17. Universal enveloping algebras

Recall that, for an associative algebra A with unity (1), a Lie algebra structure on A is given by the Lie bracket $[ab] = ab - ba$. Let $\mathcal{L}(A)$ denote this Lie algebra. Then $\mathcal{L}$ is a *functor* which converts associative algebras into Lie algebras. Every Lie algebra L has a universal enveloping algebra $\mathcal{U}(L)$ which is an associative algebra with unity. The functor $\mathcal{U}$ is "adjoint" to the functor $\mathcal{L}$. The universal enveloping algebra is defined by category theory. The Poincaré-Birkoff-Witt Theorem gives a concrete description of the elements of the elements of $\mathcal{U}(L)$ and how they are multiplied. There is also a very close relationship with the multiplication rule in the associated Lie group.

17.1. **Functors.**

I won't go through the general definition of categories and functors since we will be working with specific functors not general functors. I will just use vector spaces over a field F, Lie algebras and associative algebras (always with unity) as the main examples.

Definition 17.1.1. A *functor* from the category of vector spaces to the category of associative algebras both over F is defined to be a rule $\mathcal{F}$ which assigns to each F-vector space V an associative algebra $\mathcal{F}(V)$ over F and to each linear map $f : V \to W$, an F-algebra homomorphism $f_* : \mathcal{F}(V) \to \mathcal{F}(W)$ so that two conditions are satisfied:

(1) $(id_V)_* = id_{\mathcal{F}(V)}$
(2) $(fg)_* = f_* g_*$.

Recall that an *F-algebra* is an algebra which is also a vector space over F so that multiplication if F-bilinear. An *F-algebra homomorphism* is a ring homomorphism which is also F-linear. We say that the homomorphism is *unital* if it takes 1 to 1.

In short: a functor takes objects to objects and morphisms to morphism and satisfies the two conditions listed above.

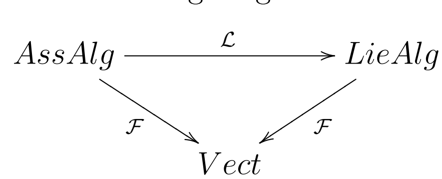
Example 17.1.2. The mapping $A \mapsto \mathcal{L}(A)$ is a functor from associative algebras to Lie algebras. For this functor, $f_* = f$ for all F-algebra homomorphisms $f : A \to B$. The reason that this works is elementary:

$$f[a, b] = f(ab - ba) = f(a)f(b) - f(b)f(a) = [f(a), f(b)]$$

We say that f_* is f considered as a homomorphism of Lie algebras $\mathcal{L}(A) \to \mathcal{L}(B)$. The two conditions are obviously satisfied and this defines a functor.

Example 17.1.3. The *forgetful functor* $\mathcal{F}$ takes an associative algebra (or Lie algebra) A to the underlying vector space. $\mathcal{F}$ is defined on morphisms by $f_* = f$. Since F-algebra homomorphisms are F-linear by definition, this defines a functor.

Exercise 17.1.4. Show that the following diagram commutes.



17.2. **Tensor and symmetric algebras.** These are two important algebras associated to any vector space. They are both graded algebras.

Definition 17.2.1. A *graded algebra* over F is an algebra A together with a direct sum decomposition:

$$A = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$$

so that $A^i A^j \subseteq A^{i+j}$. If A has unity (1) it should be in A^0. Elements in A^n are called *homogeneous of degree n*.

Example 17.2.2. The polynomial ring $P = F[X_1, \cdots, X_n]$ is a graded ring with P^k being generated by degree k monomials. The noncommutative polynomial ring $Q = F\langle X_1, \cdots, X_n \rangle$ is also a graded ring with Q^k being generated by all words of length k in the letters $X_1, \cdots, X_n$. An example of a graded Lie algebra is the standard Borel subalgebra B of any semisimple Lie algebra L. Then $B^0 = H$ is the CSA and B^k is the direct sum of all B_β where β has height k.

Exercise 17.2.3. Show that P^k has dimension $\binom{n+k-1}{k}$. For example, for $n = 2$, $\dim P^k = k + 1$ with basis elements $x^i y^{k-i}$ for $i = 0, \cdots, k$. Q^k has dimension n^k.

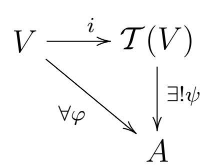
Definition 17.2.4. Given a vector space V, the *tensor algebra* $\mathcal{T}(V)$ is defined to be the vector space

$$\mathcal{T}(V) = F \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \cdots$$

with multiplication defined by tensor product (over F). This is an associative graded algebra with $\mathcal{T}^k(V) = V^{\otimes k}$, the k-fold tensor product of V with itself. Note that, in degree 1, we have $\mathcal{T}^1(V) = V$.

If $V = F^n$ then $\mathcal{T}(F^n) \cong F\langle X_1, \cdots, X_n \rangle$. For example, $V \otimes V$ is n^2 dimensional with basis given by $e_i \otimes e_j$. The tensor algebra has the following universal property.

Proposition 17.2.5. *Any linear map φ from a vector space V to an associative algebra A with unity extends uniquely to a unital algebra homomorphism $\psi : \mathcal{T}(V) \to A$:*



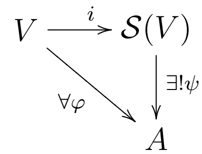
Proof. ψ is must be given by $\psi(1) = 1$, $\psi(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \varphi(v_1)\varphi(v_2)\cdots\varphi(v_k)$. □

This proposition means that $\mathcal{T}(V)$ is the universal associative algebra with unity generated by V.

Definition 17.2.6. The *symmetric algebra* $\mathcal{S}(V)$ generated by V is defined to be the quotient of $\mathcal{T}(V)$ by the ideal generated by all elements of the form $x \otimes y - y \otimes x$. This makes $\mathcal{S}(V)$ into a commutative graded algebra with unity. Since the relations are in degree 2, the degree 1 part is still the same: $\mathcal{S}^1(V) = \mathcal{T}^1(V) = V$.

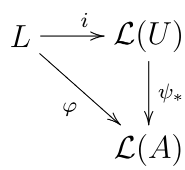
For example, when $V = F^n$ we have $\mathcal{S}(F^n) \cong F[X_1, \cdots, X_n]$.

Proposition 17.2.7. *Any linear map φ from a vector space V to a commutative algebra A with unity extends uniquely to a unital algebra homomorphism $\psi : \mathcal{S}(V) \to A$:*



17.3. **Universal enveloping algebra.** Following tradition, we define this by its desired universal property.

Definition 17.3.1. For any (possibly infinite dimensional) Lie algebra L, the *universal enveloping algebra* of L is defined to be any pair (U, i) where U is an associative algebra with unity and $i : L \to \mathcal{L}(U)$ is a Lie algebra homomorphism with the property that, for any other associative algebra with unity A and any Lie algebra homomorphism $\varphi : L \to \mathcal{L}(A)$ there is a unique unital algebra homomorphism $\psi : U \to A$ so that the following diagram commutes where $\psi_* = \psi$ considered as a homomorphism of Lie algebras.



Example 17.3.2. One important example is the case when $A = \mathrm{End}_F(V)$ is the algebra of F-linear endomorphisms of a vector space V. Then $\mathcal{L}(A) = \mathfrak{gl}(V)$ and $\varphi : L \to \mathcal{L}(A) = \mathfrak{gl}(V)$ is a representation of L making V into an L-module. The algebra homomorphism $\psi : U \to A = \mathrm{End}_F(V)$ makes V into a module over the associative algebra U. Therefore, a module over L is the same as a module over U.

Proposition 17.3.3. *The universal enveloping algebra (U, i) of L is unique up to isomorphism if it exists.*

Proof. If there is another pair (U', i') then, by the universal property, there are algebra homomorphisms $\psi : U \to U'$ and $\psi' : U' \to U$ so that $i' = \psi_* i$ and $i = \psi'_* i'$. But then $i = \psi'_* \psi_* i = (\psi'\psi)_* i$. By uniqueness, we must have $\psi'\psi = id_U$. Similarly $\psi\psi' = id_{U'}$. So, $U \cong U'$ and i, i' correspond under this isomorphism. □

The construction of U is easy when we consider the properties of an arbitrary Lie algebra homomorphism

$$\varphi : L \to \mathcal{L}(A)$$

Since φ is a linear mapping from L to A, it extends uniquely to a unital algebra homomorphism $\overline{\varphi} : \mathcal{T}(L) \to A$. Taking into account that φ is also a Lie algebra homomorphism, we see that, for any two elements $x, y \in L$, we must have

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] = f(x)f(y) - f(y)f(x) = \overline{\varphi}(x \otimes y - y \otimes x)$$

In other words, $\overline{\varphi} : \mathcal{T}(L) \to A$ has the elements

$$x \otimes y - y \otimes x - [x, y]$$

in its kernel. Let J be the two-sided ideal in $\mathcal{T}(L)$ generated by all elements of this form. Then J is in the kernel of $\overline{\varphi}$ and we have an induced unital algebra homomorphism $\psi : \mathcal{T}(L)/J \to A$.

Definition 17.3.4. $\mathcal{U}(L)$ is defined to be the quotient of $\mathcal{T}(L)$ by the ideal generated by all $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in L$. Let $i : L \to \mathcal{U}(L)$ be the inclusion map $i(x) = x$.

Note that the relations imposed on $\mathcal{U}(L)$ are the minimal ones needed to insure that $i : L \to \mathcal{L}(\mathcal{U}(L))$ is a Lie algebra homomorphism. The fact that $(\mathcal{U}(L), i)$ satisfies the definition of a universal enveloping algebra is supposed to be obvious.

Exercise 17.3.5. Show that, for any associative algebra A, there is a canonical unital algebra homomorphism $\mathcal{U}(\mathcal{L}(A)) \to A$. When is this an isomorphism? What happens when A is commutative?

In the special case that L is a graded Lie algebra, such as a standard Borel algebra, $\mathcal{T}(L)$ has another grading given by

$$\mathcal{T}(L)^k = \bigoplus_{\sum j_i = k} L^{j_1} \otimes L^{j_2} \otimes \cdots \otimes L^{j_m}$$

and the ideal J is generated by homogeneous elements. This makes $\mathcal{U}(L)$ into a graded algebra. In general, there is no graded structure on $\mathcal{U}(L)$. However, there is a filtration $\mathcal{U}_k(L)$ induced by the filtration

$$\mathcal{T}_k(L) = \mathcal{T}^0(L) \oplus \mathcal{T}^1(L) \oplus \cdots \oplus \mathcal{T}^k(L)$$

of $\mathcal{T}(L)$.

By a *filtration* of an algebra A we mean a sequence of vector subspaces

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

so that $A = \bigcup A_k$ and so that $A_j A_k \subseteq A_{j+k}$. The *associated graded algebra* is defined by $G^k(A) = A_k/A_{k-1}$ with multiplication $G^j G^k \to G^{j+k}$ induced by the filtered multiplication on A. We let $\mathcal{G}(L)$ denote the associated graded algebra of $\mathcal{U}(L)$.

Example 17.3.6. Suppose that L is abelian. Then $\mathcal{U}(L)$ is $\mathcal{T}(L)$ modulo the ideal generated by $x \otimes y - y \otimes x$ since $[x, y] = 0$. But this means $\mathcal{U}(L)$ is the symmetric algebra $\mathcal{S}(L)$. If $\dim L = n$ then, in filtration k, we have $\mathcal{U}_k(L) = \mathcal{S}_k(L)$ which is equivalent to the vector space of polynomials of degree $\leq k$ in n variables, or equivalently, homogeneous polynomials of degree equal to k in $n + 1$ variables. Thus

$$\dim \mathcal{U}_k(L) = \dim \mathcal{S}_k(F^n) = \dim \mathcal{S}^k(F^{n+1}) = \binom{n+k}{k}$$

In this example, $\mathcal{U}(L) \cong \mathcal{G}(L) \cong \mathcal{S}(L)$.

17.4. **Poincaré-Birkov-Witt.** The PBW Theorem gives a detailed description of the structure of the universal enveloping algebra $\mathcal{U}(L)$. It gives a formula for a vector space basis and how they are multiplied. One of the amazing features of the theorem is that it says that the dimension of $\mathcal{U}_k(L)$ depends only on the dimension of L. I.e., the dimension is always $\binom{n+k}{k}$. For each $k \geq 0$ consider the composition:

$$\varphi^k : \mathcal{T}^k(L) = L^{\otimes k} \hookrightarrow \mathcal{T}_k(L) \twoheadrightarrow \mathcal{U}_k(L) \twoheadrightarrow \mathcal{G}^k(L) = \mathcal{U}_k(L)/\mathcal{U}_{k-1}(L)$$

Lemma 17.4.1. *$\varphi = \sum \varphi^k : \mathcal{T}(L) \to \mathcal{G}(L)$ is a graded algebra epimorphism which induced a graded algebra epimorphism $\mathcal{S}(L) \twoheadrightarrow \mathcal{G}(L)$.*

Proof. The morphism φ is multiplicative by definition. So, it is an algebra epimorphism. Elements of the form $x \otimes y - y \otimes x \in \mathcal{T}^2(L)$ are sent to $[x, y] \in \mathcal{U}_1(L)$ which is zero in $G^2(L)$. Therefore, there is an induced algebra epimorphism $\mathcal{S}(L) \twoheadrightarrow \mathcal{G}(L)$. □

Theorem 17.4.2 (PBW)**.** *The natural graded epimorphism $\mathcal{S}(L) \to \mathcal{G}(L)$ is always an isomorphism.*

Corollary 17.4.3 (PBW basis)**.** *If $x_1, \cdots, x_n$ is a vector space basis for L then a vector space basis for $\mathcal{U}_k(L)$ is given by all monomials of length $\leq k$ of the form*

$$x_{j_1} x_{j_2} \cdots x_{j_\ell}$$

where $j_1 \leq j_2 \leq j_3 \leq \cdots \leq j_\ell$ plus 1 (given by the empty word). In particular, $i : L \to \mathcal{U}(L)$ is a monomorphism.

Proof. Monomials as above (with nondecreasing indices) of length equal to k form a basis for $\mathcal{S}^k(L)$ and therefore give a basis for $\mathcal{U}_k(L)$ modulo $\mathcal{U}_{k-1}$. □

Corollary 17.4.4. *If H is any subalgebra of L then the inclusion $H \hookrightarrow L$ extends to a monomorphism $\mathcal{U}(H) \hookrightarrow \mathcal{U}(L)$. Furthermore $\mathcal{U}(L)$ is a free $\mathcal{U}(H)$-modules.*

Proof. Extend an ordered basis of H to an ordered basis for L and use PBW bases. □

We will skip the proof of the PBW Theorem in the lectures. So, you need to read the proof. There is a purely algebraic proof in the book. Here I will give a proof using combinatorial group theory.