



Article

# Distance Measures between the Interval-Valued Complex Fuzzy Sets

Songsong Dai <sup>1</sup> , Lvqing Bi <sup>2,\*</sup>  and Bo Hu <sup>3</sup>

<sup>1</sup> School of Electronics and Information Engineering, Taizhou University, Taizhou 318000, China; ssdai@stu.xmu.edu.cn

<sup>2</sup> School of Physics and Telecommunication Engineering, Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, China

<sup>3</sup> School of Mechanical and Electrical Engineering, Guizhou Normal University, Guiyang 550025, China; hubo@gznu.edu.cn

\* Correspondence: bilvqing@stu.xmu.edu.cn

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**Abstract:** Complex fuzzy set (CFS) is a recent development in the field of fuzzy set (FS) theory. The significance of CFS lies in the fact that CFS assigned membership grades from a unit circle in the complex plane, i.e., in the form of a complex number whose amplitude term belongs to a  $[0, 1]$  interval. The interval-valued complex fuzzy set (IVCFS) is one of the extensions of the CFS in which the amplitude term is extended from the real numbers to the interval-valued numbers. The novelty of IVCFS lies in its larger range comparative to CFS. We often use fuzzy distance measures to solve some problems in our daily life. Hence, this paper develops some series of distance measures between IVCFSs by using Hamming and Euclidean metrics. The boundaries of these distance measures for IVCFSs are obtained. Finally, we study two geometric properties include rotational invariance and reflectional invariance of these distance measures.

**Keywords:** Interval-valued complex fuzzy sets; distance measures; rotational invariance; reflectional invariance

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## 1. Introduction

Complex fuzzy set [1] and some extensions of complex fuzzy set such as complex intuitionistic fuzzy sets [2], complex neutrosophic set [3], interval complex neutrosophic set [4], complex multi-fuzzy soft set [5], complex interval-valued intuitionistic fuzzy sets [6], complex vague soft sets [7] and interval-valued complex fuzzy sets [8,9] have been introduced and used in a variety of fields such as signal processing [10–12], decision-making [6,13–16], time series prediction [17–20] and image restoration [21].

The concept of distance measure can be defined as a numerical description of the difference between two objects. There are many distance measures introduced by numerous scholars to solve some problems in many fields to solve some problems and discover several applications. Among these distances, Euclidean distance and Hamming distance are widely used in several fields. It is well known that distance measure is also an important issue on fuzzy set and its extensions [22–29]. As defined in [1], each complex membership grade includes an amplitude term and a phase term. Hence, many researchers defined distances of CFSs by combining the difference between the phase terms and the difference between the amplitude terms. In this way, Zhang et al. [11] defined a distance measure of CFSs. Hu et al. [30] and Alkouri et al. [31] also introduced some distances for CFSs. These distance measures in [30,31] are defined by using Hamming and

Euclidean metrics. Recently, several concepts including the parallelity [32,33] and entropy measures [34] of complex fuzzy sets and distances of complex intuitionistic fuzzy sets [35], complex vague soft sets [7,36,37] and complex multi-fuzzy soft set [5] have been developed. Since Greenfield et al. [8,9] introduced the concept of an interval-valued complex fuzzy sets (IVCFSs), which is a generalization of the complex fuzzy set introduced by Ramot et al. [1], some progress has been made in various subjects related to the IVCFS. How to measure the difference between IVCFSs is an interesting subject that may be used as a useful tool for decision-making, predictions and pattern recognition in the form of IVCFSs. This paper considers the distance measures between interval-valued complex fuzzy sets.

In [38], Dick proposed the concept of rotational invariance for complex fuzzy operations, which is an intuitive and desirable feature. Then, this feature is examined for interval-valued complex fuzzy operations in [9], complex fuzzy aggregation operators in [15] and entropy measures of CFSs in [34]. Moreover, Bi et al. [15] proposed a novel feature called reflectional invariance for complex fuzzy aggregation operators. These two feature criteria serve as two criteria for choices of complex fuzzy operations and complex fuzzy aggregation operators in some real applications. After giving the distance measures for IVCFSs, we then examine the rotational invariance and reflectional invariance for distance measures of IVCFSs. In addition, the results proposed in this paper might serve as criteria for choices of distance measures of CFSs and IVCFSs.

The rest of the chapter is organized in the following way: In Section 2, we review the basic concepts relates to interval-valued complex fuzzy sets. In Section 3, we develop some distance measures of interval-valued complex fuzzy sets based on the Hamming distance and Euclidean distance. In Section 4, we study rotational invariance and reflectional invariance of these distance measures. In Section 5, we apply these distance measures to solve a decision-making problem with interval-valued complex fuzzy information. Conclusions are given in Section 6.

### 2. Preliminaries

Firstly, let us review the basic concepts relates to interval-valued complex fuzzy sets [8,9].

Let  $D$  be the set of values on complex unit disk, i.e.,  $D = \{a \in \mathbb{C} \mid |a| \leq 1\}$ . Suppose  $U = \{x_1, x_2, \dots, x_n\}$  is a fixed universe. A mapping  $A : U \rightarrow D$  is called a complex fuzzy set on  $U$ .

Let  $\mathbb{I}^{[0,1]}$  be the set of all closed subintervals of  $[0, 1]$  and  $\dot{D}$  be the boundary set of  $D$  i.e.,  $\dot{D} = \{a \in \mathbb{C} \mid |a| = 1\}$ . Suppose  $X, Y$  are two classical sets, and their dot product set  $X \cdot Y$  is denoted as:

$$X \cdot Y = \{\alpha \cdot \beta \mid \alpha \in X \wedge \beta \in Y\}. \tag{1}$$

A mapping  $A : U \rightarrow \mathbb{I}^{[0,1]} \cdot \dot{D}$  is called an interval-valued complex fuzzy set [8,9] on  $U$ . For any  $x \in U$ , the membership function  $A(x)$  is

$$\mu_A(x) = \left[ \underline{r}_A(x), \overline{r}_A(x) \right] \cdot e^{j\nu_A(x)}, \tag{2}$$

where  $j = \sqrt{-1}$ ,  $\left[ \underline{r}_A(x), \overline{r}_A(x) \right] \in \mathbb{I}^{[0,1]}$  is the interval-valued amplitude part and  $\nu_A(x) \in [0, 2\pi)$  is the phase part. We use the notation  $IVCF(U)$  to denote the set of all IVCFSs of  $U$ .

### 3. Distance Measures between IVCFSs

In this section, we introduce several distances in interval-valued complex fuzzy sets case. First, we give the definition of distance between interval-valued complex fuzzy sets as follows:

**Definition 1.** A distance of interval-valued complex fuzzy sets is a function  $d : (IVCF(U) \times IVCF(U)) \rightarrow \mathbb{R}^+ \cup \{0\}$ , which satisfies the following conditions: for any  $A, B, C \in IVCF(U)$ :

- (i)  $d(A, B) \geq 0$  and  $d(A, B) = 0$  if and only if  $A = B$ ,
- (ii)  $d(A, B) = d(B, A)$ ,
- (iii)  $d(A, B) + d(A, C) \geq d(B, C)$ .

Now, we define the Hamming, Euclidean, Normalized Hamming, and Normalized Euclidean distances in interval-valued complex fuzzy sets case as follows: for any  $A, B \in IVCF(U)$ ,

- The Hamming distance:

$$d_H(A, B) = \frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |r_A(x_i) - r_B(x_i)| + \frac{1}{2} |\overline{r_A(x_i)} - \overline{r_B(x_i)}| + \frac{1}{2\pi} |v_A(x_i) - v_B(x_i)| \right); \quad (3)$$

- The Euclidean distance:

$$d_E(A, B) = \sqrt{\frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |r_A(x_i) - r_B(x_i)|^2 + \frac{1}{2} |\overline{r_A(x_i)} - \overline{r_B(x_i)}|^2 + \frac{1}{4\pi^2} |v_A(x_i) - v_B(x_i)|^2 \right)}; \quad (4)$$

- The normalized Hamming distance:

$$d_{nH}(A, B) = \frac{1}{2n} \sum_{i=1}^n \left( \frac{1}{2} |r_A(x_i) - r_B(x_i)| + \frac{1}{2} |\overline{r_A(x_i)} - \overline{r_B(x_i)}| + \frac{1}{2\pi} |v_A(x_i) - v_B(x_i)| \right); \quad (5)$$

- The normalized Euclidean distance:

$$d_{nE}(A, B) = \sqrt{\frac{1}{2n} \sum_{i=1}^n \left( \frac{1}{2} |r_A(x_i) - r_B(x_i)|^2 + \frac{1}{2} |\overline{r_A(x_i)} - \overline{r_B(x_i)}|^2 + \frac{1}{4\pi^2} |v_A(x_i) - v_B(x_i)|^2 \right)}. \quad (6)$$

Clearly, it is easy to notice that the above Equations (3)–(6) have the following results:

$$\begin{aligned} 0 \leq d_H(A, B) \leq n, \quad 0 \leq d_{nH}(A, B) \leq 1, \\ 0 \leq d_E(A, B) \leq \sqrt{n}, \quad 0 \leq d_{nE}(A, B) \leq 1. \end{aligned}$$

**Theorem 1.** All functions defined in Equations (3)–(6) are distance measures of IVCFs.

**Proof.** It is easy to see that all the distance Equations (3)–(6) satisfy conditions (i) and (ii). Thus, we just go to prove the condition number (iii), i.e., the triangular inequality for  $d_H(A, B), d_E(A, B), d_{nH}(A, B), d_{nE}(A, B)$ .

Let  $A, B, C \in IVCF(U)$ ; for the Hamming distance, we have

$$\begin{aligned}
 & d_H(A, C) + d_H(C, B) \\
 = & \frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |\underline{r}_A(x_i) - \underline{r}_C(x_i)| + \frac{1}{2} |\overline{r}_A(x_i) - \overline{r}_C(x_i)| + \frac{1}{2\pi} |v_A(x_i) - v_C(x_i)| \right) \\
 & + \frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |\underline{r}_C(x_i) - \underline{r}_B(x_i)| + \frac{1}{2} |\overline{r}_C(x_i) - \overline{r}_B(x_i)| + \frac{1}{2\pi} |v_C(x_i) - v_B(x_i)| \right) \\
 = & \frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} (|\underline{r}_A(x_i) - \underline{r}_C(x_i)| + |\underline{r}_C(x_i) - \underline{r}_B(x_i)|) \right. \\
 & + \frac{1}{2} (|\overline{r}_A(x_i) - \overline{r}_C(x_i)| + |\overline{r}_C(x_i) - \overline{r}_B(x_i)|) \\
 & \left. + \frac{1}{2\pi} (|v_A(x_i) - v_C(x_i)| + |v_C(x_i) - v_B(x_i)|) \right) \\
 \geq & \frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} (|\underline{r}_A(x_i) - \underline{r}_B(x_i)|) + \frac{1}{2} (|\overline{r}_A(x_i) - \overline{r}_B(x_i)|) + \frac{1}{2\pi} (|v_A(x_i) - v_B(x_i)|) \right) \\
 = & d_H(A, B).
 \end{aligned}$$

For the Euclidean distance, we have

$$\begin{aligned}
 & d_E(A, C) + d_E(C, B) \\
 = & \sqrt{\frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |\underline{r}_A(x_i) - \underline{r}_C(x_i)|^2 + \frac{1}{2} |\overline{r}_A(x_i) - \overline{r}_C(x_i)|^2 + \frac{1}{4\pi^2} |v_A(x_i) - v_C(x_i)|^2 \right)} \\
 & + \sqrt{\frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |\underline{r}_C(x_i) - \underline{r}_B(x_i)|^2 + \frac{1}{2} |\overline{r}_C(x_i) - \overline{r}_B(x_i)|^2 + \frac{1}{4\pi^2} |v_C(x_i) - v_B(x_i)|^2 \right)} \\
 \geq & \left( \frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |\underline{r}_A(x_i) - \underline{r}_C(x_i) + \underline{r}_C(x_i) - \underline{r}_B(x_i)|^2 + \frac{1}{2} |\overline{r}_A(x_i) - \overline{r}_C(x_i) + \overline{r}_C(x_i) - \overline{r}_B(x_i)|^2 \right. \right. \\
 & \left. \left. + \frac{1}{4\pi^2} |v_A(x_i) - v_C(x_i) + v_C(x_i) - v_B(x_i)|^2 \right) \right)^{\frac{1}{2}} \text{ (Using Minkowski inequality)} \\
 = & \sqrt{\frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |\underline{r}_A(x_i) - \underline{r}_B(x_i)|^2 + \frac{1}{2} |\overline{r}_A(x_i) - \overline{r}_B(x_i)|^2 + \frac{1}{4\pi^2} |v_A(x_i) - v_B(x_i)|^2 \right)} \\
 = & d_E(A, B),
 \end{aligned}$$

analogously, for the proof of the normalized Hamming  $d_{nH}(A, B)$  and the normalized Euclidean  $d_{nE}(A, B)$  distances.  $\square$

**Example 1.** Let  $A$  and  $B$  be two IVCFSs on  $X = \{x_1, x_2, x_3, x_4, x_5\}$ , which are given as follows:

$$\begin{aligned}
 A &= \frac{[0.4, 0.5]e^{j0.3\pi}}{x_1} + \frac{[0.6, 0.9]e^{j0.5\pi}}{x_2} + \frac{[0.6, 0.9]e^{j0.1\pi}}{x_3} + \frac{[0.7, 0.9]e^{j0.2\pi}}{x_4} + \frac{[0.4, 0.7]e^{j0.5\pi}}{x_5}, \\
 B &= \frac{[0.3, 0.4]e^{j1.3\pi}}{x_1} + \frac{[0.2, 0.4]e^{j0.4\pi}}{x_2} + \frac{[0.5, 0.8]e^{j0.5\pi}}{x_3} + \frac{[0.4, 0.6]e^{j0.1\pi}}{x_4} + \frac{[0.8, 0.9]e^{j0.7\pi}}{x_5}.
 \end{aligned}$$

Using Equations (3)–(6), we have the following results:

$$\begin{aligned}
 d_H(A, B) &= \frac{1}{2} \left( \frac{1}{2} |0.4 - 0.3| + \frac{1}{2} |0.5 - 0.4| + \frac{1}{2\pi} |0.3\pi - 1.3\pi| \right) \\
 &\quad + \frac{1}{2} \left( \frac{1}{2} |0.6 - 0.2| + \frac{1}{2} |0.9 - 0.4| + \frac{1}{2\pi} |0.5\pi - 0.4\pi| \right) \\
 &\quad + \frac{1}{2} \left( \frac{1}{2} |0.6 - 0.5| + \frac{1}{2} |0.9 - 0.8| + \frac{1}{2\pi} |0.1\pi - 0.5\pi| \right) \\
 &\quad + \frac{1}{2} \left( \frac{1}{2} |0.7 - 0.4| + \frac{1}{2} |0.9 - 0.6| + \frac{1}{2\pi} |0.2\pi - 0.1\pi| \right) \\
 &\quad + \frac{1}{2} \left( \frac{1}{2} |0.4 - 0.8| + \frac{1}{2} |0.7 - 0.9| + \frac{1}{2\pi} |0.5\pi - 0.7\pi| \right) \\
 &= 0.3 + 0.25 + 0.15 + 0.175 + 0.2 \\
 &= 1.075,
 \end{aligned}$$

$$\begin{aligned}
 d_E(A, B) &= \left( \frac{1}{2} \left( \frac{1}{2} |0.4 - 0.3|^2 + \frac{1}{2} |0.5 - 0.4|^2 + \frac{1}{4\pi^2} |0.3\pi - 1.3\pi|^2 \right) \right. \\
 &\quad + \frac{1}{2} \left( \frac{1}{2} |0.6 - 0.2|^2 + \frac{1}{2} |0.9 - 0.4|^2 + \frac{1}{4\pi^2} |0.5\pi - 0.4\pi|^2 \right) \\
 &\quad + \frac{1}{2} \left( \frac{1}{2} |0.6 - 0.5|^2 + \frac{1}{2} |0.9 - 0.8|^2 + \frac{1}{4\pi^2} |0.1\pi - 0.5\pi|^2 \right) \\
 &\quad + \frac{1}{2} \left( \frac{1}{2} |0.7 - 0.4|^2 + \frac{1}{2} |0.9 - 0.6|^2 + \frac{1}{4\pi^2} |0.2\pi - 0.1\pi|^2 \right) \\
 &\quad \left. + \frac{1}{2} \left( \frac{1}{2} |0.4 - 0.8|^2 + \frac{1}{2} |0.7 - 0.9|^2 + \frac{1}{4\pi^2} |0.5\pi - 0.7\pi|^2 \right) \right)^{1/2} \\
 &= (0.13 + 0.10375 + 0.025 + 0.0925 + 0.055)^{1/2} \\
 &\approx 0.6374,
 \end{aligned}$$

and

$$d_{nH}(A, B) = 0.215, \quad d_{nE}(A, B) \approx 0.285.$$

However, in practice, the different sets may have taken different weights i.e.,  $w_i \geq 0, i = 1, 2, \dots, n$ , and  $\sum_{i=1}^n w_i = 1$ , for each element  $x_i \in U$ . Thus, we introduce the normalized weighted Hamming and Euclidean distance measures for IVCFSs.

- The normalized weighted Hamming distance:

$$d_{nwH}(A, B) = \frac{1}{2} \sum_{i=1}^n w_i \left( \frac{1}{2} |\underline{r}_A(x_i) - \underline{r}_B(x_i)| + \frac{1}{2} |\overline{r}_A(x_i) - \overline{r}_B(x_i)| + \frac{1}{2\pi} |\nu_A(x_i) - \nu_B(x_i)| \right); \quad (7)$$

- The normalized weighted Euclidean distance:

$$d_{nwE}(A, B) = \sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left( \frac{1}{2} |\underline{r}_A(x_i) - \underline{r}_B(x_i)|^2 + \frac{1}{2} |\overline{r}_A(x_i) - \overline{r}_B(x_i)|^2 + \frac{1}{4\pi^2} |\nu_A(x_i) - \nu_B(x_i)|^2 \right)}. \quad (8)$$

**Theorem 2.** All functions defined in Equations (7) and (8) are distance measures of IVCFSs.

**Proof.** The proof is similar to that of Theorem 1. □

Obviously, when  $w_i = 1/n$  for all  $i = 1, 2, \dots, n$ , then Equations (7) and (8) reduce to Equations (5) and (6), respectively. In addition, the above distance Equations (7) and (8) have the following results:

$$0 \leq d_{nwH}(A, B) \leq 1,$$

$$0 \leq d_{nwE}(A, B) \leq 1.$$

**Example 2.** Let  $A$  and  $B$  be two IVCFSSs defined in Example 1 and the weight of  $x_1, x_2, \dots, x_5$  is  $w = (0.1, 0.1, 0.2, 0.2, 0.4)$ .

Using Equations (7) and (8), we have the following results:

$$\begin{aligned} d_{nwH}(A, B) &= \frac{0.1}{2} \left( \frac{1}{2} |0.4 - 0.3| + \frac{1}{2} |0.5 - 0.4| + \frac{1}{2\pi} |0.3\pi - 1.3\pi| \right) \\ &\quad + \frac{0.1}{2} \left( \frac{1}{2} |0.6 - 0.2| + \frac{1}{2} |0.9 - 0.4| + \frac{1}{2\pi} |0.5\pi - 0.4\pi| \right) \\ &\quad + \frac{0.2}{2} \left( \frac{1}{2} |0.6 - 0.5| + \frac{1}{2} |0.9 - 0.8| + \frac{1}{2\pi} |0.1\pi - 0.5\pi| \right) \\ &\quad + \frac{0.2}{2} \left( \frac{1}{2} |0.7 - 0.4| + \frac{1}{2} |0.9 - 0.6| + \frac{1}{2\pi} |0.2\pi - 0.1\pi| \right) \\ &\quad + \frac{0.4}{2} \left( \frac{1}{2} |0.4 - 0.8| + \frac{1}{2} |0.7 - 0.9| + \frac{1}{2\pi} |0.5\pi - 0.7\pi| \right) \\ &= 0.03 + 0.025 + 0.03 + 0.035 + 0.08 \\ &= 0.2, \end{aligned}$$

$$\begin{aligned} d_{nwE}(A, B) &= \left( \frac{0.1}{2} \left( \frac{1}{2} |0.4 - 0.3|^2 + \frac{1}{2} |0.5 - 0.4|^2 + \frac{1}{4\pi^2} |0.3\pi - 1.3\pi|^2 \right) \right. \\ &\quad + \frac{0.1}{2} \left( \frac{1}{2} |0.6 - 0.2|^2 + \frac{1}{2} |0.9 - 0.4|^2 + \frac{1}{4\pi^2} |0.5\pi - 0.4\pi|^2 \right) \\ &\quad + \frac{0.2}{2} \left( \frac{1}{2} |0.6 - 0.5|^2 + \frac{1}{2} |0.9 - 0.8|^2 + \frac{1}{4\pi^2} |0.1\pi - 0.5\pi|^2 \right) \\ &\quad + \frac{0.2}{2} \left( \frac{1}{2} |0.7 - 0.4|^2 + \frac{1}{2} |0.9 - 0.6|^2 + \frac{1}{4\pi^2} |0.2\pi - 0.1\pi|^2 \right) \\ &\quad \left. + \frac{0.4}{2} \left( \frac{1}{2} |0.4 - 0.8|^2 + \frac{1}{2} |0.7 - 0.9|^2 + \frac{1}{4\pi^2} |0.5\pi - 0.7\pi|^2 \right) \right)^{1/2} \\ &= (0.013 + 0.010375 + 0.005 + 0.0185 + 0.022)^{1/2} \\ &\approx 0.2624. \end{aligned}$$

By the relations among IVCFSS, CFS, interval-valued fuzzy sets (IVFS) and FS, we established the comparison of the proposed distance measures of IVCFSSs with CFS, IVFS and FS. It is proposed that these distance measures reduce the environments of CFS, IVFS and FS. The comparison is demonstrated in Remarks 1–3.

**Remark 1.** The distance measures proposed in Equations (3)–(6) for IVCFSSs reduce to the environment of CFSs as defined in [31].

**Remark 2.** The distance measures proposed in Equations (3)–(6) for IVCFs reduce to the environment of IVFSs if we considered the phase term as zero as defined in [24].

**Remark 3.** The distance measures proposed in Equations (7) and (8) for IVCFs reduce to the environment of CFSs as defined in Equations (9) and (10):

$$D_{nwH}(A, B) = \frac{1}{2} \sum_{i=1}^n w_i \left( |r_A(x_i) - r_B(x_i)| + \frac{1}{2\pi} |v_A(x_i) - v_B(x_i)| \right), \tag{9}$$

$$D_{nwE}(A, B) = \sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left( |r_A(x_i) - r_B(x_i)|^2 + \frac{1}{4\pi^2} |v_A(x_i) - v_B(x_i)|^2 \right)}. \tag{10}$$

Here, we demonstrate the advantages of the proposed distance measures of IVCFs. Firstly, the distance measures of CFSs, IVFSs, and FS could not handle the information provided in the form of IVCFs. On the other hand, from Remarks 1 and 2, the proposed distance measures in Equations (3)–(6) can solve some problem that lies in the region of CFSs, IVFSs, and FS. Moreover, the distance measures proposed in Remark 3 for CFSs are two new distance Equations for CFSs.

#### 4. Rotational Invariance and Reflectional Invariance

Let  $A$  be a IVCF on  $U$  with membership function  $[r_A(x), \overline{r_A(x)}] \cdot e^{jv_A(x)}$ , and then we define the following two set operations for IVCFs:

The rotation of an IVCF  $A$  by  $\theta$  radians, denoted  $Rot_\theta(A)$  is defined as

$$Rot_\theta(\mu_A(x)) = [r_A(x), \overline{r_A(x)}] \cdot e^{j(v_A(x)+\theta)}. \tag{11}$$

The reflection of an IVCF  $A$ , denoted  $Ref(A)$  is defined as

$$Ref(\mu_A(x)) = [r_A(x), \overline{r_A(x)}] \cdot e^{j(2\pi-v_A(x))}. \tag{12}$$

Moreover, we define the following:

**Definition 2.** A distance  $d$  on IVCF is rotationally invariant if

$$d(Rot_\theta(A), Rot_\theta(B)) = d(A, B) \tag{13}$$

for any  $\theta$  and  $A, B \in IVCF(U)$ .

**Definition 3.** A distance  $d$  on IVCF is reflectionally invariant if

$$d(Ref(A), Ref(B)) = d(A, B) \tag{14}$$

for any  $A, B \in IVCF(U)$ .

Definitions 2 and 3 give two intuitive properties of distance measures of IVCFs, named rotational invariance and reflectional invariance, respectively. They mean that the distance measure is invariant under a rotation or a reflection. More specifically, if two IVCFs are rotated by a fixed value, rotational invariance states that the distance between two new IVCFs will not change. If two IVCFs are reflected,

reflectional invariance states that the distance between two new IVCFSs will not change as well, which are respectively shown in Figure 1,  $d(A', B')$  is equal to  $d(A, B)$  after reflecting or rotating.

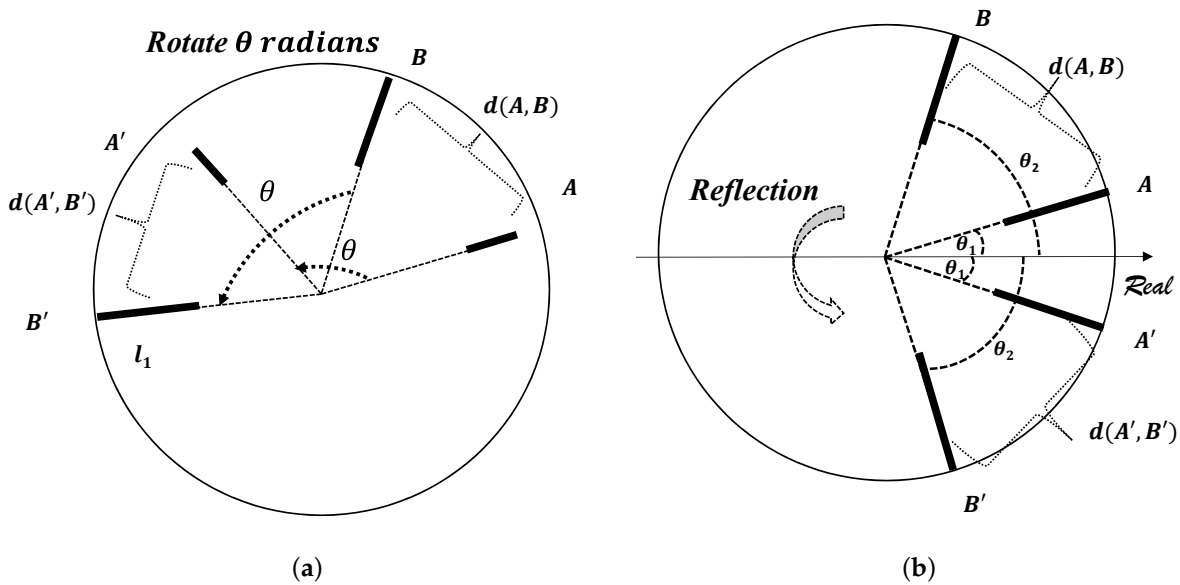


Figure 1. (a) reflectional invariance and (b) rotational invariance.

**Theorem 3.** All functions defined in Equations (3)–(8) are reflectionally invariant.

**Proof.** It is easy from that  $|(2\pi - \nu_A(x_i)) - (2\pi - \nu_B(x_i))| = |\nu_A(x_i) - \nu_B(x_i)|$  for any  $x_i \in U$ .  $\square$

**Theorem 4.** All functions defined in Equations (3)–(8) are not rotationally invariant.

**Proof.** Since the domain of the phase term is  $\nu_A(x) \in [0, 2\pi)$ , so the above rotation operation is addition modulo  $2\pi$ . Consider two IVCFSs on the set  $U = \{x_1, x_2\}$ ; let  $\mu_A(x) \equiv 1 \cdot e^{j\frac{\pi}{4}}$  and  $\mu_B(x) \equiv 1 \cdot e^{j\frac{7\pi}{4}}$  for any  $x \in U$ . Then,  $Rot_{\frac{\pi}{2}}(A) \equiv 1 \cdot e^{j\frac{3\pi}{4}}$  and  $Rot_{\frac{\pi}{2}}(B) \equiv 1 \cdot e^{j\frac{\pi}{4}}$ . By using Equations (3)–(8), we have

$$\begin{aligned}
 d_H(A, B) &= \frac{3}{4}, & d_H(Rot_{\frac{\pi}{2}}(A), Rot_{\frac{\pi}{2}}(B)) &= \frac{1}{4}, \\
 d_E(A, B) &= \frac{3}{4}, & d_H(Rot_{\frac{\pi}{2}}(A), Rot_{\frac{\pi}{2}}(B)) &= \frac{1}{4}, \\
 d_{nH}(A, B) &= \frac{3}{8}, & d_{nH}(Rot_{\frac{\pi}{2}}(A), Rot_{\frac{\pi}{2}}(B)) &= \frac{1}{8}, \\
 d_{nE}(A, B) &= \frac{3\sqrt{2}}{8}, & d_H(Rot_{\frac{\pi}{2}}(A), Rot_{\frac{\pi}{2}}(B)) &= \frac{\sqrt{2}}{8}, \\
 d_{nwH}(A, B) &= \frac{3}{8}, & d_{nwH}(Rot_{\frac{\pi}{2}}(A), Rot_{\frac{\pi}{2}}(B)) &= \frac{1}{8}, \\
 d_{nwE}(A, B) &= \frac{3\sqrt{2}}{8}, & d_{nwH}(Rot_{\frac{\pi}{2}}(A), Rot_{\frac{\pi}{2}}(B)) &= \frac{\sqrt{2}}{8}.
 \end{aligned}$$

Thus, the distance functions defined in Equations (3)–(8) are not rotationally invariant.  $\square$



Obviously, the concepts of rotational invariance and reflectional invariance can be examined for distance measures of CFSs when IVCFSs are reduced to CFSs. Then, we can study the rotational invariance and reflectional invariance of distance measures of CFSs.

**Remark 4.** *The concepts of rotational invariance and reflectional invariance proposed respectively in Definitions 2 and 3 for distance measures of IVCFSs reduce to the environment of CFSs.*

With the combination of Remark 1, we have the following results for the distance measures of CFSs.

**Corollary 1.** *Distance measures of CFSs defined in [31] are reflectionally invariant.*

**Corollary 2.** *Distance measures of CFSs defined in [31] are not rotationally invariant.*

With the combination of Remark 3, we have the following results for the distance measures of CFSs.

**Corollary 3.** *Distance functions of CFSs defined in Equations (9) and (10) are reflectionally invariant.*

**Corollary 4.** *Distance functions of CFSs defined in Equations (9) and (10) are not rotationally invariant.*

### 5. Numerical Example for Decision-Making

In this section, we consider a decision-making problem under an interval-valued complex fuzzy environment.

Assume that a customer decides to purchase a new computer for his own private use. There are four alternatives ( $A_1, A_2, A_3, A_4$ ) with different production dates. The customer considers five attributes, namely  $C_1$ : Performance,  $C_2$ : Appearance,  $C_3$ : Service,  $C_4$ : Experience and  $C_5$ : Price with the weight vectors  $w = (0.2, 0.2, 0.1, 0.1, 0.4)$  for selecting computer. The corresponding rating values of attributes of alternatives are interval-valued complex fuzzy values as shown in Table 1. For instance, the rating value of the alternative  $A_1$  under  $C_1$  attribute is the interval-valued complex fuzzy value  $[0.8, 0.9] \cdot e^{j2\pi(0.8)}$ , which means that the performance of the computer  $A_1$  at  $C_1$  is marked by an interval value  $[0.8, 0.9]$ , and the customer satisfied 80% with the suitability of production date of the computer  $A_1$  at  $C_1$ ; this explanation of the phase part is referred to Garg et al.’s works on complex intuitionistic fuzzy aggregation operators [13,14]. The customer wants to select a computer among the four alternatives above. Here, we use the principle of minimum degree of difference between IVCFSs to solve the given problem.

**Table 1.** Feature values of the facial expressions.

	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$
$A_1$	$[0.8, 0.9] \cdot e^{j2\pi(0.8)}$	$[0.7, 0.8] \cdot e^{j2\pi(0.8)}$	$[0.8, 0.9] \cdot e^{j2\pi(0.9)}$	$[0.7, 0.9] \cdot e^{j2\pi(0.9)}$	$[0.8, 0.9] \cdot e^{j2\pi(0.6)}$
$A_2$	$[0.8, 0.9] \cdot e^{j2\pi(0.7)}$	$[0.7, 0.8] \cdot e^{j2\pi(0.9)}$	$[0.7, 0.8] \cdot e^{j2\pi(0.7)}$	$[0.7, 0.8] \cdot e^{j2\pi(0.7)}$	$[0.6, 0.7] \cdot e^{j2\pi(0.8)}$
$A_3$	$[0.6, 0.7] \cdot e^{j2\pi(0.7)}$	$[0.8, 0.9] \cdot e^{j2\pi(0.7)}$	$[0.7, 0.8] \cdot e^{j2\pi(0.7)}$	$[0.6, 0.7] \cdot e^{j2\pi(0.8)}$	$[0.7, 0.8] \cdot e^{j2\pi(0.8)}$
$A_4$	$[0.8, 0.9] \cdot e^{j2\pi(0.8)}$	$[0.7, 0.8] \cdot e^{j2\pi(0.9)}$	$[0.6, 0.7] \cdot e^{j2\pi(0.9)}$	$[0.8, 0.9] \cdot e^{j2\pi(0.8)}$	$[0.8, 0.9] \cdot e^{j2\pi(0.9)}$

Now, we calculate the distance between the ideal choice (each rating value is  $1 \cdot e^{j2\pi(1)}$ ) and  $A_i$  ( $i = 1, 2, 3, 4$ ) with the weighting vector  $w$ . We consider the distances as defined in Equations (3)–(8) and obtain the results as shown in Table 2.

**Table 2.** Distance results.

	$d_H$	$d_E$	$d_{nH}$	$d_{nE}$	$d_{nwH}$	$d_{nwE}$
$A_1$	0.95	0.4743	0.19	0.2121	0.2175	0.2424
$A_2$	1.225	0.5766	0.245	0.2579	0.245	0.2598
$A_3$	1.325	0.6144	0.265	0.2748	0.255	0.2646
$A_4$	0.875	0.433	0.175	0.1937	0.16	0.1761

From Table 2, we get an ordering of the choices for each case as shown in Table 3. Clearly, the best alternative is  $A_4$ , which is the one with the lowest distance to the ideal choice.

**Table 3.** Ordering of the alternatives.

Ordering	
$d_H$	$A_4 \succ A_1 \succ A_2 \succ A_3$
$d_E$	$A_4 \succ A_1 \succ A_2 \succ A_3$
$d_{nH}$	$A_4 \succ A_1 \succ A_2 \succ A_3$
$d_{nE}$	$A_4 \succ A_1 \succ A_2 \succ A_3$
$d_{nwH}$	$A_4 \succ A_1 \succ A_2 \succ A_3$
$d_{nwE}$	$A_4 \succ A_1 \succ A_2 \succ A_3$

### 6. Conclusions

In this paper, we have presented several distance measures in the case of the interval-valued complex fuzzy sets, which are very useful to deal with the decision information represented in interval-valued complex fuzzy values under uncertain situations. These distance measures include a family of Hamming and Euclidean distances such as the Hamming distance, the normalized Hamming distance, the normalized weighted Hamming distance, the Euclidean distance, the normalized Euclidean distance and the normalized weighted Euclidean distance. Furthermore, these distance measures are reflectionally invariant but not rotationally invariant. Finally, based on these measures, we presented an illustrative example for decision-making under interval-valued complex fuzzy information.

As future work, we can consider distance measures for interval-valued complex fuzzy sets that are both reflectionally invariant and rotationally invariant.

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