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Algorithm for T-Spherical Fuzzy Multi-Attribute Decision Making Based on Improved Interactive Aggregation Operators

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Abstract: The objective of this manuscript is to present some new, improved aggregation operators for the T-spherical fuzzy sets, which is an extension of the several existing sets, such as intuitionistic fuzzy sets, picture fuzzy sets, neutrosophic sets, and Pythagorean fuzzy sets. In it, some new, improved operational laws and their corresponding properties are studied. Further, based on these laws, we propose some geometric aggregation operators and study their various relationships. Desirable properties, as well as some special cases of the proposed operators, are studied. Then, based on these proposed operators, we present a decision-making approach to solve the multi-attribute decision-making problems. The reliability of the presented decision-making method is explored with the help of a numerical example and the proposed results are compared with several prevailing studies' results. Finally, the superiority of the proposed approach is explained with a counter example to show the advantages of the proposed work.

Keywords: multi-attribute decision making; aggregation operators; spherical fuzzy sets; interactive geometric operators

1. Introduction

The term fuzzy set (FS) was developed by Zadeh [1] based on a characteristic function that described the degree of membership of an element. Atanassov [2] established the theory of intuitionistic fuzzy set (IFS) as a generalization of FS with the help of two characteristic functions, known as membership and non-membership functions, describing the positive and negative aspects of an element or object. In the framework of IFSs, there was a constraint on two characteristic functions, in that their sum must not exceed the unit interval, which restricted the selection of membership and non-membership grades. Accordingly, Atanassov and Gargov [3] extended the IFS to the interval-valued intuitionistic fuzzy sets (IVIFSs), which contain the degrees of agreeing and disagreeing as interval values instead of single digits. Keeping in mind the constraint on IFSs, Yager [4,5] introduced a new generalization of IFSs, known as Pythagorean fuzzy set (PyFS), with a condition that the sum of squares of membership and non-membership grades must not exceed the unit interval.

The frameworks of IFSs and PyFSs have importance in situations where the structure of FSs fails to be applied. But these structures have their own limitations, as in the circumstances of voting where opinion cannot be restricted to “yes” or “no” but some refusal degree and abstinence is also involved. Therefore, Cuong [6,7] developed a novel concept of picture fuzzy sets (PFSs), which

is based on four characteristic functions known as membership, non-membership, abstinence, and refusal grades. Cuong's structure of PFSs is diverse in nature but, similar to IFSs, there is also a restriction in PFSs that the sum of all three membership grades must not exceed the unit interval. In the above-stated environments, various researchers have constructed their methodologies for solving the multi-attribute decision-making (MADM) problems, focusing on information measures, aggregation operators, etc. For instance, Xu [8] presented some weighted averaging aggregation operators (AOs) for intuitionistic fuzzy numbers (IFNs). Garg [9,10] presented some improved interactive AOs for IFNs. Wang and Liu [11] gave interval-valued intuitionistic fuzzy hybrid weighted AOs based on Einstein operations. Wang and Liu [12] presented some hybrid weighted AOs using Einstein norm operations for IFNs. Wang et al. [13] presented some AOs to aggregate various interval-valued intuitionistic fuzzy (IVIF) numbers (IVIFNs). Garg [14,15] presented generalized AOs using Einstein norm operations for Pythagorean fuzzy sets. Xu and Xia [16] proposed induced generalized aggregation tools and applied them in MADM. Garg and Kumar [17] presented some new similarity measures for IVIFNs based on the connection number of the set pair analysis theory. However, apart from these, a comprehensive overview of the different approaches under the IFSs, IVIFs, PyFSs, etc., to solve MCDM problems are summarized in [18–33].

Apart from the above theories, the concept of the spherical fuzzy sets (SFSs) has been introduced by Mahmood et al. [34], which consists of three membership degrees with a condition that the sum of squares of all degrees must not exceeds one. Further, the concepts of SFSs are extended to T-spherical fuzzy sets T-SFSs, where there are no restrictions on their constants and, hence, T-SFSs can handle all the situations where the frameworks of FSs, IFSs, PyFSs, PFSs, and SFSs failed. For this environment, Mahmood et al. [34] presented some aggregation operators for T-SFSs. Later on, Ullah et al. [35] presented the concept of the symmetry measures for handling the uncertainties under the T-SFSs environment, and applied it to solve the decision-making problems. However, from the existing work, it is noticeable that the existing AOs under the IFSs, PFSs, etc., have failed to handle the situations under some certain cases. For instance, under the IFS environment, if we consider the two IFNs, $A = (0, n_A)$ and $B = (m_B, n_B)$ where m_B, n_A, n_B represented degrees of membership grades that lies between zero and one, by applying the geometric AOs, as defined in [36], to such numbers we then get the aggregated numbers as $(0, n_A + n_B - n_A n_B)$. Thus, the final aggregated value of membership degree is zero, irrespective of value of m_B . Similarly, for T-SFSs, if we assume $A = (m_A, 0, n_A)$ and $B = (m_B, i_B, n_B)$ then, using the geometric aggregation operators of PFSs [37,38] and T-SFSs [34], we obtain the result of type $(\text{some value}, 0, \text{some value})$. This shows that the abstinence value of B is not accounted for in aggregation. Further, by taking $A = (0, 0, n_A)$ and $B = (m_B, i_B, n_B)$, then using the operators defined in [34,37,38], we get $(\text{some value}, 0, \text{some value})$. This shows that the membership and abstinence value is not accounted for in aggregation. These examples clearly point out the shortcomings that exist in the aggregation operators of PFSs and T-SFSs.

In order to overcome such shortcomings, and by utilizing the advantages of the T-SFSs over the several other existing theories, in this manuscript we have presented some new, improved geometric interactive aggregation operators. For it, firstly, we define some new operational laws by adding the degree of the hesitation into the operations. To do this, the concept of probability membership, non-membership, and heterogeneous are introduced and then some of their desirable properties are studied. Then, based on these proposed operational laws, some weighted, ordered weighted, and hybrid geometric aggregation operators, namely, T-spherical fuzzy weighted geometric interaction averaging (T-SFWGIA), T-spherical fuzzy ordered weighted geometric interaction averaging (T-SFOWGIA), and T-spherical fuzzy hybrid geometric interaction averaging (T-SFHGIA) operators are introduced in the paper. The desirable properties of such operators are investigated in detail. Then, based on such operators, we developed an algorithm for solving the decision-making problem under the T-SFS environment. The practical utility of the proposed approach is demonstrated through a numerical example, and comparative studies investigate the superiority of the approach. Finally, a counter example is provided to show the supremacy of the proposed operators with respect

to the existing operators. Therefore, motivated from it, the objectives of the paper are summarized as follows:

- (1) To propose some new operational laws based on the probability membership, non-membership, and heterogeneous laws.
- (2) To define some new, improved weighted geometric aggregation operators under the T-SFSs environment.
- (3) To develop an algorithm for solving the multi-attribute decision-making problems based on the proposed operators.
- (4) To check numerical applicability of the approach to a real-life case, and to compare the outcomes with prevailing approaches.

To do so, the organization of this manuscript is summarized as follows: Section 2 gives a basic overview of the basic concepts of IFSs, PFSs, SFSs, and T-SFSs; Section 3 deals with some new multiplication operations laws and their corresponding weighed geometric AOs; in Section 4, we present a MADM approach for solving the decision-making problem by using the proposed AOs (here the preferences related to each alternative are summarized in the form of T-SFS information); Section 5 presents a numerical example to illustrate the proposed approach and the comparative analysis; and finally, Section 6 concludes the paper with some concluding remarks.

2. Preliminaries

In this section, we present some basic concepts related to IFS, PyFS, PFS, SFS, and T-SFS over the universal set X .

Definition 1. [2] An IFS on X consists of membership and non-membership functions defined as

$$P = \{ \langle x, m(x), n(x) \rangle \mid x \in X \}$$

such that $m, n : X \rightarrow [0, 1]$ with a condition $0 \leq m(x) + n(x) \leq 1 \forall x \in X$ Further, the degree of refusal of x in P is $r(x) = 1 - (m(x) + n(x))$ and the pair (m, n) is regarded as an IFN.

Definition 2. [4] A Pythagorean fuzzy set (PyFS) on X consists of membership and non-membership functions defined as

$$P = \{ \langle x, m(x), n(x) \rangle \mid x \in X \}$$

such that $m, n : X \rightarrow [0, 1]$ with a condition that $0 \leq m^2(x) + n^2(x) \leq 1 \forall x \in X$. Further, the degree of refusal of x in P is $r(x) = \sqrt{1 - (m^2(x) + n^2(x))}$ and the pair (m, n) is regarded as a Pythagorean fuzzy number (PyFN).

Definition 3. [6] A picture fuzzy set (PFS) on X consists of membership, abstinence, and non-membership functions defined as

$$P = \{ \langle x, m(x), i(x), n(x) \rangle \mid x \in X \}$$

such that $m, i, n : X \rightarrow [0, 1]$ with a condition that $0 \leq m(x) + i(x) + n(x) \leq 1 \forall x \in X$ Further, the degree of refusal of x in P is $r(x) = 1 - (m(x) + i(x) + n(x))$ and (m, i, n) is regarded as a picture fuzzy number (PFN).

Definition 4. [34] A spherical fuzzy set (SFS) on X consists of membership, abstinence, and non-membership functions defined as

$$P = \{ \langle x, m(x), i(x), n(x) \rangle \mid x \in X \}$$

such that $m, i, n : X \rightarrow [0, 1]$ with a condition that $0 \leq m^2(x) + i^2(x) + n^2(x) \leq 1 \forall x \in X$ Further, the degree of refusal of x in P is $r(x) = \sqrt{1 - (m^2(x) + i^2(x) + n^2(x))}$ and (m, i, n) is regarded as a spherical fuzzy number (SFN).

Definition 5. [34] A T-SFS on X consists of membership, abstinence, and non-membership functions defined as

$$P = \{ \langle x, m(x), i(x), n(x) \rangle \mid x \in X \}$$

such that $m, i, n : X \rightarrow [0, 1]$ with a condition that $0 \leq m^t(x) + i^t(x) + n^t(x) \leq 1 \forall x \in X \ t = 1, 2, \dots, k$. Further, the degree of refusal of x in P is $r(x) = \sqrt[t]{1 - (m^t(x) + i^t(x) + n^t(x))}$ and (m, i, n) is regarded as a T-spherical fuzzy number (T-SFN).

Definition 6. [34] Let $P = (m, i, n)$ be a T-SFS. Then the score value of P is defined as

$$SC(P) = m^t - n^t$$

and accuracy function is defined as

$$AC(P) = m^t + i^t + n^t$$

The one which has a greater score is the superior value. If the score of two T-SFNs is equal, then we rank them using the accuracy value, and a number is called superior if it has greater accuracy. If again accuracy values of two T-SFNs become equal, then both numbers are considered as similar.

Definition 7. [39] Let $P = (m_P, n_P)$ and $P' = (m_{P'}, n_{P'})$ be two IFNs. Then the existing operational laws between them are defined as

- (1) $P \otimes P' = ((1 - n_P)(1 - n_{P'}) - (1 - m_P - n_P)(1 - m_{P'} - n_{P'}), 1 - (1 - n_P)(1 - n_{P'}))$
- (2) $P^\lambda = ((1 - n_P)^\lambda - (1 - m_P - n_P)^\lambda, 1 - (1 - n_P)^\lambda)$.

Definition 8. For any collection of T-SFNs $P_j = \langle m_j, i_j, n_j \rangle \ (j = 1, 2, 3, \dots, k)$, [34] defined the T-spherical fuzzy weighted geometric aggregation operator (T-SFWGA) as

$$T - SFWGA_w(P_1, P_2, \dots, P_k) = \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^k (m_j^t + i_j^t)^{w_j} - \prod_{j=1}^k (i_j^t)^{w_j}, \prod_{j=1}^k (i_j^t)^{w_j}} \\ \sqrt[t]{1 - \prod_{j=1}^k (1 - n_j^t)^{w_j}} \end{array} \right) \tag{1}$$

where $w = (w_1, w_2, \dots, w_k)^T$ be the weighting vector of T-SFNs P_j with $w_j \in (0, 1]$ and $\sum_{j=1}^k w_j = 1$ and $t = 1, 2, \dots, k$.

3. Proposed Operational Laws and Aggregation Operators

This section is divided into two subsections. One presents the improved operations laws for the T-SFSs, while other presents some improved geometric AOs under the T-SFS environment.

3.1. Improved Operational Laws

In this section, we present some new, improved operations laws by incorporating the features of the degree of refusal into the analysis.

Definition 9. Let $P_1 = (m_{P_1}, i_{P_1}, n_{P_1})$ and $P_2 = (m_{P_2}, i_{P_2}, n_{P_2})$ be two T-SFNs. Then, the proposed operational laws are defined as

- (1) $P_1 \otimes P_2 = \left(\begin{array}{c} \sqrt[t]{(1 - n_{P_1}^t)(1 - n_{P_2}^t) - (1 - m_{P_1}^t - i_{P_1}^t - n_{P_1}^t)(1 - m_{P_2}^t - i_{P_2}^t - n_{P_2}^t) - i_{P_1}^t i_{P_2}^t} \\ \sqrt[t]{1 - (1 - i_{P_1}^t)(1 - i_{P_2}^t)}, \sqrt[t]{1 - (1 - n_{P_1}^t)(1 - n_{P_2}^t)} \end{array} \right)$
- (2) $P^\lambda = \left(\sqrt[t]{(1 - n_P^t)^\lambda - (1 - m_P^t - i_P^t - n_P^t)^\lambda - i_P^{t\lambda}}, \sqrt[t]{1 - (1 - i_P^t)^\lambda}, \sqrt[t]{1 - (1 - n_P^t)^\lambda} \right)$

For two T-SFNs, $P_1 = (m_{P_1}, i_{P_1}, n_{P_1})$ and $P_2 = (m_{P_2}, i_{P_2}, n_{P_2})$, new operations of multiplication can be construed from four aspects, such as between:

- (1) Two non-membership functions of different T-SFNs.
- (2) Two membership functions of different T-SFNs.
- (3) Membership and non-membership functions of different T-SFNs.
- (4) Two neutral functions of different T-SFNs.

These multiplication rules are of the form:

1. $E(n_{P_1}, n_{P_2}) = n_{P_1} \cdot n_{P_2}$. Therefore, $n_{P_1 \otimes P_2} = \sqrt[t]{(n_{P_1}^t + n_{P_2}^t - n_{P_1}^t n_{P_2}^t)}$ is considered as a probability non-membership (PN) function operator, that is,

$$PN(n_{P_1}, n_{P_2}) = \sqrt[t]{n_{P_1}^t + n_{P_2}^t - n_{P_1}^t n_{P_2}^t}$$

2. $E(m_{P_1}, m_{P_2}) = (m_{P_1} + i_{P_1}) \cdot (m_{P_2} + i_{P_2})$. Therefore, $m_{P_1 \otimes P_2} = \sqrt[t]{1 - (1 - (m_{P_1}^t + i_{P_1}^t))(1 - (m_{P_2}^t + i_{P_2}^t))}$ is considered as a probability membership (PM) function operator, that is,

$$PM(m_{P_1}, m_{P_2}) = \sqrt[t]{1 - (1 - m_{P_1}^t - i_{P_1}^t)(1 - m_{P_2}^t - i_{P_2}^t)}$$

3. $I(n_{P_1}, m_{P_2}) = \sqrt[t]{(m_{P_2}^t + i_{P_2}^t)n_{P_1}^t} \cdot I(n_{P_1}, m_{P_2})$ is considered as a probability heterogeneous (PH) function operator, that is,

$$PH(n_{P_1}, m_{P_2}) = \sqrt[t]{m_{P_2}^t n_{P_1}^t + i_{P_2}^t n_{P_1}^t}$$

4. $I(i_{P_1}, i_{P_2}) = i_{P_1} \cdot i_{P_2}$. Therefore, $i_{P_1 \otimes P_2} = \sqrt[t]{(i_{P_1}^t + i_{P_2}^t - i_{P_1}^t i_{P_2}^t)} \cdot i_{P_1 \otimes P_2}$ is considered as a probability neutral (PNe) function operator, that is,

$$PNe(i_{P_1}, i_{P_2}) = \sqrt[t]{i_{P_1}^t + i_{P_2}^t - i_{P_1}^t i_{P_2}^t}$$

From the proposed laws, it is observed that the several existing laws can be considered as a special case of it. For instance,

- (i) For $t = 2$, above operations become valid for SFNs.
- (ii) For $t = 1$, above operations become valid for PFNs.
- (iii) For $t = 2$ and $i = 0$, above operations become valid for PyFNs.
- (iv) For $t = 1$ and $i = 0$, above operations become valid for IFNs.

Further, it is observed that for the above defined PN, PH satisfies the following properties:

Theorem 1. Let $P = \langle m_P, i_P, n_P \rangle, Q = \langle m_Q, i_Q, n_Q \rangle, R = \langle m_R, i_R, n_R \rangle$ and $D = \langle m_D, i_D, n_D \rangle$ be four T-SFNs. Then, we have:

- (1) *Boundedness:* $PN(1,1) = 1, PN(0,0) = 0, 0 \leq PN(n_P, n_Q) \leq 1$.
- (2) *Monotonicity:* If $n_P \leq n_R$ and $n_Q \leq n_D$. Then $PN(n_P, n_Q) \leq PN(n_R, n_D)$.
- (3) *Commutativity:* $PN(n_P, n_Q) = PN(n_Q, n_P)$.

Proof.

(1) For two T-SFNs, P and Q, and by definition of PN, we have $PN(n_P, n_Q) = \sqrt[t]{n_P^t + n_Q^t - n_P^t n_Q^t}$. Thus, we have $PN(1,1) = 1$ and $PN(0,0) = 0$. Further, since $n_P, n_Q \in [0, 1]$ and $t \in \mathbb{Z}$, which implies that $n_P^t + n_Q^t - n_P^t n_Q^t = 1 - (1 - n_P^t)(1 - n_Q^t) \leq 1$. Also, $PN(n_P, n_Q) \geq 0$. Therefore, $0 \leq PN(n_P, n_Q) \leq 1$.

(2) Since $n_P \leq n_R$ and $n_Q \leq n_D$. Thus, for any $t \in Z$, we get $1 - n_P^t \geq 1 - n_R^t$ and $1 - n_Q^t \geq 1 - n_D^t$, and hence $1 - (1 - n_P^t)(1 - n_Q^t) \leq 1 - (1 - n_R^t)(1 - n_D^t)$. Thus, $PN(n_P, n_Q) \leq PN(n_R, n_D)$ holds.

(3) Holds trivial. \square

Theorem 2. Let $P = \langle m_P, i_P, n_P \rangle, Q = \langle m_Q, i_Q, n_Q \rangle, R = \langle m_R, i_R, n_R \rangle$ and $S = \langle m_S, i_S, n_S \rangle$ be four T-SFN. Then:

- (1) Boundedness: $PH(1, 0, 1) = 1, PH(0, 0, 0) = 0, 0 \leq PH(m_P, i_P, n_P) \leq 1$.
- (2) Monotonicity: If $m_P \leq m_R, i_P \leq i_R$ and $n_Q \leq n_S$. Then $PH(m_P, i_P, n_Q) \leq PH(m_R, i_R, n_S)$ and if $n_P \leq n_R, i_Q \leq i_S$ and $m_Q \leq m_S$. Then $PH(n_P, i_Q, n_Q) \leq PH(n_R, i_S, m_S)$
- (3) Commutativity: $PH(m_P, i_P, n_P) = PH(n_P, i_P, m_P)$.

Proof. Similar to Theorem 1, so we omit here.

Theorem 3. If P and Q are two T-SFNs and $\lambda > 0$ is a real number, then $P \otimes Q$ and P^λ are also T-SFNs.

Proof. Follows from the definition easily, so we omit here.

Theorem 4. Let $P_1 = \langle m, i, n \rangle, P_2 = \langle m', i', n' \rangle$ be a T-SFNs, $\lambda, \lambda_1, \lambda_2 > 0$ be real numbers. Then we have

- (1) $P_1 \otimes P_2 = P_2 \otimes P_1$
- (2) $(P_1 \otimes P_2)^\lambda = P_1^\lambda \otimes P_2^\lambda$
- (3) $P_1^{\lambda_1} \otimes P_1^{\lambda_2} = P_1^{\lambda_1 + \lambda_2}$.

Proof. Follows from the definition easily, so we omit here.

3.2. Aggregation Operators

In this section, based on the above proposed operational laws, we have proposed some series of geometric interactive improved AOs, namely, T-SFWGIA, T-SFOWGIA, and T-SFHGIA, under the T-SFS environment.

Definition 10. For any collection, $P_j = \langle m_j, i_j, n_j \rangle (j = 1, 2, 3, \dots, k)$ of T-SFNs. If the mapping

$$T - SFWGIA_w(P_1, P_2, \dots, P_k) = \otimes_{j=1}^k P_j^{w_j} \tag{2}$$

then $T - SFWGIA_w$ is called a T-Spherical fuzzy weighted geometric interactive averaging (T-SFWGIA) operator, where $w = (w_1, w_2, \dots, w_k)^T$ is the weighting vector of P_j with $w_j \in (0, 1]$ and $\sum_{j=1}^k w_j = 1$.

Theorem 5. For any collection of T-SFNs, $P_j = \langle m_j, i_j, n_j \rangle (j = 1, 2, 3, \dots, k)$, the aggregated values obtained by using Definition 10 is still T-SFNs and is given by:

$$T - SFWGIA_w(P_1, P_2, \dots, P_k) = \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^k (1 - n_j^t)^{w_j} - \prod_{j=1}^k (1 - m_j^t - i_j^t - n_j^t)^{w_j} - \prod_{j=1}^k (i_j^t)^{w_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^k (1 - i_j^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_j^t)^{w_j}} \end{array} \right)$$

Proof. For any collection of T-SFNs, $P_j = \langle m_j, i_j, n_j \rangle (j = 1, 2, 3, \dots, k)$, we shall proof the result by induction on k .

For $k = 1$, we have:

$$T - SFWGIA_w(P_1) = P_1^{w_1} = (m_1, i_1, n_1) \\ = \left(\sqrt[t]{(1 - n_1^t)^1 - (1 - (m_1^t + i_1^t + n_1^t))^1 - (i_1^t)^1}, \sqrt[t]{1 - 1 + (i_1^t)^1}, \sqrt[t]{1 - 1 + (n_1^t)^1} \right)$$

Thus, hold for $k = 1$. Now, the result holds for $n = m$:

$$T - SFWGIA_w(P^1, P^2, \dots, P_m) = \left(\frac{\sqrt[t]{\prod_{j=1}^m (1 - n_j^t)^{w_j} - \prod_{j=1}^m (1 - m_j^t - i_j^t - n_j^t)^{w_j} - \prod_{j=1}^m (i_j^t)^{w_j}}}{\sqrt[t]{1 - \prod_{j=1}^m (1 - i_j^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^m (1 - n_j^t)^{w_j}} \right)$$

Then for $k = m + 1$, we have:

$$T - SFWGIA_w(P_1, P_2, \dots, P_{m+1}) = \otimes_{j=1}^{m+1} P_j^{w_j} \\ = T - SFWGIA_w(P_1, P_2, \dots, P_m) \otimes P_{m+1}^{w_{m+1}} \\ = \left(\frac{\sqrt[t]{\prod_{j=1}^m (1 - n_j^t)^{w_j} - \prod_{j=1}^m (1 - m_j^t - i_j^t - n_j^t)^{w_j} - \prod_{j=1}^m (i_j^t)^{w_j}}}{\sqrt[t]{1 - \prod_{j=1}^m (1 - i_j^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^m (1 - n_j^t)^{w_j}} \right) \\ \otimes \left(\frac{\sqrt[t]{(1 - n_{m+1}^t)^{w_{m+1}} - (1 - m_{m+1}^t - i_{m+1}^t - n_{m+1}^t)^{w_{m+1}} - (i_{m+1}^t)^{w_{m+1}}}}{\sqrt[t]{1 - (1 - i_{m+1}^t)^{w_{m+1}}}, \sqrt[t]{1 - (1 - n_{m+1}^t)^{w_{m+1}}}} \right) \\ = \left(\frac{\sqrt[t]{\prod_{j=1}^{m+1} (1 - n_j^t)^{w_j} - \prod_{j=1}^{m+1} (1 - m_j^t - i_j^t - n_j^t)^{w_j} - \prod_{j=1}^{m+1} (i_j^t)^{w_j}}}{\sqrt[t]{1 - \prod_{j=1}^{m+1} (1 - i_j^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^{m+1} (1 - n_j^t)^{w_j}}} \right)$$

So, the result holds for $k = m + 1$. Therefore, by the principle of mathematical induction, the result holds for all $k \in \mathbb{Z}^+$. \square

Theorem 6. If $P_j = (m_j, i_j, n_j), j = 1, \dots, k$ are T-SFNs. Then the aggregated value using the T-SFWGIA operator is also T-SFN.

Proof. Since $P_j = (m_j, i_j, n_j)$ is a T-SFN, $j = 1, \dots, k$, we have $0 \leq m_j, i_j, n_j \leq 1$. So $0 \leq m_j^t, i_j^t, n_j^t \leq 1$ and $0 \leq m_j^t + i_j^t + n_j^t \leq 1$. Then:

$$\leq \prod_{j=1}^k (1 - n_j^t)^{w_j} - \prod_{j=1}^k (1 - m_j^t - i_j^t - n_j^t)^{w_j} - \prod_{j=1}^k (i_j^t)^{w_j} \leq 1 \\ 0 \leq 1 - \prod_{j=1}^k (1 - i_j^t)^{w_j} \leq 1 \\ 0 \leq 1 - \prod_{j=1}^k (1 - n_j^t)^{w_j} \leq 1$$

Now:

$$\sqrt[t]{\frac{\prod_{j=1}^k (1 - n_j^t)^{w_j} - \prod_{j=1}^k (1 - (m_j^t + i_j^t + n_j^t))^{w_j} - \prod_{j=1}^k (i_j^t)^{w_j} + 1 - \prod_{j=1}^k (1 - i_j^t)^{w_j} + 1 - \prod_{j=1}^k (1 - n_j^t)^{w_j}}{2 - \prod_{j=1}^k (1 - (m_j^t + i_j^t + n_j^t))^{w_j} - \prod_{j=1}^k (i_j^t)^{w_j} - \prod_{j=1}^k (1 - i_j^t)^{w_j}}} \in [0, 1]$$

Thus, $T - SFWGIA_w(P_1, \dots, P_k)$ is T-SFN.

Further, it is observed that the proposed operator satisfies certain properties, which are listed as follows: \square

Theorem 7. If all T-SFNs, $P_j (j = 1, 2, \dots, k)$, are equal to P_0 , where P_0 is another T-SFN, then

$$T - SFWGIA_w(P_1, \dots, P_k) = P_0$$

Proof. Assume that $P_j = P_0 = (m_0, i_0, n_0)$ is a T-SFN $\forall j$. Then, by definition of T-SFWGIA operator, we have:

$$\begin{aligned} T - SFWGIA_w(P_1, P_2, \dots, P_k) &= \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^k (1 - n_j^t)^{w_j} - \prod_{j=1}^k (1 - (m_j^t + i_j^t + n_j^t))^{w_j} - \prod_{j=1}^k (i_j^t)^{w_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^k (1 - i_j^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_j^t)^{w_j}} \end{array} \right) \\ &= \left(\begin{array}{c} \sqrt[t]{(1 - n_j^t)^{\sum_{j=1}^k w_j} - (1 - (m_j^t + i_j^t + n_j^t))^{\sum_{j=1}^k w_j} - (i_j^t)^{\sum_{j=1}^k w_j}}, \\ \sqrt[t]{1 - (1 - i_j^t)^{\sum_{j=1}^k w_j}}, \sqrt[t]{1 - (1 - n_j^t)^{\sum_{j=1}^k w_j}} \end{array} \right) \\ &= (m_0, i_0, n_0) \\ &= P_0 \end{aligned}$$

Theorem 8. If $P_j = (m_j, i_j, n_j)$ is a T-SFN and

$$\begin{aligned} P^L &= (\max\{0, (\min(m_j + i_j + n_j) - \min i_j - \max n_j)\}, \min i_j, \max n_j), \\ P^U &= (\max(m_j + i_j + n_j) - \max i_j - \min n_j, \max i_j, \min n_j). \end{aligned}$$

Then, we have

$$P^L \leq T - SFWGIA_w(P_1, \dots, P_k) \leq P^U$$

Proof is straightforward.

Theorem 9. For a collection of two different T-SFNs, $A_j = (m_{A_j}, i_{A_j}, n_{A_j})$, ($j = 1, 2, \dots, k$) and $B_j = (m_{B_j}, i_{B_j}, n_{B_j})$, ($j = 1, 2, \dots, k$), which satisfy the following inequalities if $n_{A_j} \geq n_{B_j}$, $i_{A_j} \geq i_{B_j}$ and $m_{A_j}^t + i_{A_j}^t + n_{A_j}^t \leq m_{B_j}^t + i_{B_j}^t + n_{B_j}^t \forall j$, then we have

$$T - SFWGIA_w(A_1, A_2, \dots, A_k) \leq T - SFWGIA_w(B_1, B_2, \dots, B_k)$$

Proof. Since $n_{A_j} \geq n_{B_j}$, we have:

$$\sqrt[t]{1 - \prod_{j=1}^k (1 - n_{A_j}^t)^{w_j}} \geq \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{B_j}^t)^{w_j}}$$

and $i_{A_j} \geq i_{B_j}$

$$\sqrt[t]{1 - \prod_{j=1}^k (1 - i_{A_j}^t)^{w_j}} \geq \sqrt[t]{1 - \prod_{j=1}^k (1 - i_{B_j}^t)^{w_j}}$$

As, $n_{A_j} \geq n_{B_j}$, $m_{A_j}^t + i_{A_j}^t + n_{A_j}^t \leq m_{B_j}^t + i_{B_j}^t + n_{B_j}^t \forall j$ we have:

$$\left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^k (1 - n_{A_j}^t)^{w_j} - \prod_{j=1}^k (1 - (m_{A_j}^t + i_{A_j}^t + n_{A_j}^t))^{w_j} - \prod_{j=1}^k (i_{A_j}^t)^{w_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^k (1 - i_{A_j}^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{A_j}^t)^{w_j}} \end{array} \right) \leq \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^k (1 - n_{B_j}^t)^{w_j} - \prod_{j=1}^k (1 - (m_{B_j}^t + i_{B_j}^t + n_{B_j}^t))^{w_j} - \prod_{j=1}^k (i_{B_j}^t)^{w_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^k (1 - i_{B_j}^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{B_j}^t)^{w_j}} \end{array} \right)$$

Therefore, we have:

$$T - SFWGIA_w(A_1, A_2, \dots, A_k) \leq T - SFWGIA_w(B_1, B_2, \dots, B_k)$$

Definition 11. [34] For any collection, $P_j = (m_j, i_j, n_j)$ ($j = 1, 2, \dots, k$) of T-SFNs. The $T - SFOWGA_w : \Omega^n \rightarrow \Omega$ is a mapping defined as

$$T - SFOWGA_w(P_1, P_2, \dots, P_k) = \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^k (m_{\sigma(j)}^t + i_{\sigma(j)}^t)^{w_j} - \prod_{j=1}^k (i_{\sigma(j)}^t)^{w_j}, \prod_{j=1}^k (i_{\sigma(j)}^t)^{w_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{\sigma(j)}^t)^{w_j}} \end{array} \right) \tag{3}$$

where Ω is the collection of all T-SFNs, then $T - SFOWGA_w$ is called a T-SFOWGA operator with weighting vector $w = (w_1, w_2, \dots, w_k)^T$ of P_j with $w_j \in (0, 1]$ and $\sum_{j=1}^k w_j = 1$.

Definition 12. For any collection, $P_j = (m_j, i_j, n_j)$, ($j = 1, 2, \dots, k$) of T-SFNs. The $T - SFOWGIA_w : \Omega^n \rightarrow \Omega$ is a mapping defined as:

$$T - SFOWGIA_w(P_1, P_2, \dots, P_k) = \otimes_{j=1}^k P_{\sigma(j)}^{w_j} \tag{4}$$

then $T - SFOWGIA_w$ is called T-SFOWGIA operator, where $w = (w_1, w_2, \dots, w_k)^T$ is the weighting vector of P_j with $w_j \in (0, 1]$ and $\sum_{j=1}^k w_j = 1$ and σ is the permutation of $\{1, 2, \dots, k\}$, such that $\sigma(j - 1) \geq \sigma(j)$.

Theorem 10. For any collection $P_j = (m_j, i_j, n_j)$, ($j = 1, 2, \dots, k$) of T-SFNs. Then

$$T - SFOWGIA_w(P_1, P_2, \dots, P_k) = \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^k (1 - n_{\sigma(j)}^t)^{w_j} - \prod_{j=1}^k (1 - (m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t))^{w_j} - \prod_{j=1}^k (i_{\sigma(j)}^t)^{w_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^k (1 - i_{\sigma(j)}^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{\sigma(j)}^t)^{w_j}} \end{array} \right)$$

Proof is similar to Theorem 5.

Theorem 11. If $P_j = (m_j, i_j, n_j)$ is a T-SFN, $j = 1, \dots, k$. Then the aggregated value using the T-SFOWGIA operator is also T-SFN.

Proof. Since $P_{\sigma(j)} = (m_{\sigma(j)}, i_{\sigma(j)}, n_{\sigma(j)})$ is a T-SFN, $j = 1, \dots, k$, we have $0 \leq m_{\sigma(j)}, i_{\sigma(j)}, n_{\sigma(j)} \leq 1$. So $0 \leq m_{\sigma(j)}^t, i_{\sigma(j)}^t, n_{\sigma(j)}^t \leq 1$ and $0 \leq m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t \leq 1$. Then:

$$\begin{aligned} 0 &\leq \prod_{j=1}^k (1 - n_{\sigma(j)}^t)^{w_j} - \prod_{j=1}^k (1 - (m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t))^{w_j} - \prod_{j=1}^k (i_{\sigma(j)}^t)^{w_j} \leq 1 \\ 0 &\leq 1 - \prod_{j=1}^k (1 - i_{\sigma(j)}^t)^{w_j} \leq 1 \\ 0 &\leq 1 - \prod_{j=1}^k (1 - n_{\sigma(j)}^t)^{w_j} \leq 1 \end{aligned}$$

Now:

$$\begin{aligned} &\sqrt[t]{\prod_{j=1}^k (1 - n_{\sigma(j)}^t)^{w_j} - \prod_{j=1}^k (1 - (m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t))^{w_j} - \prod_{j=1}^k (i_{\sigma(j)}^t)^{w_j} + 1 - \prod_{j=1}^k (1 - i_{\sigma(j)}^t)^{w_j} \\ &\quad + 1 - \prod_{j=1}^k (1 - n_{\sigma(j)}^t)^{w_j}} \\ &= \sqrt[t]{2 - \prod_{j=1}^k (1 - (m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t))^{w_j} - \prod_{j=1}^k (i_{\sigma(j)}^t)^{w_j} - \prod_{j=1}^k (1 - i_{\sigma(j)}^t)^{w_j}} \in [0, 1] \end{aligned}$$

Thus, $T - SFOWGIA_w(P_1, \dots, P_k)$ is T-SFN. \square

Theorem 12. $T - SFOWGIA_w(P_1, \dots, P_k) = P_0$ if $P_j = P_0 = (m_j, i_j, n_j)$ is a T-SFN $\forall j$.

Proof. We have:

$$\begin{aligned} &T - SFOWGIA_w(P_1, \dots, P_k) = \\ &\left(\begin{array}{l} \sqrt[t]{\prod_{j=1}^k (1 - n_{\sigma(j)}^t)^{w_j} - \prod_{j=1}^k (1 - (m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t))^{w_j} - \prod_{j=1}^k (i_{\sigma(j)}^t)^{w_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^k (1 - i_{\sigma(j)}^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{\sigma(j)}^t)^{w_j}} \end{array} \right) \\ &= \left(\begin{array}{l} \sqrt[t]{(1 - n_{\sigma(j)}^t)^{\sum_{j=1}^k w_j} - (1 - (m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t))^{\sum_{j=1}^k w_j} - (i_{\sigma(j)}^t)^{\sum_{j=1}^k w_j}}, \\ \sqrt[t]{1 - (1 - i_{\sigma(j)}^t)^{\sum_{j=1}^k w_j}}, \sqrt[t]{1 - (1 - n_{\sigma(j)}^t)^{\sum_{j=1}^k w_j}} \end{array} \right) \\ &= (m_{\sigma(0)}, i_{\sigma(0)}, n_{\sigma(0)}) \\ &= P_0 \end{aligned}$$

\square

Theorem 13. If $P_j = (m_j, i_j, n_j)$ is a T-SFN and

$$\begin{aligned} P^L &= (\max\{0, (\min(m_j + i_j + n_j) - \min i_j - \max n_j)\}, \min i_j, \max n_j), \\ P^U &= (\max(m_j + i_j + n_j) - \max i_j - \min n_j), \max i_j, \min n_j). \text{ Then} \\ P^L &\leq T - SFOWGIA_w(P_1, \dots, P_k) \leq P^U \end{aligned}$$

Proof is straightforward.

Theorem 14. $T - SFOWGIA_w(B_1 B_2, \dots, B_k) = T - SFOWGIA_w(A_1, \dots, A_k)$ if $B_j = (m_{B_j}, i_{B_j}, n_{B_j})$ is any permutation of $A_j = (m_{A_j}, i_{A_j}, n_{A_j})$ where $j = 1, \dots, k$.

Proof.

$$\begin{aligned}
 & T - SFOWGIA_w(B_1, B_2, \dots, B_k) = \\
 & \left(\frac{\sqrt[t]{\prod_{j=1}^k (1 - n_{B_{\sigma(j)}}^t)^{w_j} - \prod_{j=1}^k (1 - (m_{B_{\sigma(j)}}^t + i_{B_{\sigma(j)}}^t + n_{B_{\sigma(j)}}^t))^{w_j} - \prod_{j=1}^k (i_{B_{\sigma(j)}}^t)^{w_j}}}{\sqrt[t]{1 - \prod_{j=1}^k (1 - i_{B_{\sigma(j)}}^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{B_{\sigma(j)}}^t)^{w_j}}} \right) \\
 & T - SFOWGIA_w(A_1, A_2, \dots, A_k) = \\
 & \left(\frac{\sqrt[t]{\prod_{j=1}^k (1 - n_{A_{\sigma(j)}}^t)^{w_j} - \prod_{j=1}^k (1 - (m_{A_{\sigma(j)}}^t + i_{A_{\sigma(j)}}^t + n_{A_{\sigma(j)}}^t))^{w_j} - \prod_{j=1}^k (i_{A_{\sigma(j)}}^t)^{w_j}}}{\sqrt[t]{1 - \prod_{j=1}^k (1 - i_{A_{\sigma(j)}}^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{A_{\sigma(j)}}^t)^{w_j}}} \right)
 \end{aligned}$$

If $B_j = (m_{B_j}, i_{B_j}, n_{B_j})$ is any permutation of $A_j = (m_{A_j}, i_{A_j}, n_{A_j})$ then we have $B_{\sigma(j)} = A_{\sigma(j)}$. Thus, $T - SFOWGIA_w(B_1, \dots, B_k) = T - SFOWGIA_w(A_1, \dots, A_k)$. \square

Definition 13. For any collection, $P_j = \langle m_j, i_j, n_j \rangle$ of T-SFNs ($j = 1, 2, 3, \dots, k$). If the mapping

$$T - SFHGA_{\omega, w}(P_1, P_2, \dots, P_k) = \otimes_{j=1}^k (\tilde{P}_{\sigma(j)})^{w_j} \tag{5}$$

then $T - SFHGA_{\omega, w}$ is called a T-SFHGA operator, where $\tilde{P}_j = (P_j)^{n_{\omega_j}}$ and $\omega = (\omega_1, \dots, \omega_k)^T$ is the weighting vector of P_j with $\omega_j \in (0, 1]$ and $\sum_{j=1}^k \omega_j = 1$.

Theorem 15. [34] For any collection, $P_j = \langle m_j, i_j, n_j \rangle$ ($j = 1, 2, 3, \dots, k$) of T-SFNs. If

$$T - SFHGA_{\omega, w}(P_1, P_2, \dots, P_k) = \left(\frac{\sqrt[t]{\prod_{j=1}^k (m_{\tilde{P}_{\sigma(j)}}^t + i_{\tilde{P}_{\sigma(j)}}^t)^{w_j} - \prod_{j=1}^k (i_{\tilde{P}_{\sigma(j)}}^t)^{w_j}}}{\prod_{j=1}^k (i_{\tilde{P}_{\sigma(j)}}^t)^{w_j}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{\tilde{P}_{\sigma(j)}}^t)^{w_j}}} \right)$$

then $T - SFHGA_{\omega, w}$ is called a T-SFHGA operator with weighting vector $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$ of P_j with $\omega_j \in (0, 1]$ and $\sum_{j=1}^k \omega_j = 1$.

Definition 14. For any collection, $P_j = \langle m_j, i_j, n_j \rangle$ ($j = 1, 2, 3, \dots, k$) of T-SFNs. If the mapping

$$T - SFHGIA_{\omega, w}(P_1, P_2, \dots, P_k) = \otimes_{j=1}^k \tilde{P}_{\sigma(j)}^{w_j} \tag{6}$$

then $T - SFHGIA_{\omega, w}$ is called a T-SFHGIA operator, where $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$ is the weighting vector of P_j with $\omega_j \in [0, 1]$ and $\sum_{j=1}^k \omega_j = 1$.

Theorem 16. For any collection, $P_j = \langle m_j, i_j, n_j \rangle$ ($j = 1, 2, 3, \dots, k$) of T-SFNs. Then

$$\begin{aligned}
 & T - SFHGIA_{\omega, w}(P_1, P_2, \dots, P_k) \\
 & = \left(\frac{\sqrt[t]{\prod_{j=1}^k (1 - n_{\tilde{P}_{\sigma(j)}}^t)^{w_j} - \prod_{j=1}^k (1 - (m_{\tilde{P}_{\sigma(j)}}^t + i_{\tilde{P}_{\sigma(j)}}^t + n_{\tilde{P}_{\sigma(j)}}^t))^{w_j} - \prod_{j=1}^k (i_{\tilde{P}_{\sigma(j)}}^t)^{w_j}}}{\sqrt[t]{1 - \prod_{j=1}^k (1 - i_{\tilde{P}_{\sigma(j)}}^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_{\tilde{P}_{\sigma(j)}}^t)^{w_j}}} \right)
 \end{aligned}$$

The following example demonstrates these aggregation operators:

Example 1. Let $P_1 = (0.3, 0.8, 0.1)$, $P_2 = (0.4, 0.3, 0.6)$, $P_3 = (0.7, 0.1, 0.5)$, $P_4 = (0.9, 0.4, 0.1)$ and $P_5 = (0.2, 0.6, 0.7)$ are T-SFN. The weight vector for P_i ($i = 1, 2, \dots, 5$) is $\omega = (0.18, 0.22, 0.16, 0.21, 0.23)^T$. With loss of generality, we use $t = 2$ for all calculations.

Firstly, we utilized T-SFHGIA operators on this data to aggregate it.

$$\begin{aligned}
 P_1 &= \left(\begin{array}{c} \sqrt{(1 - 0.1^2)^{5 \times 0.18} - (1 - (0.3^2 + 0.8^2 + 0.1^2))^{5 \times 0.18} - (0.8^2)^{5 \times 0.18}} \\ \sqrt{1 - (1 - 0.8^2)^{5 \times 0.18}}, \sqrt{1 - (1 - 0.1^2)^{5 \times 0.18}} \end{array} \right) \\
 &= (0.1559, 0.7754, 0.0949) \\
 P_2 &= \left(\begin{array}{c} \sqrt{(1 - 0.6^2)^{5 \times 0.22} - (1 - (0.4^2 + 0.3^2 + 0.6^2))^{5 \times 0.22} - (0.3^2)^{5 \times 0.22}} \\ \sqrt{1 - (1 - 0.3^2)^{5 \times 0.22}}, \sqrt{1 - (1 - 0.6^2)^{5 \times 0.22}} \end{array} \right) \\
 &= (0.4317, 0.3139, 0.6228) \\
 P_3 &= \left(\begin{array}{c} \sqrt{(1 - 0.5^2)^{5 \times 0.16} - (1 - (0.7^2 + 0.1^2 + 0.5^2))^{5 \times 0.16} - (0.1^2)^{5 \times 0.16}} \\ \sqrt{1 - (1 - 0.1^2)^{5 \times 0.16}}, \sqrt{1 - (1 - 0.5^2)^{5 \times 0.16}} \end{array} \right) \\
 &= (0.6629, 0.0895, 0.4534) \\
 P_4 &= \left(\begin{array}{c} \sqrt{(1 - 0.1^2)^{5 \times 0.21} - (1 - (0.9^2 + 0.4^2 + 0.1^2))^{5 \times 0.21} - (0.4^2)^{5 \times 0.21}} \\ \sqrt{1 - (1 - 0.4^2)^{5 \times 0.21}}, \sqrt{1 - (1 - 0.1^2)^{5 \times 0.21}} \end{array} \right) \\
 &= (0.9094, 0.4090, 0.1024) \\
 P_5 &= \left(\begin{array}{c} \sqrt{(1 - 0.7^2)^{5 \times 0.23} - (1 - (0.2^2 + 0.6^2 + 0.7^2))^{5 \times 0.23} - (0.6^2)^{5 \times 0.23}} \\ \sqrt{1 - (1 - 0.6^2)^{5 \times 0.23}}, \sqrt{1 - (1 - 0.7^2)^{5 \times 0.23}} \end{array} \right) \\
 &= (0.2705, 0.6336, 0.7342)
 \end{aligned}$$

The score values corresponding to these aggregated numbers were obtained as $SC(P_1) = 0.0153$, $SC(P_2) = -0.2016$, $SC(P_3) = 0.2338$, $SC(P_4) = 0.8166$, $SC(P_5) = -0.4658$. Based on the score values, we had the following arrangement of data:

$$P_{\sigma(1)} = (0.9094, 0.4090, 0.1024), P_{\sigma(2)} = (0.6629, 0.0895, 0.4534), P_{\sigma(3)} = (0.1559, 0.7754, 0.0949), \\
 P_{\sigma(4)} = (0.4317, 0.3139, 0.6228), P_{\sigma(5)} = (0.2705, 0.6336, 0.7342)$$

By using the normal distribution-based method, we found $w = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$ and by the definition of T-SFHGIA operator we had

$$T - SFHGIA_{\omega, w}(P_1, P_2, P_3, P_4, P_5) = (0.4688, 0.5643, 0.4792)$$

Theorem 17. If $P_j = (m_j, i_j, n_j)$ is a T-SFN, $j = 1, \dots, k$, then the aggregated value using the T-SFHGIA operator is also T-SFN.

Proof is similar as in Theorem 11.

Theorem 18. $T - SFHGIA_{\omega, w}(P_1, P_2, \dots, P_k) = P_0$ if $P_j = P_0 = (m_j, i_j, n_j)$ is a T-SFN $\forall j$. Proof is similar as in Theorem 12.

Theorem 19. If $P_j = (m_j, i_j, n_j)$ is a T-SFN and

$$\begin{aligned}
 P^L &= (\max\{0, (\min(m_j + i_j + n_j) - \min i_j - \max n_j)\}, \min i_j, \max n_j), \\
 P^U &= (\max(m_j + i_j + n_j) - \max i_j - \min n_j, \max i_j, \min n_j). \text{ Then} \\
 P^L &\leq T - SFHGIA_{\omega, w}(P_1, \dots, P_k) \leq P^U
 \end{aligned}$$

Proof is straightforward.

Theorem 20. $T - SFHGIA_{\omega,w}(B_1, \dots, B_k) = T - SFHGIA_{\omega,w}(A_1, \dots, A_k)$ if $B_j = (m_{B_j}, i_{B_j}, n_{B_j})$ is any permutation of $A_j = (m_{A_j}, i_{A_j}, n_{A_j})$ where $j = 1, \dots, k$.

Proof is similar as Theorem 14.

Whenever membership and neutral number of one T-SFN become zero then the membership and abstinence value is not accounted for in the aggregation [34]. However, the geometric interaction averaging operators that are developed in our manuscript overcome this problem. The example below will describe this more clearly.

Example 2. Let $P_1 = (0.7, 0.5, 0.6)$, $P_2 = (0.9, 0.5, 0.4)$, $P_3 = (0, 0, 0.1)$, $P_4 = (0.5, 0.3, 0.4)$ and $P_5 = (0.6, 0.4, 0.5)$ are T-SFN. The weight vector for P_i ($i = 1, 2, \dots, 5$) is $\omega = (0.18, 0.22, 0.16, 0.21, 0.23)^T$.

For the solution, first we will find the T-SFHGA operator.

As, $0.7 + 0.5 + 0.6 = 1.8 \notin [0, 1]$, $0.7^2 + 0.5^2 + 0.6^2 = 1.1 \notin [0, 1]$ but $0.7^3 + 0.5^3 + 0.6^3 = 0.684 \in [0, 1]$. Similarly, P_2 and P_4 satisfy the condition for $t = 3$.

$$\begin{aligned}\tilde{P}_1 &= \left(\sqrt[3]{(0.7^3 + 0.5^3)^{5 \times 0.18} - (0.5^3)^{5 \times 0.18}}, 0.5^{5 \times 0.18}, \sqrt[3]{1 - (1 - 0.6^3)^{5 \times 0.18}} \right) \\ &= (0.7054, 0.5359, 0.5816) \\ \tilde{P}_2 &= \left(\sqrt[3]{(0.9^3 + 0.5^3)^{5 \times 0.22} - (0.5^3)^{5 \times 0.22}}, 0.5^{5 \times 0.22}, \sqrt[3]{1 - (1 - 0.4^3)^{5 \times 0.22}} \right) \\ &= (0.9041, 0.4665, 0.4125) \\ \tilde{P}_3 &= \left(\sqrt[3]{(0^3 + 0^3)^{5 \times 0.16} - (0^3)^{5 \times 0.16}}, 0^{5 \times 0.16}, \sqrt[3]{1 - (1 - 0.1^3)^{5 \times 0.16}} \right) \\ &= (0, 0, 0.0928) \\ \tilde{P}_4 &= \left(\sqrt[3]{(0.5^3 + 0.3^3)^{5 \times 0.21} - (0.3^3)^{5 \times 0.21}}, 0.3^{5 \times 0.21}, \sqrt[3]{1 - (1 - 0.4^3)^{5 \times 0.21}} \right) \\ &= (0.4874, 0.2885, 0.4063) \\ \tilde{P}_5 &= \left(\sqrt[3]{(0.6^3 + 0.4^3)^{5 \times 0.23} - (0.4^3)^{5 \times 0.23}}, 0.4^{5 \times 0.23}, \sqrt[3]{1 - (1 - 0.5^3)^{5 \times 0.23}} \right) \\ &= (0.5738, 0.3486, 0.5221)\end{aligned}$$

Scores values for these aggregated numbers were obtained as $SC(\tilde{P}_1) = 0.1543$, $SC(\tilde{P}_2) = 0.6689$, $SC(\tilde{P}_3) = -0.0008$, $SC(\tilde{P}_4) = 0.0487$, $SC(\tilde{P}_5) = 0.0466$, and, based on these score values, we had

$$\tilde{P}_{\sigma(1)} = (0.9041, 0.4665, 0.4125), \tilde{P}_{\sigma(2)} = (0.7054, 0.5359, 0.5816), \tilde{P}_{\sigma(3)} = (0.4874, 0.2885, 0.4063), \tilde{P}_{\sigma(4)} = (0.5738, 0.3486, 0.5221), \tilde{P}_{\sigma(5)} = (0, 0, 0.0928)$$

By using the normal distribution-based method, we found $w = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$, and, by the definition of T-SFHGA operator, we found

$$T - SFHGA_{\omega,w}(P_1, P_2, P_3, P_4, P_5) = (0, 0, 0.4803) \quad (7)$$

This type of aggregated value seems meaningless, as whenever the membership and abstinence value is zero in any one of the T-SFN it will make the value of the membership and non-membership as zero in the whole aggregated value. This shows that the geometric aggregation operator of T-SFSs [34] does not possess the ability to aggregate such types of information effectively.

On the other hand, the proposed new geometric interactive aggregation operators can process any type of information effectively. Now, the Example 2 was solved using the proposed new

aggregation operators in order to justify its effectiveness. For it, we aggregated the data using the T-SFHGIA operator:

$$\begin{aligned}
 P_1 &= \left(\begin{array}{c} \sqrt[3]{(1 - 0.6^3)^{5 \times 0.18} - (1 - (0.7^3 + 0.5^3 + 0.6^3))^{5 \times 0.18} - (0.5^3)^{5 \times 0.18}}, \\ \sqrt[3]{1 - (1 - 0.5^3)^{5 \times 0.18}}, \sqrt[3]{1 - (1 - 0.6^3)^{5 \times 0.18}} \end{array} \right) \\
 &= (0.6656, 0.5359, 0.5816) \\
 P_2 &= \left(\begin{array}{c} \sqrt[3]{(1 - 0.4^3)^{5 \times 0.22} - (1 - (0.9^3 + 0.5^3 + 0.4^3))^{5 \times 0.22} - (0.5^3)^{5 \times 0.22}}, \\ \sqrt[3]{1 - (1 - 0.5^3)^{5 \times 0.22}}, \sqrt[3]{1 - (1 - 0.4^3)^{5 \times 0.22}} \end{array} \right) \\
 &= (0.9144, 0.4665, 0.4125) \\
 P_3 &= \left(\begin{array}{c} \sqrt[3]{(1 - 0.1^3)^{5 \times 0.16} - (1 - (0^3 + 0^3 + 0.1^3))^{5 \times 0.16} - (0^3)^{5 \times 0.16}}, \\ \sqrt[3]{1 - (1 - 0^3)^{5 \times 0.16}}, \sqrt[3]{1 - (1 - 0.1^3)^{5 \times 0.16}} \end{array} \right) \\
 &= (0, 0, 0.0928) \\
 P_4 &= \left(\begin{array}{c} \sqrt[3]{(1 - 0.4^3)^{5 \times 0.21} - (1 - (0.5^3 + 0.3^3 + 0.4^3))^{5 \times 0.21} - (0.3^3)^{5 \times 0.21}}, \\ \sqrt[3]{1 - (1 - 0.3^3)^{5 \times 0.21}}, \sqrt[3]{1 - (1 - 0.4^3)^{5 \times 0.21}} \end{array} \right) \\
 &= (0.5141, 0.2885, 0.4063) \\
 P_5 &= \left(\begin{array}{c} \sqrt[3]{(1 - 0.5^3)^{5 \times 0.23} - (1 - (0.6^3 + 0.4^3 + 0.5^3))^{5 \times 0.23} - (0.4^3)^{5 \times 0.23}}, \\ \sqrt[3]{1 - (1 - 0.4^3)^{5 \times 0.23}}, \sqrt[3]{1 - (1 - 0.5^3)^{5 \times 0.23}} \end{array} \right) \\
 &= (0.6422, 0.3486, 0.5221)
 \end{aligned}$$

The score values of these numbers were obtained as $SC(P_1) = 0.0981$, $SC(P_2) = 0.6943$, $SC(P_3) = -0.0008$, $SC(P_4) = 0.0688$, $SC(P_5) = 0.1225$, and, based on score values, we had the following arrangement:

$$\begin{aligned}
 P_{\sigma(1)} &= (0.9144, 0.4665, 0.4125), \\
 P_{\sigma(2)} &= (0.6422, 0.3486, 0.5221), P_{\sigma(3)} = (0.6656, 0.5359, 0.5816), \\
 P_{\sigma(4)} &= (0.5141, 0.2885, 0.4063), P_{\sigma(5)} = (0, 0, 0.0928)
 \end{aligned}$$

Now, by using the definition of the T-SFHGIA operator, we found

$$T - SFHGIA_{\omega, w}(P_1, P_2, P_3, P_4, P_5) = (0.8375, 0.4223, 0.4928) \tag{8}$$

Clearly, the aggregated value obtained in Equation (8) was an improvement of the one obtained in Equation (7), as it incorporated the zero values occurring in the membership and abstinence of T-SFNs efficiently. The analysis of Equations (7) and (8) proved the significance of proposed aggregation operators.

4. MADM Approach Based on Proposed Operators

Consider a decision-making problem which consists of a set of alternatives ($Y = \{y_1, y_2, \dots, y_l\}$) and set of attributes ($Z = \{z_1, z_2, \dots, z_q\}$) associated with weighted vector ($w = (w_1, w_2, \dots, w_q)^T$), where $w_k \in (0, 1]$ and $\sum_{k=1}^q w_k = 1$. Suppose every alternative (y_j) is represented by T-SFNs ($P_{jk} = \langle m_{jk}, i_{jk}, n_{jk} \rangle$), which show by which degree alternatives satisfy, neutral, and not satisfy the given attribute. Then, the following steps of the MADM approach, based on the proposed operators, are summarized as follows:

Step 1 Find the value of t for which the information of the decision matrix lies in the T-spherical fuzzy environment.

Step 2 Assume the weighting vector $\omega = (\omega_1, \dots, \omega_q)^T$ of $P_{j1}, P_{j2}, \dots, P_{jq}$. where $\omega_k \in (0, 1]$ and $\sum_{k=1}^q \omega_k = 1$ we get $P_{jk} = P_{jk}^{l\omega_k}$.

Step 3 By calculating the scores of each attribute of all alternatives, we find:

$$P_{\sigma(j1)}, P_{\sigma(j2)}, \dots, P_{\sigma(jk)}$$

Step 4 By using the normal-distribution based method we find w and then aggregate the data using the T-SFHGIA operator.

Step 5 Find the scores of all alternatives.

Step 6 With the help of score values, we find the best option.

5. Numerical Example

The above-mentioned approach has been illustrated with a real-life decision-making problem under the T-SFS environment, and obtained results have been compared with the other existing results.

5.1. Case Study

Jharkhand is the eastern state of India, which has 40 percent of the mineral resources of the country, and is the second leading state in terms of mineral wealth, after Chhattisgarh state. It is also known for its vast forest resources. Jamshedpur, Bokaro, and Dhanbad, cities in Jharkhand, are famous for industries from all over the world. After that, it is known as being the state in India that has widespread poverty state, because it is primarily a rural state, as 76 percent of the population lives in villages that depend on agriculture and wages from agriculture. Only 30 percent of the villages are connected by roads, and only 55 percent of the villages have access to electricity and other facilities. But in the today’s life, many are looking for ways to make changes in order to better their lives, and, accordingly, many move to the urban cities for better jobs. To stop this emigration, the Jharkhand government wants to set up agricultural-based industries in the rural areas. For this, the government organized the “Momentum Jharkhand” global investor summit 2017, in Ranchi, to invite companies to invest in the rural areas. The government announced the various facilities that were available to be set up as five food processing plants in the rural areas, and the five attributes required for selection of the companies to set them up, namely, project cost (Q_1), technical capability (Q_2), financial status (Q_3), company background (Q_4), and other factors (Q_5). The three companies that were interested in this projects, Surya Food and Agro Pvt. Ltd. (s_1), Mother Dairy Fruit and Vegetable Pvt. Ltd. (s_2), and Parle Products Ltd. (s_3), were taken as in the form of the alternatives. Then, the main object of the government was to choose the best company among them for the task. In order to fulfill this, a decision maker evaluated these and gave their preferences in the term of T-SFS, and their preference values were summarized in the form of a decision-matrix, shown in Table 1 as follows.

Table 1. Input information related to each alternative.

	Q_1	Q_2	Q_3	Q_4	Q_5
s_1	(0.7, 0.5, 0.6)	(0.9, 0.5, 0.4)	(0.4, 0.2, 0.1)	(0.5, 0.3, 0.4)	(0.6, 0.4, 0.5)
s_2	(0.5, 0.4, 0.6)	(0.7, 0.2, 0.3)	(0.5, 0.3, 0.6)	(0.4, 0.1, 0.6)	(0.5, 0.2, 0.4)
s_3	(0.4, 0.1, 0.2)	(0.5, 0.4, 0.1)	(0, 0, 0.5)	(0.6, 0.2, 0.2)	(0.6, 0.1, 0.5)

The given problem was solved using two approaches. First it was solved using new interactive operators showing their applicability. Then it was solved using geometric aggregation operators proposed in [34], showing their failure.

Solution using proposed operators:

Step 1 With some calculations, it was found that all the values in Table 1 were T-SFNs for $t = 3$.

Step 2 By taking $\omega = (0.18, 0.22, 0.16, 0.21, 0.23)^T$ we found P_{jk} and their values were summarized as below.

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	$\begin{pmatrix} 0.6656, \\ 0.5359, \\ 0.5816 \end{pmatrix}$	$\begin{pmatrix} 0.9144, \\ 0.4665, \\ 0.4125 \end{pmatrix}$	$\begin{pmatrix} 0.3333, \\ 0.2759, \\ 0.0928 \end{pmatrix}$	$\begin{pmatrix} 0.5141, \\ 0.2825, \\ 0.4063 \end{pmatrix}$	$\begin{pmatrix} 0.6422, \\ 0.3486, \\ 0.5221 \end{pmatrix}$
j = 2	$\begin{pmatrix} 0.4520, \\ 0.4384, \\ 0.5816 \end{pmatrix}$	$\begin{pmatrix} 0.7194, \\ 0.1703, \\ 0.3095 \end{pmatrix}$	$\begin{pmatrix} 0.4212, \\ 0.3817, \\ 0.5614 \end{pmatrix}$	$\begin{pmatrix} 0.4053, \\ 0.0891, \\ 0.6086 \end{pmatrix}$	$\begin{pmatrix} 0.5264, \\ 0.1571, \\ 0.4184 \end{pmatrix}$
j = 3	$\begin{pmatrix} 0.3843, \\ 0.1259, \\ 0.1931 \end{pmatrix}$	$\begin{pmatrix} 0.5397, \\ 0.3650, \\ 0.1032 \end{pmatrix}$	$\begin{pmatrix} 0, \\ 0, \\ 0.4662 \end{pmatrix}$	$\begin{pmatrix} 0.6104, \\ 0.1845, \\ 0.2033 \end{pmatrix}$	$\begin{pmatrix} 0.6209, \\ 0.0708, \\ 0.5221 \end{pmatrix}$

Step 3 Now we had to find the score of each attribute of all alternatives, and their computed values were given as below

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	0.0981	0.6943	0.0362	0.0688	0.1225
j = 2	-0.1043	0.3426	-0.1021	-0.1589	0.0726
j = 3	0.0495	0.1561	-0.1013	0.2190	0.0970

By comparing the score values, we had

$$\begin{aligned}
 SC(P_{12}) &> SC(P_{15}) > SC(P_{11}) > SC(P_{14}) > SC(P_{13}) \\
 SC(P_{22}) &> SC(P_{25}) > SC(P_{23}) > SC(P_{21}) > SC(P_{24}) \\
 SC(P_{34}) &> SC(P_{32}) > SC(P_{35}) > SC(P_{31}) > SC(P_{33})
 \end{aligned}$$

Based on above score analysis, we found $P_{\sigma(jk)}$ and summarized them as

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	$\begin{pmatrix} 0.9144, \\ 0.5150, \\ 0.4125 \end{pmatrix}$	$\begin{pmatrix} 0.6422, \\ 0.9857, \\ 0.5221 \end{pmatrix}$	$\begin{pmatrix} 0.6656, \\ 0.4838, \\ 0.5816 \end{pmatrix}$	$\begin{pmatrix} 0.5141, \\ 0.3048, \\ 0.4063 \end{pmatrix}$	$\begin{pmatrix} 0.3333, \\ 0.1857, \\ 0.0928 \end{pmatrix}$
j = 2	$\begin{pmatrix} 0.7194, \\ 0.2064, \\ 0.3095 \end{pmatrix}$	$\begin{pmatrix} 0.5264, \\ 0.9987, \\ 0.4184 \end{pmatrix}$	$\begin{pmatrix} 0.4212, \\ 0.2787, \\ 0.5614 \end{pmatrix}$	$\begin{pmatrix} 0.4520, \\ 0.9804, \\ 0.5816 \end{pmatrix}$	$\begin{pmatrix} 0.4053, \\ 0.1016, \\ 0.6086 \end{pmatrix}$
j = 3	$\begin{pmatrix} 0.6104, \\ 0.2033, \\ 0.2033 \end{pmatrix}$	$\begin{pmatrix} 0.5397, \\ 0.4125, \\ 0.1032 \end{pmatrix}$	$\begin{pmatrix} 0.6209, \\ 0.9999, \\ 0.5221 \end{pmatrix}$	$\begin{pmatrix} 0.3843, \\ 0.0966, \\ 0.1931 \end{pmatrix}$	$\begin{pmatrix} 0, \\ 0, \\ 0.4662 \end{pmatrix}$

Step 4 By using the normal distribution-based method, we got $w = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$, and by using the defined aggregation operators, we had

$$\begin{aligned}
 P_1 &= T - SFHGIA_{\omega,w}(P_{11}, P_{12}, P_{13}, P_{14}, P_{15}) \\
 &= \left(\frac{\sqrt[3]{\prod_{j=1}^5 \left(1 - n_{P_{\sigma(1k)}}^3\right)^{w_j} - \prod_{j=1}^5 \left(1 - \left(m_{P_{\sigma(1k)}}^3 + i_{P_{\sigma(1k)}}^3 + n_{P_{\sigma(1k)}}^3\right)\right)^{w_j} - \prod_{j=1}^5 \left(i_{P_{\sigma(1k)}}^3\right)^{w_j}}{\sqrt[3]{1 - \prod_{j=1}^5 \left(1 - i_{P_{\sigma(1k)}}^3\right)^{w_j}}, \sqrt[3]{1 - \prod_{j=1}^5 \left(1 - n_{P_{\sigma(1k)}}^3\right)^{w_j}}} \right) \\
 &= (0.9380, 0.4264, 0.4928) \\
 P_2 &= T - SFHGIA_{\omega,w}(P_{21}, P_{22}, P_{23}, P_{24}, P_{25}) \\
 &= \left(\frac{\sqrt[3]{\prod_{j=1}^5 \left(1 - n_{P_{\sigma(2k)}}^3\right)^{w_j} - \prod_{j=1}^5 \left(1 - \left(m_{P_{\sigma(2k)}}^3 + i_{P_{\sigma(2k)}}^3 + n_{P_{\sigma(2k)}}^3\right)\right)^{w_j} - \prod_{j=1}^5 \left(i_{P_{\sigma(2k)}}^3\right)^{w_j}}{\sqrt[3]{1 - \prod_{j=1}^5 \left(1 - i_{P_{\sigma(2k)}}^3\right)^{w_j}}, \sqrt[3]{1 - \prod_{j=1}^5 \left(1 - n_{P_{\sigma(2k)}}^3\right)^{w_j}}} \right) \\
 &= (0.9420, 0.3390, 0.5296)
 \end{aligned}$$

$$\begin{aligned}
 P_3 &= T - SFHGA_{\omega,w}(P_{31}, P_{32}, P_{33}, P_{34}, P_{35}) \\
 &= \left(\frac{\sqrt[3]{\prod_{j=1}^5 \left(1 - n_{P_{\sigma(3k)}}^3\right)^{w_j} - \prod_{j=1}^5 \left(1 - \left(m_{P_{\sigma(3k)}}^3 + i_{P_{\sigma(3k)}}^3 + n_{P_{\sigma(3k)}}^3\right)\right)^{w_j} - \prod_{j=1}^5 \left(i_{P_{\sigma(3k)}}^3\right)^{w_j}}{\sqrt[3]{1 - \prod_{j=1}^5 \left(1 - i_{P_{\sigma(3k)}}^3\right)^{w_j}}, \sqrt[3]{1 - \prod_{j=1}^5 \left(1 - n_{P_{\sigma(3k)}}^3\right)^{w_j}}} \right) \\
 &= (0.9779, 0.9713, 0.3906)
 \end{aligned}$$

Step 5 The score values of three alternatives based on their aggregated values were computed as $SC(P_1) = 0.7056$, $SC(P_2) = 0.6874$, and $SC(P_3) = 0.8813$.

Step 6 By comparing score values, we got

$$SC(P_3) > SC(P_1) > SC(P_2)$$

The comparison of score values indicated that P_3 had a greater score value. So, the third company was the best option. Thus, by using the new geometric interaction averaging operators a MADM problem was successfully solved.

Solution using aggregation operators proposed in [34]:

Step 1 The input preferences related to each alternative was summarized in Table 1 for $t = 3$.

Step 2 By using weight vector $\omega = (0.18, 0.22, 0.16, 0.21, 0.23)^T$ we found P'_{jk} as follows

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	$\begin{pmatrix} 0.7054, \\ 0.5359, \\ 0.5816 \end{pmatrix}$	$\begin{pmatrix} 0.9041, \\ 0.4665, \\ 0.4125 \end{pmatrix}$	$\begin{pmatrix} 0.4655, \\ 0.2759, \\ 0.0928 \end{pmatrix}$	$\begin{pmatrix} 0.4874, \\ 0.2825, \\ 0.4063 \end{pmatrix}$	$\begin{pmatrix} 0.5738, \\ 0.3486, \\ 0.5221 \end{pmatrix}$
j = 2	$\begin{pmatrix} 0.5180, \\ 0.4384, \\ 0.5816 \end{pmatrix}$	$\begin{pmatrix} 0.6776, \\ 0.1703, \\ 0.3095 \end{pmatrix}$	$\begin{pmatrix} 0.7330, \\ 0.3817, \\ 0.5614 \end{pmatrix}$	$\begin{pmatrix} 0.3826, \\ 0.0891, \\ 0.6086 \end{pmatrix}$	$\begin{pmatrix} 0.4553, \\ 0.1517, \\ 0.4184 \end{pmatrix}$
j = 3	$\begin{pmatrix} 0.4370, \\ 0.1259, \\ 0.1931 \end{pmatrix}$	$\begin{pmatrix} 0.4811, \\ 0.3650, \\ 0.1032 \end{pmatrix}$	$\begin{pmatrix} 0, \\ 0, \\ 0.4662 \end{pmatrix}$	$\begin{pmatrix} 0.5863, \\ 0.1845, \\ 0.2033 \end{pmatrix}$	$\begin{pmatrix} 0.5563, \\ 0.0708, \\ 0.5221 \end{pmatrix}$

Step 3 Now, we had to find the score of each attribute of all alternatives.

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	0.1543	0.6689	0.1000	0.0487	0.0466
j = 2	-0.0577	0.2815	0.2169	-0.1695	0.0212
j = 3	0.0762	0.1103	-0.1013	0.1932	0.0298

By comparing the score values, we had

$$\begin{aligned}
 SC(P'_{12}) &> SC(P'_{11}) > SC(P'_{13}) > SC(P'_{14}) > SC(P'_{15}) \\
 SC(P'_{22}) &> SC(P'_{23}) > SC(P'_{25}) > SC(P'_{21}) > SC(P'_{24}) \\
 SC(P'_{34}) &> SC(P'_{32}) > SC(P'_{31}) > SC(P'_{35}) > SC(P'_{33})
 \end{aligned}$$

Based on above score analysis, we found $P'_{\sigma(jk)}$

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	$\begin{pmatrix} 0.9041, \\ 0.4665, \\ 0.4125 \end{pmatrix}$	$\begin{pmatrix} 0.7054, \\ 0.5359, \\ 0.5816 \end{pmatrix}$	$\begin{pmatrix} 0.4655, \\ 0.2759, \\ 0.0928 \end{pmatrix}$	$\begin{pmatrix} 0.4874, \\ 0.2825, \\ 0.4063 \end{pmatrix}$	$\begin{pmatrix} 0.5738, \\ 0.3486, \\ 0.5221 \end{pmatrix}$
j = 2	$\begin{pmatrix} 0.6776, \\ 0.1703, \\ 0.3095 \end{pmatrix}$	$\begin{pmatrix} 0.7330, \\ 0.3817, \\ 0.5614 \end{pmatrix}$	$\begin{pmatrix} 0.4553, \\ 0.1571, \\ 0.4184 \end{pmatrix}$	$\begin{pmatrix} 0.5180, \\ 0.4384, \\ 0.5816 \end{pmatrix}$	$\begin{pmatrix} 0.3826, \\ 0.0891, \\ 0.6086 \end{pmatrix}$
j = 3	$\begin{pmatrix} 0.5863, \\ 0.1845, \\ 0.2033 \end{pmatrix}$	$\begin{pmatrix} 0.4811, \\ 0.3650, \\ 0.1032 \end{pmatrix}$	$\begin{pmatrix} 0.4370, \\ 0.1259, \\ 0.1931 \end{pmatrix}$	$\begin{pmatrix} 0.5563, \\ 0.0708, \\ 0.5221 \end{pmatrix}$	$\begin{pmatrix} 0, \\ 0, \\ 0.4662 \end{pmatrix}$

Step 4 By using the normal distribution-based method, we got $w = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$, and by using the defined aggregation operators, we had

$$\begin{aligned}
 P'_1 &= T - SFHGIA_{\omega, w}(P'_{11}, P'_{12}, P'_{13}, P'_{14}, P'_{15}) \\
 &= \left(\sqrt[3]{\prod_{j=1}^5 \left(m_{\tilde{P}_{\sigma(1k)}}^3 + i_{\tilde{P}_{\sigma(1k)}}^3 \right)^{w_j}} - \prod_{j=1}^5 \left(i_{\tilde{P}_{\sigma(1k)}}^3 \right)^{w_j}, \prod_{j=1}^5 \left(i_{\tilde{P}_{\sigma(1k)}} \right)^{w_j}, \sqrt[3]{1 - \prod_{j=1}^5 \left(1 - n_{\tilde{P}_{\sigma(1k)}}^3 \right)^{w_j}} \right) \\
 &= (0.5750, 0.3533, 0.4473) \\
 P'_2 &= T - SFHGIA_{\omega, w}(P'_{21}, P'_{22}, P'_{23}, P'_{24}, P'_{25}) \\
 &= \left(\sqrt[3]{\prod_{j=1}^5 \left(m_{\tilde{P}_{\sigma(2k)}}^3 + i_{\tilde{P}_{\sigma(2k)}}^3 \right)^{w_j}} - \prod_{j=1}^5 \left(i_{\tilde{P}_{\sigma(2k)}}^3 \right)^{w_j}, \prod_{j=1}^5 \left(i_{\tilde{P}_{\sigma(2k)}} \right)^{w_j}, \sqrt[3]{1 - \prod_{j=1}^5 \left(1 - n_{\tilde{P}_{\sigma(2k)}}^3 \right)^{w_j}} \right) \\
 &= (0.5384, 0.1970, 0.5721) \\
 P'_3 &= T - SFHGIA_{\omega, w}(P'_{31}, P'_{32}, P'_{33}, P'_{34}, P'_{35}) \\
 &= \left(\sqrt[3]{\prod_{j=1}^5 \left(m_{\tilde{P}_{\sigma(3k)}}^3 + i_{\tilde{P}_{\sigma(3k)}}^3 \right)^{w_j}} - \prod_{j=1}^5 \left(i_{\tilde{P}_{\sigma(3k)}}^3 \right)^{w_j}, \prod_{j=1}^5 \left(i_{\tilde{P}_{\sigma(3k)}} \right)^{w_j}, \sqrt[3]{1 - \prod_{j=1}^5 \left(1 - n_{\tilde{P}_{\sigma(3k)}}^3 \right)^{w_j}} \right) \\
 &= (0, 0, 0.3692)
 \end{aligned}$$

This seems meaningless because membership and abstinence of only one T-SFN is zero, but existing operators make a whole aggregated value zero.

Step 5 This step involved the computation of score values:

$$\begin{aligned}
 SC(P_1) &= 0.1006 \\
 SC(P_2) &= -0.0312 \\
 SC(P_3) &= -0.0503
 \end{aligned}$$

Step 6 By comparing score values, we got

$$SC(P_1) > SC(P_2) > SC(P_3)$$

From the above example, the applicability of the proposed operators could easily be checked by comparing the results obtained using new and existing geometric aggregation operators. It was noticed that whenever membership and abstinence of one T-SFN became zero, then the aggregated value using existing aggregation operators seemed impractical. However, the aggregated value using new geometric interactive aggregation operators seemed significant and consistent.

5.2. Advantages of the Proposed Work

In this section, we prove the generalization of proposed work over the existing literature. Here we observed that under some certain conditions the proposed aggregation operators became the

existing aggregation operators under different environment, which shows the superiority of our proposed work.

Consider the T-SFWGIA operator defined as

$$T-SFWGIA_w(P_1, P_2, \dots, P_k) = \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^k (1 - n_j^t)^{w_j} - \prod_{j=1}^k (1 - (m_j^t + i_j^t + n_j^t))^{w_j} - \prod_{j=1}^k (i_j^t)^{w_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^k (1 - i_j^t)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^k (1 - n_j^t)^{w_j}} \end{array} \right) \tag{9}$$

- (1) If we take $t = 2$, the Equation (9) becomes spherical fuzzy weighted geometric interaction averaging operator (SFWGIA operator) and we have

$$SFWGIA_w(P_1, P_2, \dots, P_k) = \left(\begin{array}{c} \sqrt{\prod_{j=1}^k (1 - n_j^2)^{w_j} - \prod_{j=1}^k (1 - (m_j^2 + i_j^2 + n_j^2))^{w_j} - \prod_{j=1}^k (i_j^2)^{w_j}}, \\ \sqrt{1 - \prod_{j=1}^k (1 - i_j^2)^{w_j}}, \sqrt{1 - \prod_{j=1}^k (1 - n_j^2)^{w_j}} \end{array} \right)$$

- (2) If we take $t = 1$, the Equation (9) becomes picture fuzzy weighted geometric interaction averaging operator (PFWGIA operator) and we have

$$PFWGIA_w(P_1, P_2, \dots, P_k) = \left(\begin{array}{c} \prod_{j=1}^k (1 - n_j)^{w_j} - \prod_{j=1}^k (1 - (m_j + i_j + n_j))^{w_j} - \prod_{j=1}^k (i_j)^{w_j}, \\ 1 - \prod_{j=1}^k (1 - i_j)^{w_j}, 1 - \prod_{j=1}^k (1 - n_j)^{w_j} \end{array} \right)$$

- (3) If we take $t = 2$ and $i = 0$, the Equation (9) becomes Pythagorean fuzzy weighted geometric interaction averaging operator (PyFWGIA operator) and we have

$$PyFWGIA_w(P_1, P_2, \dots, P_k) = \left(\begin{array}{c} \sqrt{\prod_{j=1}^k (1 - n_j^2)^{w_j} - \prod_{j=1}^k (1 - (m_j^2 + n_j^2))^{w_j}}, \\ \sqrt{1 - \prod_{j=1}^k (1 - n_j^2)^{w_j}} \end{array} \right)$$

- (4) If we take $t = 1$ and $i = 0$, the Equation (9) becomes intuitionistic fuzzy weighted geometric interaction averaging operator (IFWGIA operator) and we have

$$IFWGIA_w(P_1, P_2, \dots, P_k) = \left(\begin{array}{c} \prod_{j=1}^k (1 - n_j)^{w_j} - \prod_{j=1}^k (1 - (m_j + n_j))^{w_j}, \\ 1 - \prod_{j=1}^k (1 - n_j)^{w_j} \end{array} \right)$$

Similarly, T-SFOWGIA and T-SFHGIA operators can be converted to the existing operators. All of this clearly indicated that our proposed work could be used in the problems described in existing literature, but the operators of existing literature are unable to deal with problems of T-spherical fuzzy information. For example, if we look at Example 2, it can be seen that none of the existing operators can be applied to such problems where information is in the form of T-SFNs.

5.3. Comparative Analysis

The significance of the proposed new geometric operators lies in the fact that the result obtained by using these operations were more justifiable than those developed earlier (i.e., [34,37,38]). Such

operators could not deal with situations where if membership and abstinence value of any number becomes zero then the membership and abstinence value of their aggregated value is also zero. Hence the existing operators of PFSs and T-SFSs did possess the capability of dealing with any kinds of information. But, on the other hand, the new geometric operators of T-SFSs can deal with any type of data justifiably. This point is demonstrated in the case study described in Section 5.1.

The second main advantage of our proposed work is that it has the ability to aggregate the data available in the form of IFSs, PyFSs, PFSs, and SFSs. But, conversely, the existing operators could not handle the data provided in the T-spherical fuzzy environment. For example, if we look at Example 2, its data is purely in the form of T-SFNs based on four grades, being membership, abstinence, non-membership, and refusal degree with $t = 3$, which shows that the aggregation operators of IFSs, PyFSs, PFSs, and SFSs could not aggregate this data. But if we look at Example 3, its data is in the form of IFNs, and our proposed operators easily aggregated this type of data with $t = 1$ and $i = 0$.

Hence, by all means, the proposed work had superiority over the existing work.

Example 3. Let $P_1 = (0, 0.5)$, $P_2 = (0.5, 0.4)$, $P_3 = (0.4, 0.2)$, $P_4 = (0.3, 0.3)$ and $P_5 = (0.7, 0.1) \in IFN$. The weight vector for $P_i (i = 1, 2, \dots, 5)$ is $\omega = (0.18, 0.22, 0.16, 0.21, 0.23)^T$.

$$\begin{aligned} P_1 &= \left((1 - 0.5)^{5 \times 0.18} - (1 - (0 + 0.5))^{5 \times 0.18}, 1 - (1 - 0.5)^{5 \times 0.18} \right) \\ &= (0, 0.5796) \\ P_2 &= \left((1 - 0.4)^{5 \times 0.22} - (1 - (0.5 + 0.4))^{5 \times 0.22}, 1 - (1 - 0.4)^{5 \times 0.22} \right) \\ &= (0.5039, 0.3183) \\ P_3 &= \left((1 - 0.2)^{5 \times 0.16} - (1 - (0.4 + 0.2))^{5 \times 0.16}, 1 - (1 - 0.2)^{5 \times 0.16} \right) \\ &= (0.4000, 0.2000) \\ P_4 &= \left((1 - 0.3)^{5 \times 0.21} - (1 - (0.3 + 0.3))^{5 \times 0.21}, 1 - (1 - 0.3)^{5 \times 0.21} \right) \\ &= (0.2870, 0.2746) \\ P_5 &= \left((1 - 0.1)^{5 \times 0.23} - (1 - (0.7 + 0.1))^{5 \times 0.23}, 1 - (1 - 0.1)^{5 \times 0.23} \right) \\ &= (0.7203, 0.1094) \end{aligned}$$

Scores values were

$$SC(P_1) = -0.5796, SC(P_2) = 0.1856, SC(P_3) = 0.2000, SC(P_4) = 0.0125, SC(P_5) = 0.6109.$$

Thus, $SC(P_5) > SC(P_3) > SC(P_2) > SC(P_4) > SC(P_1)$ and we had

$$\begin{aligned} P_{\sigma(1)} &= (0.7203, 0.1094) \\ P_{\sigma(2)} &= (0.4000, 0.2000) \\ P_{\sigma(3)} &= (0.5039, 0.3183) \\ P_{\sigma(4)} &= (0.2870, 0.2746) \\ P_{\sigma(5)} &= (0, 0.5796) \end{aligned}$$

By using the normal distribution-based method, we found $w = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$.

Now, by using the definition of the T-SFHGIA operator, we found

$$T - SFHGIA_{\omega, w}(P_1, P_2, P_3, P_4, P_5) = (0.4093, 0.2919)$$

Here we got the same result as in [9,10,39]. Thus, the proposed new operators had the capability to solve the problems that lie in the existing structures.

6. Conclusions

In this manuscript, we utilized the concept of T-SFS to handle the uncertainty in the data, so as to capture the information with some more degree of freedom. For it, we defined some new, improved interactive aggregation operations by adding the degree of refusal into the analysis. Then, we studied some basic properties of them. Based on these operational laws, we defined some new weighted geometric aggregation operators and studied their desirable properties. Some of the counter examples were also provided, which showed that the proposed operators worked well in all cases where the existing ones failed to classify the objects. In addition to this, in a comprehensive scrutiny of T-SFSs and the decision-maker preferences, a MADM approach was presented, based on the proposed operator, to select the best alternatives among the feasible ones. Finally, the presented decision-making approach was explained with the help of a numerical example, and an extensive comparative analysis was conducted in relation to the existing decision-making theories. Additionally, the advantages as well as the superiority of the approach was tested with some examples. The advantages of the proposed operators were that a decision maker could choose the required operator in order to optimize their desired goals with more confidence level as compared the existing operators. Furthermore, it was concluded that the several existing operators could be deduced from the proposed one and, hence, the presented operators and algorithm were more generalized. In the future, there is the scope to extend the proposed method to some different environments, and to extend its application in various fields related to decision-theory [40–47].

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