynamics Laboratory* data concerning a mid-size male aviators population ([14], page 44) the following mid experimental estimations of A and B and the corresponding λ – ratio value :

$$A_{\exp}\approx 28.6\,\text{cm},\; B_{\exp}\approx 34.6\,\text{cm},\; \lambda_{\exp}\approx 0.45 \tag{28}$$

A more accurate estimation of the optimal theoretical λ-ratio value would necessitate to take into account each specific joint velocity performance. This is not made in our analysis which is essentially dedicated to highlight the potentiality of our manipulability theorem.



(a)



(b)

Fig.6. Determination and comparison of maximum manipulabilities and corresponding optimal elbow flexion-extension (θ_4) angle of the non-redundant 3R kinematic model and the redundant 4R kinematic model (for $\theta_3 = 0$), (a) Maximum manipulability versus λ-ratio, (b) Corresponding optimum θ_4 angle versus λ-ratio.

If we pursue now our theoretical analysis, it results in the case of our 0.4 optimum λ value a 20 % manipulability increase of the maximum manipulability in comparison with the non-redundant 3R kinematic structure. Moreover it is interesting to note that, due to the global bell-shape of the maximum manipulability/λ-ratio curve, the choice of the λ value maximizing the maximum manipulability increases all the 'manipulability surface' as illustrated in Fig. 7 determined for A=0.4, B=0.6 which can be compared with Fig. 4.a determined for A=0.5, B=0.5.

Furthermore, as illustrated in Fig. 6.b., the optimal elbow flexion-extension angle maximizing the manipulability is close to 80° when the corresponding 3R model optimal θ_4 flexion-extension angle is close to 70°. It is interesting to compare this result with the manipulability expression of the very simple model of the human arm limited to shoulder and elbow flexion-extension defining a end-point movement in the vertical plane ($\boldsymbol{O}_{\text{base}}$, $\boldsymbol{X}_{\text{base}}$, $\boldsymbol{Z}_{\text{base}}$) :

$$\omega_{2R} = ABS_4 = (A+B)^2 S_4 \lambda (1-\lambda) \qquad (29)$$

In this plane model, the elbow joint optimal value is evidently equal to 90°; this value corresponds in appearance to our day-life use of machine control lever or wheel. Now this value surprisingly decreases to 70° in the non-redundant spatial 3R model but the redundant 4R model leads to a more 'natural' 80° optimal value.

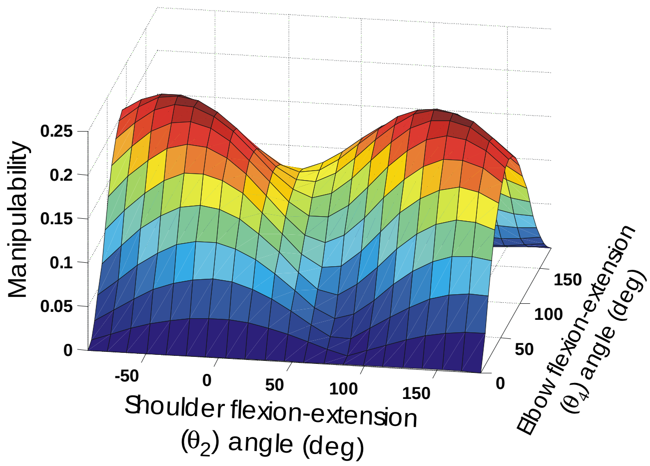


Fig. 7. Manipulability of the redundant 4R kinematic structure with 'optimal' λ-ratio equal to 0.4 (A = 0.4, B = 0.6) determined at θ_3 = 0 : in comparison with Fig. 4.a graph all points are raised until 20%.

IV. Conclusion

We have proposed a new approach for helping the manipulability analysis of redundant serial kinematic chains – artificial manipulators as natural limbs. From a closed-form resolution of the det($\boldsymbol{JJ}^T$) computation we have proposed to interpret, according to Yoshikawa's manipulability notion, any redundant n d.o.f. robot operating in a m-dimensional operational space ($m < n$) as the set of the $\binom{n}{m}$ non-redundant m d.o.f. kinematic chains obtained in freezing ($n - m$) d.o.f. of the original redundant robot. If we call subrobots of the redundant robot these subkinematic structures - whose determinants of their corresponding Jacobian matrices are the m-minors squared of ($\boldsymbol{JJ}^T$) – the following theorem results : the manipulability square of a redundant robot is the sum of the manipulabilities squares of the subrobots constituting it. It is then substituted to the n-dimensional manipulability problem associated to a n d.o.f. redundant robot $\binom{n}{m}$ m-dimensional manipulability problems associated to m d.o.f. non-redundant robots. This substitution can largely help the manipulability analysis of redundant kinematic serial structure in particular to determine their singularities and the optimal geometric parameters maximizing the robot manipulability.

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