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Spectra of Self-Similar Measures

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Abstract: This paper is devoted to the characterization of spectrum candidates with a new tree structure to be the spectra of a spectral self-similar measure $\mu_{N,D}$ generated by the finite integer digit set D and the compression ratio N^{-1}. The tree structure is introduced with the language of symbolic space and widens the field of spectrum candidates. The spectrum candidate considered by Łaba and Wang is a set with a special tree structure. After showing a new criterion for the spectrum candidate with a tree structure to be a spectrum of $\mu_{N,D}$, three sufficient and necessary conditions for the spectrum candidate with a tree structure to be a spectrum of $\mu_{N,D}$ were obtained. This result extends the conclusion of Łaba and Wang. As an application, an example of spectrum candidate $\Lambda(N,\mathcal{B})$ with the tree structure associated with a self-similar measure is given. By our results, we obtain that $\Lambda(N,\mathcal{B})$ is a spectrum of the self-similar measure. However, neither the method of Łaba and Wang nor that of Strichartz is applicable to the set $\Lambda(N,\mathcal{B})$.

Keywords: spectrality; tree structure; self-similar measure; orthogonal basis

MSC: 42C05; 42A65; 28A78; 28A80

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1. Introduction

Let μ be a probability measure on $\mathbb{R}^d$ with compact support K. We say that μ is a spectral measure if there exists a countable set $\Lambda \subset \mathbb{R}^d$ such that the set of exponential functions $E_\Lambda := \{\exp 2\pi i\langle \lambda, x\rangle : \lambda \in \Lambda\}$ is an orthogonal basis of $L^2(\mu)$. In this case, Λ is called a spectrum of μ and (μ, Λ) is called a spectral pair. In particular, if μ is the normalized Lebesgue measure restricted on K, we say K is a spectral set.

In [1], Fuglede introduced the notion of a spectral set in the study of the extendability of the commuting partial differential operators and raised the famous conjecture: K is a spectral set if and only if K is a translational tile. Although the conjecture was finally disproven for the case that $K \subset \mathbb{R}^d$ with $d \geq 3$ and is still open for $\mathbb{R}^d$ with $d \leq 2$, it has led to the development of harmonic analysis, operator theory, tiling theory, convex geometry, etc.

In 1998, Jorgensen and Pedersen [2] discovered the first singular, non-atomic spectral measure—the middle-forth Cantor measure—and proved the middle-third Cantor measure is not a spectral measure. Following this discovery, there has been much research on the spectrality of self-similar (or self-affine) measures and Moran-type self-similar (or self-affine) measures (see for example [3–21] and the references therein).

Consider the iterated function system (IFS) $\{\phi_j\}_{j=1}^q$ given by

$$\phi_j(x) = \frac{1}{N}(x + d_j),$$

where N is an integer with $|N| > 1$ and $D = \{d_j\}_{j=1}^{q}$ is a finite subset of $\mathbb{R}$. It is well known (see [22] or [23]) that there exists a unique probability measure $\mu_{N,D}$ satisfying

$$\mu_{N,D}(E) = \frac{1}{q}\sum_{j=1}^{q}\mu_{N,D}(\phi_j^{-1}(E)), \text{ for Borel set } E \text{ of } \mathbb{R}.$$

The measure $\mu_{N,D}$ is called the self-similar measure of the IFS $\{\phi_j\}_{j=1}^{q}$ and is supported on the set

$$T(N,D) = \left\{ \sum_{k=1}^{\infty} d_k N^{-k} : d_k \in D, k \geq 1 \right\},$$

which is the attractor of $\{\phi_j\}_{j=1}^{q}$. Given a finite set $S \subset \mathbb{Z}$ with $\sharp S = \sharp D$, we say $(\frac{1}{N}D, S)$ is a compatible pair if the matrix $[\frac{1}{\sqrt{q}}\exp(2\pi i \frac{d}{N}s)]_{d\in D, s\in S}$ is a unitary matrix. In other words, $(\delta_{\frac{1}{N}D}, S)$ is a spectral pair. For a finite set A in $\mathbb{R}$,

$$\delta_A := \frac{1}{\sharp A}\sum_{a\in A}\delta_a,$$

where δ_a is the Dirac measure at a. Write

$$\Lambda(N,S) = \left\{ \sum_{j=0}^{k} s_j N^j : k \geq 0, s_j \in S \right\}.$$

Using the dominated convergence theorem, Strichartz [24] proved that $\mu_{N,D}$ is a spectral measure with a spectrum $\Lambda(N,S)$ under the conditions that $(\delta_{\frac{1}{N}D}, S)$ is a spectral pair with $0 \in S$ and the Fourier transform of $\delta_{\frac{1}{N}D}$ does not vanish on $T(N,S)$. By using the *Ruelle* transfer operator, Łaba and Wang in [3] removed the condition that the Fourier transform of $\delta_{\frac{1}{N}D}$ does not vanish on $T(N,S)$. Furthermore, they obtained the following conclusion:

Theorem 1. (Łaba and Wang). *Let $N \in \mathbb{N}$ with $|N| > 1$, $D \subset \mathbb{Z}$ with $0 \in D$, and $\gcd(D) = 1, 0 \in S \subset \mathbb{Z}$. If $(\frac{1}{N}D, S)$ is a compatible pair, then $(\mu_{N,D}, \Lambda(N,S))$ is not a spectral pair if and only if there exist integers $m \geqslant 1$, $\{s_j\}_{j=0}^{m-1} \subset S$ and $\{\eta_j\}_{j=0}^{m-1} \subset \mathbb{Z}\backslash\{0\}$ such $\eta_{j+1} = N^{-1}(\eta_j + s_j)$ for $0 \leqslant j \leqslant m-1$, where $\eta_m := \eta_0, s_m := s_0$.*

It is well known that to prove the spectrality of the invariant measure $\mu_{N,D}$, the first key step is to construct a suitable spectrum candidate. In this process, the set $\Lambda(N,S) = S + NS + N^2S + \cdots$ (finite sum) is the natural spectrum candidate to be considered. Form Theorem 1, we conclude that $\Lambda(N,S)$ is not a spectrum of $\mu_{N,D}$ if and only if there is a periodic orbit $\{\eta_j\}_{j=0}^{m-1} \subset \mathbb{Z}\backslash\{0\}$ under the dual IFS $\{\psi_i(x) = \frac{1}{N}(x + s_i) : s_i \in S\}$. The following example implies that the natural spectrum candidate has a weak point. When $D = \{0,1\}$, the invariant measure $\mu_{2,D}$ is just the Lebesgue measure on the unit interval with the unique spectrum $\mathbb{Z}$. However, $\Lambda(2,\{0,1\}) = \mathbb{N} \neq \mathbb{Z}$ in this case. In other words, the natural candidate $\Lambda(2,\{0,1\})$ is not a spectrum of $\mu_{2,D}$. Actually, any set with form $S + 2S + 2^2S + \cdots$ (finite sum) is not a spectrum of $\mu_{2,D}$. In this case, one needs to consider the spectrum candidate with a more general form $S_1 + NS_2 + N^2S_3 + \cdots$ (finite sum), where $(\frac{1}{N}D,\ S_i)$ are compatible pairs. Moreover, it is well known that a spectral self-similar (or self-affine) measure has more than one spectrum in general. The results in [7,9–11] show that one may consider spectrum candidates with a tree structure. It is worth mentioning that Li [16] obtained a simplified form of Theorem 1. To the best of our understanding, partial results have been obtained in the case of a higher-dimensional space. Developing the method in [3], Dutkay and Jorgensen [14] obtained a sufficient condition for the spectral pair of self-affine measures, and Li [19] obtained a necessary condition for the natural spectrum candidate to be a spectrum of a self-affine measure.

Motivated by the above results, we considered a class of spectrum candidates with a tree structure (defined in Section 2) and obtained three necessary and sufficient conditions for such spectrum candidates not to be the spectra of $\mu_{N,D}$ (Theorem 2), which generalizes Łaba and Wang's result.

The most difficult part of the proof of Theorem 2 is that the first statement implies the second. For this purpose, we show a new criterion for Λ to be a spectrum of $\mu_{N,D}$. As an application, we give an example involving a self-similar measure μ and a spectrum candidate $\Lambda(N, \mathcal{B})$ with a tree structure in Section 4. By Theorem 2, we obtain $(\mu, \Lambda(N, \mathcal{B}))$ is a spectral pair. However, neither the criterion of Łaba and Wang (Theorem 1) nor that of Strichartz [24] is applicable to this set $\Lambda(N, \mathcal{B})$.

2. Preliminaries

In this section, we shall recall some basic properties of spectral measures and introduce the tree structure using symbolic space.

Let μ be a probability measure on $\mathbb{R}$. The Fourier transform of μ is defined by

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} \, \mathrm{d}\mu(x), \ \ x \in \mathbb{R}.$$

We write $\mathcal{Z}(\hat{\mu}) = \{\xi : \hat{\mu}(\xi) = 0\}$. For a discrete set $\Lambda \subset \mathbb{R}$, write $E_\Lambda = \{\exp(2\pi i x \lambda) : \lambda \in \Lambda\}$ for a family of exponential functions in $L^2(\mu)$. Then, E_Λ is an orthogonal family of $L^2(\mu)$ if and only if

$$\Lambda - \Lambda \subset \mathcal{Z}(\hat{\mu}) \cup \{0\}.$$

Define

$$Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}(\lambda + \xi)|^2, \ x \in \mathbb{R}.$$

By using the Parseval identity, Jorgenson and Pederson ([2]) obtained the following basic criterion for the orthogonality of E_Λ in $L^2(\mu)$.

Proposition 1. *The exponential function set E_Λ is an orthogonal set of $L^2(\mu)$ if and only if $Q_\Lambda(\xi) \le 1$ for all $\xi \in \mathbb{R}$, and E_Λ is an orthogonal basis of $L^2(\mu)$ if and only if $Q_\Lambda(\xi) = 1$ for all $\xi \in \mathbb{R}$.*

Given a finite set $D \subset \mathbb{R}$, we call

$$m_D(\xi) = \frac{1}{\sharp D} \sum_{d \in D} \exp(2\pi i \xi d), \qquad \xi \in \mathbb{R}$$

the mask of D. It is clear that it is just the Fourier transformation of the uniform probability measure on D.

Definition 1. *For two finite subsets D and S of $\mathbb{R}$ with the same cardinality m, we say (D, S) is a compatible pair if*

$$\left[\frac{1}{\sqrt{m}} \exp(2\pi i d s)\right]_{d \in D, s \in S}$$

is a unitary matrix.

The following conclusion is well known.

Lemma 1. *For two finite subsets D and S of $\mathbb{R}$ with the same cardinality m, the following statements are equivalent:*

(i). *(D, S) is a compatible pair;*
(ii). *$m_D(s_1 - s_2) = 0$ for any $s_1 \neq s_2 \in S$;*
(iii). *$\sum_{s \in S} |m_D(\xi + s)|^2 = 1$ for any $\xi \in \mathbb{R}$.*

In other words, (D, S) is a compatible pair if and only if S is a spectrum of the uniform probability measure on D.

Let N be an integer with $|N| > 1$ and $D = \{d_j\}_{j=1}^q$ a finite subset of $\mathbb{Z}$ with $0 \in D$. We denote by $\mu_{N,D}$ the unique invariant measure with respect to the IFS $\{\phi_j(x) = \frac{1}{N}(x + d_j) : 1 \le j \le q\}$ with equal probability weights, i.e.,

$$\mu_{N,D} = \frac{1}{q}\sum_{j=1}^{q} \mu_{N,D} \circ \phi_j^{-1}.$$

In the sequel, we write $\mu = \mu_{N,D}$ for simplicity. Thus, we have

$$\hat{\mu}(\xi) = \prod_{j=1}^{\infty} m_D(N^{-j}\xi), \quad \xi \in \mathbb{R}.$$

For $k \ge 1$, we write

$$\hat{\mu}_k(\xi) = \prod_{j=1}^{k} m_D(N^{-j}\xi), \quad \xi \in \mathbb{R}. \tag{1}$$

Write $Y(m_D) = \{\xi \in \mathbb{R} : m_D(\xi) = 1\}$. When $gcd(D) = 1$, we have

$$Y(m_D) = \{\xi \in \mathbb{R} : |m_D(\xi)| = 1\} = \mathbb{Z}. \tag{2}$$

Now, we introduce the tree structure. First, we recall some basic notation of symbolic space. Given a positive integer $q > 1$, write $\Sigma_q = \{0, 1, \cdots, q-1\}$. Let $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma_q^n$ stand for the set of all finite words, where $\Sigma_q^0 = \{\vartheta\}$ denotes the set of empty words. The length of a finite word σ is the number of symbols it contains and is denoted by $|\sigma|$. The concatenation of two finite words σ and σ' is written as $\sigma\sigma'$. We say σ is a prefix of $\sigma\sigma'$. Given $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \Sigma^*$ and $1 \le k \le n$, write $\sigma|_k = \sigma_1\cdots\sigma_k$. The following definition will bring convenience to us.

Definition 2. *A sequence of finite words $\{I_n\}_{n\ge1} \subset \Sigma^*$ is called increasing if for any $n \ge 1$, I_n is a prefix of I_{n+1} and $|I_{n+1}| = |I_n| + 1$.*

Let $\mathcal{C}$ be a mapping from Σ^* to $\mathbb{Z}$ satisfying $\mathcal{C}(\vartheta) = 0$ and $\mathcal{C}(I) = 0$ if I ends with the symbol 0. It induces a family of mapping $\mathcal{F} = \{F_I\}_{I\in\Sigma^*}$ defined by

$$\begin{aligned} F_I :\ &\Sigma^* \longrightarrow \mathbb{Z}, \\ &J \longmapsto \mathcal{C}(IJ|_1) + N\mathcal{C}(IJ|_2) + \cdots + N^{|J|-1}\mathcal{C}(IJ), \end{aligned}$$

where $IJ|_i$ is the concatenation of I and $J|i$ for $1 \le i \le |J|$. We write $F(J) = F_\vartheta(J)$ for convenience. By a simple deduction, we have the following consistency: for any $I, J, K \in \Sigma^*$,

$$\begin{aligned} &F_I(J) + N^{|J|}F_{IJ}(K) \\ =&\mathcal{C}(IJ|_1) + \cdots + N^{|J|-1}\mathcal{C}(IJ) + N^{|J|}\mathcal{C}(IJK|_1) + \cdots + N^{|JK|-1}\mathcal{C}(IJK) \\ =&F_I(JK). \end{aligned}$$

Definition 3. *We say a countable set $\Lambda \subset \mathbb{R}$ has a $(\mathcal{C}, \mathcal{F})$ tree structure if there exists a mapping $\mathcal{C}$ and an associated family of mappings $\mathcal{F}$ defined in the above paragraph such that*

$$\Lambda = \bigcup_{I\in\Sigma^*} \{F(I)\}.$$

For $I \in \Sigma^*$, let $S_I = \{\mathcal{C}(Ii) : i \in \Sigma_q\}$. According to the definition of the mapping $\mathcal{C}$, we have $\mathcal{C}(I0) = 0 \in S_I$.

Remark 1. *Given a sequence of finite sets $\mathcal{S} = \{S_n\}_{n \geq 1}$, if $S_I = S_{n+1}$ for any $I \in \Sigma^n (n \geq 0)$, we obtain*

$$\Lambda = S_1 + NS_2 + N^2 S_3 \cdots .$$

In particular, if $S_n = S$ for $n \geq 1$, we obtain

$$\Lambda = S + NS + N^2 S \cdots ,$$

which is just the case considered by Łaba and Wang in [3].

In this paper, we consider a countable set Λ as a spectrum candidate satisfying the following three conditions:
(C1). Λ has a $(\mathcal{C}, \mathcal{F})$ tree structure.
(C2). For any $I \in \Sigma^*$, $(\frac{1}{N}D, S_I)$ is a compatible pair.
(C3). The set $\widetilde{S} = \bigcup_{I \in \Sigma^*} S_I$ is bounded.

Remark 2. *Since we only assume that $(\frac{1}{N}D, S_I)$ is a compatible pair with $S_I = \{\mathcal{C}(Ii) : i \in \Sigma_q\}$, the map $\mathcal{C}$ may not be a maximal mapping defined in [8] (Definition 2.5) even if $D = \{0, 1, \cdots, q-1\}$.*

Now, we exploit some basic properties of Λ satisfying the conditions (C1), (C2), and (C3). The first one is the uniqueness of the tree representation.

Proposition 2. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $gcd(D) = 1$. Assume that a countable set Λ satisfies the conditions (C1), (C2), and (C3). Then, for any $I \in \Sigma_q^*$ and J, $K \in \Sigma_q^n$ with $n \geqslant 0$, we have $F_I(J) = F_I(K)$ if and only if $J = K$.*

Proof. We just prove the necessity. Suppose there exist $I \in \Sigma_q^*$ and $J \neq K \in \Sigma_q^n$ with $n \geqslant 1$ such that $F_I(J) = F_I(K)$. Let l be the smallest integer with $J|_l \neq K|_l$. From $F_I(J) = F_I(K)$, it follows that

$$N^{l-1}\mathcal{C}(IJ|_l) + \cdots + N^{n-1}\mathcal{C}(IJ) = N^{l-1}\mathcal{C}(IK|_l) + \cdots + N^{n-1}\mathcal{C}(IK),$$

which implies $\mathcal{C}(IJ|_l) \equiv \mathcal{C}(IK|_l) (\mod N)$. Noting $\mathcal{C}(IJ|_l)$, $\mathcal{C}(IK|_l) \in S_{IJ|_{l-1}}$, we obtain $(\frac{1}{N}D, S_{IJ|_{l-1}})$ is not a compatible pair, which is a contradiction to the condition (C2). □

Proposition 3. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $gcd(D) = 1$. Assume that a countable set Λ satisfies the conditions (C1), (C2), and (C3). Then, $E(\Lambda)$ is an orthogonal set of $L^2(\mu)$.*

Proof. Given $\alpha \neq \beta \in \Lambda$, there exist two finite words $I, J \in \Sigma^*$ such that

$$\alpha = F(I), \quad \beta = F(J).$$

If $|I| \neq |J|$, we add symbol 0 in the end of I or J to obtain $|I| = |J|$. Without loss of generality, we assume that $I, J \in \Sigma_q^n$ for some integer n. Let l be the smallest positive integer satisfying $I|_l \neq J|_l$. Recall that $F(I|_l) = \mathcal{C}(I|_1) + N\mathcal{C}(I|_2) \cdots + N^{l-1}\mathcal{C}(I|_l)$. Then, there exists an integer z_0 such that

$$N^{-l}(F(I|_l) - F(J|_l)) = \frac{1}{N}(\mathcal{C}(I|_l) - \mathcal{C}(J|_l)) + z_0.$$

By virtue of the condition (C2), we know that $(\frac{1}{N}D, S_{I|_{l-1}})$ is a compatible pair. Noting that both $\mathcal{C}(I|_l)$ and $\mathcal{C}(J|_l)$ belong to $S_{I|_{l-1}}$, we obtain

$$m_D(N^{-l}(F(I|_l) - F(J|_l))) = m_D(\frac{1}{N}(\mathcal{C}(I|_l) - \mathcal{C}(J|_l) + z_0) = m_D(\frac{1}{N}(\mathcal{C}(I|_l) - \mathcal{C}(J|_l))) = 0.$$

This leads to

$$\begin{aligned}
&\hat{\mu}(\alpha - \beta) = \hat{\mu}(F(I) - F(J)) \\
=&\prod_{j=1}^{l-1} m_D(N^{-j}(F(I) - F(J)))m_D(N^{-l}(F(I|_l) - F(J|_l))) \prod_{j=l+1}^{\infty} (N^{-j}(F(I) - F(J))) \\
=&0.
\end{aligned}$$

□

For any $I \in \Sigma_q^*$ and $k \geqslant 1$, define

$$\Lambda_I = \{F_I(J) : J \in \Sigma^*\} \quad \text{and} \qquad \Lambda_I^k := \{F_I(J) : \ J \in \Sigma_q^k\}.$$

We write $\Lambda^k := \Lambda_\vartheta^k$ for simplicity. It is clear that

$$\Lambda_I^k \subsetneq \Lambda_I^{k+1}.$$

From the condition (C2) and Lemma 1(ii), it follows that $E(\Lambda_I^k)$ is an orthogonal set of $L^2(\mu_k)$. By (2), we obtain $\#\Lambda_I^k = q^k$. Noting the fact that $\dim(L^2(\mu_k)) = q^k$, we conclude that $E(\Lambda_I^k)$ is an orthogonal basis of $L^2(\mu_k)$. In other words, Λ_I^k is a spectrum of μ_k. By Lemma 1, we have

$$\sum_{\lambda \in \Lambda_I^k} \prod_{j=1}^{k} |m_D(N^{-j}(\xi + \lambda))|^2 = \sum_{\lambda \in \Lambda_I^k} |\hat{\mu}_k(\xi + \lambda)|^2 \equiv 1, \quad \forall \, \xi \in \mathbb{R}. \tag{3}$$

In fact, we have the following conclusion.

Proposition 4. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $gcd(D) = 1$. Assume that a countable set Λ satisfies the conditions* (C1), (C2), *and* (C3). *Then, $Q_\Lambda(\xi) \equiv 1$ if and only if $Q_{\Lambda_I}(\xi) \equiv 1$ for any $I \in \Sigma^*$,*

Proof. By virtue of $\Lambda_\vartheta = \Lambda$, the sufficiency is obvious.

Next, we prove the necessity. Given $n \geq 1$ and $I \in \Sigma_q^n$, write $B_I = \{\xi + F(I) : \xi \in [0,1]\}$ and $\widetilde{B}_I = \{N^{-n}(\xi + F(I)) : \xi \in [0,1]\}$. It is easy to see that both B_I and $\widetilde{B}_I$ are compact sets. Noting the fact that $\hat{\mu}_n$ can be extended to be an entire function on the complex plane, $\hat{\mu}_n$ has at most finitely many zero points in B_I. On the other hand, recall that

$$\Lambda = \bigcup_{I \in \Sigma_q^n} \bigcup_{J \in \Sigma^*} (F(I) + N^n F_I(J)), \quad n \geq 1.$$

Noting the fact that every integer is a period of m_D, we have $\hat{\mu}_n(\xi + F(IJ)) = \hat{\mu}_n(\xi + F(I))$ for any $I \in \Sigma_q^n$ and $J \in \Sigma_q^*$. Hence,

$$
\begin{aligned}
Q_\Lambda(\xi) &= \sum_{\lambda\in\Lambda} |\hat{\mu}_n(\xi+\lambda)|^2 |\hat{\mu}(N^{-n}(\xi+\lambda))|^2 \\
&= \sum_{I\in\Sigma_q^n} \sum_{J\in\Sigma^*} |\hat{\mu}_n(\xi+F(I))|^2 |\hat{\mu}(N^{-n}(\xi+F(I)+N^nF_I(J)))|^2 \\
&= \sum_{I\in\Sigma_q^n} |\hat{\mu}_n(\xi+F(I))|^2 \sum_{J\in\Sigma^*} |\hat{\mu}(N^{-n}(\xi+F(I))+F_I(J))|^2 \\
&= \sum_{I\in\Sigma_q^n} |\hat{\mu}_n(\xi+F(I))|^2 Q_{\Lambda_I}(N^{-n}(\xi+F(I))).
\end{aligned} \tag{4}
$$

In combination with (3), this means $Q_{\Lambda_I}(\xi)$ takes 1 on except at most finitely many points in $\widetilde{B}_I$, which implies $Q_{\Lambda_I}(\xi) \equiv 1$ by using the continuity of $Q_{\Lambda_I}(\xi)$. □

In the end of this section, we define the dual IFS $\{\Phi_s(x) = \frac{1}{N}(x+s) : s \in \widetilde{S}\}$, which plays an important role in what follows. Let T be the invariant set of the IFS, i.e.,

$$T = \bigcup_{s\in\widetilde{S}} \Phi_s(T).$$

Define $\mathcal{Z}(\hat{\mu}, T) = \mathcal{Z}(\hat{\mu}) \cap T$, which stands for the zero point set of $\hat{\mu}$ on T. It is clear that $p := \#\mathcal{Z}(\hat{\mu}, T)$ is finite.

3. Main Theorem

In this section, we will give our main results involving three equivalent statements. To prove the most difficult part of the proof, we prepared several lemmas including a new criterion for a spectrum candidate with a tree structure to be a spectrum of a self-similar measure. At the end of this section, we show that the new criterion is just a sufficient and necessary condition, which is stated as a corollary .

Theorem 2. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $\gcd(D) = 1$. Assume that a countable set Λ satisfies the conditions* (C1), (C2), *and* (C3). *Then, the following statements are equivalent:*

(i). *(μ, Λ) is not a spectral pair.*
(ii). *There exists a finite word $I \in \Sigma^*$ such that $\inf_{\xi\in T} Q_{\Lambda_I}(\xi) = 0$.*
(iii). *There exist a finite word $J \in \Sigma^*$, a sequence of nonzero integers $\{\beta_l\}_{l\geq1} \subset \mathbb{Z}, \setminus\{0\}$ and a sequence of increasing finite words $\{Ji_1\cdots i_l\}_{l\geq1} \subset \Sigma^*$, which has a prefix J such that, for any $l \geq 1$, we have $\beta_{l+1} = \frac{1}{N}(\beta_l + \mathcal{C}(Ji_1\cdots i_l))$.*

We shall divide the proof into three parts (iii) $\Rightarrow$ (i), (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii). First, we prove (iii) $\Rightarrow$ (i), which plays a key role in the proof of (i) $\Rightarrow$ (ii).

Proof of Theorem 2 (iii) $\Rightarrow$ (i)**.** We shall prove $Q_{\Lambda_J}(\beta_1) = 0$. Thus, from Proposition 4, the conclusion follows.

Given $\lambda \in \Lambda_I$, there exists a positive integer $m \geqslant 1$ and $L \in \Sigma_q^m$ such that

$$\lambda = F_I(L) \in \Lambda_I^m.$$

Since the sequence $\{\beta_l\}_{l\geq1}$ is nonzero, the sequence of integers $\{\mathcal{C}(Ji_1\cdots i_l)\}_{l\geqslant1}$ has infinitely many nonzero terms. Thus, there exist infinitely many terms l with $i_l \neq 0$. Take an integer $r > m$ with $i_r \neq 0$. Write $\lambda^* := F_J(K) \in \Lambda_J^r$. According to Proposition 2 and $i_r \neq 0$, we have $\lambda \neq \lambda^*$ and $\lambda \in \Lambda_J^m \subset \Lambda_J^r$. From $\beta_{k+1} = N^{-1}(\beta_k + \mathcal{C}(Ji_1\cdots i_k))(k \geqslant 1)$, it follows that

$$\begin{aligned}\beta_1+\lambda^* &= N(\beta_2+\mathcal{C}(JK|_2)+N\mathcal{C}(JK|_3)+\cdots+N^{r-2}\mathcal{C}(JK)) \\ &= \cdots \\ &= N^r\beta_{r+1}\in N^r\mathbb{Z},\end{aligned}$$

which implies $|\widehat{\mu_r}(\beta_1+\lambda^*)|^2=1$. Noting (3) and $\lambda\neq\lambda^*$, we have

$$1\leqslant|\widehat{\mu_r}(\beta_1+\lambda^*)|^2+|\widehat{\mu_r}(\beta_1+\lambda)|^2\leqslant\sum_{\gamma\in\Lambda_J^r}|\widehat{\mu_r}(\beta_1+\gamma)|^2=1,$$

Thus, we obtain $|\widehat{\mu_r}(\beta_1+\lambda)|=0$. Hence,

$$|\hat{\mu}(\beta_1+\lambda)|=0,\qquad\forall\,\lambda\in\Lambda_J.$$

It follows that $Q_{\Lambda_J}(\beta_1)=\sum_{\lambda\in\Lambda_J}|\hat{\mu}(\beta_1+\lambda)|^2=0$. $\square$

The following three lemmas play key roles in the proof of Theorem 2 (i) $\Rightarrow$ (ii). First, we show a new criterion for Λ to be a spectrum of μ.

Lemma 2. *Let $N\in\mathbb{Z}$ with $|N|>1$ and $D\subset\mathbb{Z}$ with $0\in D$ and $gcd(D)=1$. Assume that a countable set Λ satisfies the conditions* (C1), (C2), *and* (C3). *If there exists a positive number $c>0$ such that, for any ξ and $I\in\Sigma^*$, there is $\lambda_{\xi,I}\in\Lambda_I$ satisfying*

$$|\widehat{\mu}(N^{-|I|}(\xi+F(I))+\lambda_{\xi,I})|\geq c,$$

then (μ,Λ) is a spectral pair.

Proof. Suppose (μ,Λ) is not a spectral pair. Then, there exists $\xi_0\in T$ such that $Q_\Lambda(\xi_0)<1$.

Recall that $\Lambda^n=\{\mathcal{C}(J|_1)+N\mathcal{C}(J|_2)+\cdots+N^{n-1}\mathcal{C}(J):J\in\Sigma_q^n\}$ for $n\geq1$. We write $Q_n(\xi_0):=\sum_{\lambda\in\Lambda^n}|\hat{\mu}(\xi_0+\lambda)|^2$. By virtue of $\lim_{n\to\infty}\Lambda^n=\Lambda$ and $\Lambda_n\subset\Lambda_{n+1}$ for $n\geq1$, we obtain

$$\lim_{n\to\infty}Q_n(\xi_0)=Q_\Lambda(\xi_0)\text{ and }Q_n(\xi_0)\leqslant Q_{n+1}(\xi_0).$$

Given a positive number ε with $\varepsilon<\frac{1}{2}(1-Q_\Lambda(\xi_0))$, there exists an integer $M\geqslant1$ such that

$$Q_\Lambda(\xi_0)-\varepsilon\leqslant Q_M(\xi_0)\leqslant Q_n(\xi_0)\leqslant Q_\Lambda(\xi_0)<1,\quad\forall\,n\geqslant M.\tag{5}$$

By (1), we have

$$\lim_{m\to\infty}\hat{\mu}_m(\xi_0+\lambda)=\hat{\mu}(\xi_0+\lambda),\qquad\forall\,\lambda\in\Lambda.$$

In combination with (5), we have a positive integer $K\geqslant M+1$ such that

$$\sum_{\lambda\in\Lambda^M}|\hat{\mu}_K(\xi_0+\lambda)|^2\leqslant\sum_{\lambda\in\Lambda^M}|\hat{\mu}(\xi_0+\lambda)|^2+\varepsilon\leqslant Q_\Lambda(\xi_0)+\varepsilon.$$

According to (3), we have $\sum_{\lambda\in\Lambda^K}|\hat{\mu}_K(\xi_0+\lambda)|^2=1$. Thus,

$$\begin{aligned}\sum_{I\in\Sigma_q^K\setminus\Sigma_q^M}|\hat{\mu}_K(\xi_0+F(I))|^2 &= \sum_{\lambda\in\Lambda^K}|\hat{\mu}_K(\xi_0+\lambda)|^2-\sum_{\lambda\in\Lambda^M}|\hat{\mu}_K(\xi_0+\lambda)|^2 \\ &\geqslant 1-Q_\Lambda(\xi_0)-\varepsilon>0.\end{aligned}\tag{6}$$

For any $I\in\Sigma_q^K\setminus\Sigma_q^M$, there exists $\lambda_{\xi_0,I}\in\Lambda_I$ such that

$$|\hat{\mu}(N^{-K}(\xi_0+F(I))+\lambda_{\xi_0,I})|>c.\tag{7}$$

Write $\widetilde{\Lambda} = \{F(I) + N^K \lambda_{\xi_0,I} : I \in \Sigma_q^K \backslash \Sigma_q^M, \lambda_{\xi_0,I} \in \Lambda_I\}$. It is clear that $\widetilde{\Lambda} \subset \Lambda$. Since $(\frac{1}{N}D, S_I)$ is a compatible pair for any $I \in \Sigma^*$, $\mathcal{C}(I) = 0$ if and only if the finite word I ends with the symbol 0. Then, we have

$$\Lambda^M \cap \widetilde{\Lambda} = \varnothing.$$

In combination with (5)–(7), we obtain

$$\begin{aligned}
Q_\Lambda(\xi_0) &= \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi_0 + \lambda)|^2 \\
&\geq \sum_{\lambda \in \Lambda^M} |\hat{\mu}(\xi_0 + \lambda)|^2 + \sum_{\lambda \in \widetilde{\Lambda}} |\hat{\mu}(\xi_0 + \lambda)|^2 \\
&= \sum_{\lambda \in \Lambda^M} |\hat{\mu}(\xi_0 + \lambda)|^2 + \sum_{I \in \Sigma^K \backslash \Sigma^M} |\hat{\mu}(\xi_0 + F(I) + N^K \lambda_{\xi_0,I})|^2 \\
&= \sum_{\lambda \in \Lambda^M} |\hat{\mu}(\xi_0 + \lambda)|^2 + \sum_{I \in \Sigma^K \backslash \Sigma^M} |\hat{\mu}_K(\xi_0 + F(I))|^2 |\hat{\mu}(N^{-K}(\xi_0 + F(I)) + \lambda_{\xi_0,I}|^2 \\
&\geqslant Q_\Lambda(\xi_0) - \varepsilon + c^2 \sum_{I \in \Sigma^K \backslash \Sigma^M} |\hat{\mu}_K(\xi_0 + F(I))|^2 \\
&\geqslant Q_\Lambda(\xi_0) - \varepsilon + c^2(1 - Q_\Lambda(\xi_0) - \varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$0 \geqslant c^2(1 - Q_\Lambda(\xi_0)),$$

which is a contradiction to $Q_\Lambda(\xi_0)) < 1$. □

To use Lemma 2, we need the following lemma, which implies that, under some conditions for any point in T, there exists a path that escapes from $\mathcal{Z}(\hat{\mu}, T)$.

Lemma 3. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $\gcd(D) = 1$. Assume that a countable set Λ satisfies the conditions* (C1), (C2), *and* (C3) *and $\inf_{\xi \in T} Q_{\Lambda_I}(\xi) > 0$ for any $I \in \Sigma_q^*$. If $\mathcal{Z}(\hat{\mu}, T) \neq \varnothing$ and for any $\alpha \in \mathcal{Z}(\hat{\mu}, T)$ and $I \in \Sigma^*$, there exists no $K \in \Sigma^*$ with $\alpha + F_I(K) = 0$, then for any $\xi \in T$, there exist two nonnegative integers w and v with $1 \leqslant v \leqslant p + 1$ and a finite word $J = j_1 \cdots j_{w+v} \in \Sigma_q^*$ satisfying the following property:*

If $w = 0$, we have

$$0 < |m_D(N^{-l}(\xi + F_I(J|_l)))| < 1, \qquad 1 \leqslant l \leqslant v,$$

and $|\hat{\mu}(N^{-v}(\xi + F_I(J)))| > 0$;

If $w > 0$, we have

$$\begin{aligned}
&m_D(N^{-l}(\xi + F_I(J|_l))) = 1, \qquad 1 \leqslant l \leqslant w, \\
&0 < |m_D(N^{-l}(\xi + F_I(J|_l)))| < 1, \qquad w + 1 \leqslant l \leqslant w + v,
\end{aligned}$$

and $|\hat{\mu}(N^{-w-v}(\xi + F_I(J)))| > 0$.

Proof. First, we shall prove the existence of w. If $T \cap \mathbb{Z} = \varnothing$, we take $w = 0$. If $T \cap \mathbb{Z} \neq \varnothing$, since $(\frac{1}{N}D, S_I)$ is a compatible pair, by Lemma 1(iii), there exists $j_1 \in \Sigma_q$ such that

$$|m_D(N^{-1}(\xi + F_I(j_1)))| > 0. \tag{8}$$

If $|m_D(N^{-1}(\xi + F_I(j_1)))| < 1$, we take $w = 0$. If $|m_D(N^{-1}(\xi + F_I(j_1)))| = 1$, also by Lemma 1(iii), there exists $j_2 \in \Sigma_q$ such that

$$|m_D(N^{-2}(\xi + F_I(j_1 j_2)))| > 0.$$

If $|m_D(N^{-2}(\xi + F_I(j_1j_2)))| < 1$, we take $w = 1$. When $|m_D(N^{-2}(\xi + F_I(j_1j_2)))| = 1$, the process goes on. Under the process, we claim that there exists a finite sequence of symbols $\{j_n\}_{n=1}^{w} \subset \Sigma_q$ such that

$$m_D(N^{-l}(\xi + F_I(j_1 \cdots j_l))) = 1, \qquad \forall\, 1 \leqslant l \leqslant w,$$

and

$$0 < |m_D(N^{-w-1}(\xi + F_I(j_1 \cdots j_{w+1})))| < 1, \qquad \forall\, j_{w+1} \in \Sigma_q. \tag{9}$$

Otherwise, there exists an infinite sequence $\{j_l\}_{l \geqslant 1} \subset \Sigma_q$ such that $m_D(N^{-l}(\xi + F_I(j_1 \cdots j_l))) = 1$ for $l \geqslant 1$. By (2) and the hypothesis of the lemma, we have $N^{-l}(\xi + F_I(j_1 \cdots j_l)) \in \mathbb{Z} \backslash \{0\}$. According to the proof of Theorem 2(iii) $\Rightarrow$ (i), we obtain $Q_{\Lambda_I}(\xi) = 0$, which is a contradiction to the condition $\inf_{\xi \in T} Q_{\Lambda_I}(\xi) > 0$ for any $I \in \Lambda^*$.

Next, we shall prove the existence of v. We write $\widetilde{J} := j_1 \cdots j_w$ and $\eta := N^{-w}(\xi + F_I(\widetilde{J}))$, where $\widetilde{J} = \vartheta$, $F_I(\widetilde{J}) = 0$ and $\eta = \xi$ when $w = 0$. In what follows, we define a sequence of sets $\{Y_n\}_{n \geq 0}$ by induction on n. Define $Y_0 = \{\vartheta\}$, and

$$Y_n := \{L \in \Sigma_q^n : L|_{n-1} \in Y_{n-1}, 0 < |m_D(N^{-n}(\eta + F_{I\widetilde{J}}(L)))| < 1\}, \quad n \geq 1.$$

We have the following claim. $\square$

Claim: For $n \geq 1$, we have $\#Y_n \geqslant 2^n$.

Proof. When $n = 1$, since $(\frac{1}{N}D, S_{I\widetilde{J}})$ is a compatible pair, there exist two symbols $l_1 \neq l_2 \in \Sigma_q$ such that

$$0 < |m_D(N^{-1}(\eta + F_{I\widetilde{J}}(l_k)))| < 1, \qquad 1 \leqslant k \leqslant 2.$$

Thus, we obtain $\#Y_1 \geqslant 2$. Suppose the inequality $\#Y_n \geqslant 2^n$ holds as $n = k$. Let $n = k + 1$. For any $L \in Y_k$, it is clear $L|_1 \in Y_1$. By (9), we obtain $N^{-1}(\eta + F_{I\widetilde{J}}(L|_1)) \notin \mathbb{Z}$. Thus, $N^{-k}(\eta + F_{I\widetilde{J}}(L)) \notin \mathbb{Z}$. Since $(\frac{1}{N}D, S_{I\widetilde{J}L})$ is a compatible pair, there exist at least two symbols $l_1 \neq l_2 \in \Sigma_q$ such that

$$0 < |m_D(N^{-n-1}(\eta + F_{I\widetilde{J}}(Ll_k)))| < 1, \qquad 1 \leqslant k \leqslant 2.$$

By the arbitrariness of $L \in Y_k$, we obtain $\#Y_{n+1} \geqslant 2^{n+1}$. Hence, the claim follows by induction.

Together with Proposition 2, the above claim implies

$$\#\{N^{-p-1}(\alpha + F_{I\widetilde{J}}(L) : L \in Y_{p+1}\} = \#Y_{p+1} \geqslant 2^{p+1} > p.$$

Thus, by $p = \sharp\mathcal{Z}(\hat{\mu}, T)$, there exists a finite word $L \in Y_{p+1}$ such that

$$|\hat{\mu}(N^{-p-1}(\eta + F_{I\widetilde{J}}(L)))| > 0.$$

Let $v \geqslant 1$ be the smallest positive integer such that $|\hat{\mu}(N^{-v}(\eta + F_{I\widetilde{J}}(L)))| > 0$ for some $L = l_1 \cdots l_v$. By taking $J = \widetilde{J} l_1 \cdots l_v$, we finish the proof. $\square$

Lemma 4. *If $T \cap \mathbb{Z} \neq \varnothing$, then there exists $\alpha_1 > 0$ such that, for any integer sequence $\{\theta_i\}_{i \geqslant 1} \subset T \cap \mathbb{Z}$, we have*

$$\prod_{i=1}^{\infty} |m_D(x_i)| \geqslant \alpha_1,$$

where $x_i \in B(\theta_i, N^{-i})$.

Proof. For any $\theta \in T \cap \mathbb{Z}$, we have $m_D(\theta) = 1$. On the other hand, the mask function m_D can be extended to an entire function on the complex plane. Thus, m_D is uniformly continuous on any compact set. Hence, there exists a positive number c_1 such that

$$|1 - m_D(x)| = |m_D(\theta) - m_D(x)| \leqslant c_1|x - \theta|, \qquad \forall\, x \in \{\xi + y : \ \xi \in T, |y| \leq 1\}.$$

Given a sequence $\{\theta_i\}_{i\geqslant 1} \subset T \cap \mathbb{Z}$, we have

$$|m_D(x_i)| \geqslant 1 - c_1|x_i - \theta_i| \geqslant 1 - N^{-i}c_1, \quad \forall\, x_i \in B(\theta_i, N^{-i}),\ i \geqslant 1.$$

It is clear that there exists a positive integer $K > 0$ such that, for $k \geqslant K$, we have $N^{-k}c_1 < \frac{1}{2}$. Note an elementary inequality:

$$1 - x \geqslant e^{-2x}, \quad 0 \leq x \leqslant \frac{1}{2}.$$

Then, we have

$$\begin{aligned}
\prod_{i=1}^{\infty}|m_D(x_i)| =& \prod_{i=1}^{K}|m_D(x_i)| \prod_{i=K+1}^{\infty}|m_D(x_i)| \\
\geqslant& (\frac{1}{2})^K \prod_{i=K+1}^{\infty} e^{-2c_1N^{-i}} \\
=& (\frac{1}{2})^K e^{\sum_{i=K+1}^{\infty} -2c_1N^{-i}} \\
=& (\frac{1}{2})^K e^{-2c_1\frac{1}{N^K(N-1)}} =: \alpha_1 > 0
\end{aligned} \tag{10}$$

for all $x_i \in B(\theta_i, N^{-i})$. The proof is complete. $\square$

Proof of Theorem 2(i) $\Rightarrow$ (ii). We expect to obtain a contradiction after assuming

$$\inf_{\xi\in T} Q_{\Lambda_I}(\xi) > 0, \quad \forall\, I \in \Sigma_q^*. \tag{11}$$

We shall prove that there is a positive number $c > 0$ such that, for any $\xi \in T$ and $I \in \Sigma^*$, there exists $\lambda_{\xi,I} \in \Lambda_I$ satisfying

$$|\widehat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| \geq c.$$

If $\mathcal{Z}(\hat{\mu}, T) = \varnothing$, then $\hat{\mu}(\xi)$ has a positive lower bound on compact set T. Write $c := \inf_{\xi\in T}|\hat{\mu}(\xi)| > 0$. For any $\xi \in T$ and $I \in \Sigma_q^*$, take $\lambda_{\xi,I} = 0 \in \Lambda_I$. Noting $N^{-|I|}(\xi + F(I)) \in T$, we have

$$|\hat{\mu}(N^{-|I|}(\xi + F(I)))| \geqslant c.$$

From Lemma 2, it follows that (μ, Λ) is a spectral pair, which is a contradiction to the hypothesis. $\square$

Next, we focus on the case $\mathcal{Z}(\hat{\mu}, T) \neq \varnothing$. We shall deal with two cases.
Case i. *For any $\eta \in \mathcal{Z}(\hat{\mu}, T)$ and $I \in \Sigma^*$, there exists $J \in \Sigma^*$ such that*

$$\eta + F_I(J) = 0. \tag{12}$$

By $|\hat{\mu}(0)| = 1$, there exists a positive number δ with $0 < \delta_1 < 1$ such that

$$|\hat{\mu}(x)| > \frac{1}{2}, \qquad \forall\, x \in B(0, \delta_1). \tag{13}$$

Write $\delta := \min\{\delta_1, \frac{d}{4}\}$, where d denotes the smallest distance between different points in $\mathcal{Z}(\hat{\mu}, T) \cup (T \cap \mathbb{Z})$, i.e., $d := \min\{|x-y| : x \neq y \in \mathcal{Z}(\hat{\mu}, T) \cup T \cap \mathbb{Z}\}$.

We denote the set of points that has a positive distance from the zero points of $\hat{\mu}(\xi)$ in T by

$$P := T \backslash \left(\bigcup_{\theta \in \mathcal{Z}(\hat{\mu}, T)} B(\theta, \delta) \right).$$

It is clear that P is a compact set and $\alpha_0 := \inf_{\xi \in P} |\hat{\mu}(\xi)| > 0$. Write $\alpha := \min\{\frac{1}{2}\alpha_1, \alpha_0\}$. Given $\xi \in T$ and $I \in \Sigma^*$, define $\widetilde{\xi} = N^{-|I|}(\xi + F(I))$.

If $\widetilde{\xi} \in P$, we take $\lambda_{\xi,I} = 0$. Then,

$$|\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| = |\hat{\mu}(\widetilde{\xi})| \geq \alpha_0 \geq \alpha. \tag{14}$$

If $\widetilde{\xi} \notin P$, by the definition of P, there exists a unique $\theta \in \mathcal{Z}(\hat{\mu}, T) \subset T \backslash \{0\}$ such that $\widetilde{\xi} \in B(\theta, \delta)$. According to (12), there exists $J \in \Sigma^*$ such that

$$\theta + F_I(J) = 0. \tag{15}$$

Take $\lambda_{\xi,I} = F_I(J)$. Then, we have

$$N^{-l}(\widetilde{\xi} + F_I(J|_l)) \in B(N^{-l}(\theta + F_I(J|_l)), N^{-l}\delta), \quad 1 \leq l \leq |J|. \tag{16}$$

On the other hand, by (15), we have

$$N^{-l}(\theta + F_I(J|_l)) \in \mathbb{Z} \cap T, \quad 1 \leq l \leq |J|.$$

In combination with Lemma 4 and (16), this leads to

$$\prod_{l=1}^{|J|} |m_D(N^{-l}(\theta + F_I(J|_l)))| > \alpha_1. \tag{17}$$

Furthermore, by (16) we have

$$N^{-|J|}(\widetilde{\xi} + F_I(J)) \in B(N^{-|J|}(\theta + F_I(J)), N^{-|J|}\delta) \subset B(0, \delta_1).$$

Then, by (13), we have $|\hat{\mu}(N^{-|J|}(\widetilde{\xi} + F_I(J)))| \geq \frac{1}{2}$. Together with (17), this inequality implies

$$\begin{aligned} |\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| =& |\hat{\mu}(\widetilde{\xi} + F_I(J))| \\ =& \prod_{l=1}^{|J|} |m_D(N^{-l}(\theta + F_I(J|_l)))| |\hat{\mu}(N^{-|J|}(\widetilde{\xi} + F_I(J)))| \\ \geqslant& \frac{1}{2}\alpha_1 \\ \geqslant& \alpha. \end{aligned} \tag{18}$$

Case ii: *There exist $\eta^* \in \mathcal{Z}(\hat{\mu}, T)$ and $I \in \Sigma^*$ such that, for any $J \in \Sigma^*$, we have*

$$\eta^* + F_I(J) \neq 0. \tag{19}$$

Recall that $\widetilde{S} = \bigcup_{I \in \Sigma^*} S_I$ and $p = \sharp \mathcal{Z}(\hat{\mu}, T)$. Let

$$U := \bigcup_{l=1}^{p+1} \left\{ N^{-l}(\theta + \lambda) : \lambda \in \widetilde{S} + N\widetilde{S} + \cdots + N^{l-1}\widetilde{S}, \theta \in \mathcal{Z}(\hat{\mu}, T) \cup (T \cap \mathbb{Z}) \right\}.$$

Furthermore, we write

$$V = \{x \in U : \ |m_D(x)| \neq 0\} \text{ and } W = \{x \in V : |\hat{\mu}(x)| \neq 0\}.$$

It is clear $W \subset V \subset U \subset T$. Since $(\frac{1}{N}D, S_I)$ is a compatible pair for any $I \in \Sigma^*$, we obtain $V \neq \varnothing$.

Next, we shall prove $W \neq \varnothing$.

Claim 1: There exists $a \in \widetilde{S}$ such that

$$0 < |m_D(N^{-1}(\eta^* + a))| < 1.$$

Proof. If $T \cap \mathbb{Z} = \varnothing$, then we have $\sup\{|m_D(\eta)| : \eta \in T\} < 1$ by noting that T is compact. A trivial fact that $N^{-1}(\eta^* + a) \in T$ for any $a \in \widetilde{S}$ implies the claim is true.

When $T \cap \mathbb{Z} \neq \varnothing$, suppose the claim is false. Since $(\frac{1}{N}D, S_I)$ is a compatible pair, by Lemma 1(iii) for $\eta^* \in \mathcal{Z}(\hat{\mu}, T)$, there exists $j_1 \in \Sigma_q$ such that $m_D(N^{-1}(\eta^* + F_I(j_1))) = 1$. By (2) and (19), we obtain $N^{-1}(\eta^* + F_I(j_1)) \in (T \cap \mathbb{Z})\backslash\{0\}$. Furthermore, there exists $j_2 \in \Sigma_q$ such that $m_D(N^{-2}(\eta^* + F_I(j_1 j_2))) = 1$, which implies $N^{-2}(\eta^* + F_I(j_1 j_2)) \in (T \cap \mathbb{Z})\backslash\{0\}$. Repeating this process, we obtain a sequence of symbols $\{j_l\}_{l \geqslant 1} \subset \Sigma_q$ such that

$$N^{-l}(\eta^* + F_I(j_1 \cdots j_l)) \in (T \cap \mathbb{Z})\backslash\{0\}, \quad l \geqslant 1.$$

By a similar argument in the proof of Theorem 2(iii) $\Rightarrow$ (i), we obtain $Q_{\Lambda_I}(\eta^*) = 0$, which implies a contradiction to (11). The claim is proven.

Next, we define a sequence of set $\{Y_n\}_{n \geq 0}$ by induction on n. Let $Y_0 := \{\eta^*\}$, and

$$Y_n := \{N^{-1}(\eta + a) : 0 < |m_D(N^{-1}(\eta + a))| < 1, \ \eta \in Y_{n-1}, \ a \in \widetilde{S}\}, \quad n \geqslant 1.$$

By a similar argument in the proof of the claim in Lemma 3, we obtain $\#Y_n \geq 2^n$ for $1 \leq n \leq p+1$. On the other hand, for any $\eta \in Y_{p+1}$, there exists $\lambda \in \widetilde{S} + N\widetilde{S} + \cdots + N^p\widetilde{S}$ such that $\eta = N^{-p-1}(\eta^* + \lambda)$ and $0 < |m_D(\eta)| < 1$, which implies $Y_{p+1} \subset V$. Then, we conclude

$$\#V \geqslant \#Y_{p+1} \geqslant 2^{p+1} > p.$$

Recall that p is the number of zero points of $\hat{\mu}(\xi)$ on compact T. Then, we obtain $W \neq \varnothing$.

Noting that $W \subset V \subset U$ and U is a finite set, it is obvious that both W and V are finite sets. Write

$$\begin{aligned} \alpha_2 &:= \min\{|m_D(\eta))| \neq 0 : \eta \in V\} > 0, \\ \alpha_3 &:= \min\{|\hat{\mu}(\eta)| \neq 0 : \eta \in W\} > 0. \end{aligned}$$

Then, there exists a positive number $\delta_2 > 0$ such that, for any $\eta \in V$ and $\omega \in W$, we have

$$|m_D(x)| > \frac{1}{2}\alpha_2, \qquad \forall\, x \in B(\eta, \delta_2), \tag{20}$$

$$|\hat{\mu}(x)| > \frac{1}{2}\alpha_3, \qquad \forall\, x \in B(\omega, \delta_2). \tag{21}$$

Write $\widetilde{\delta} := \min\left\{\delta_1, \delta_2, \frac{d}{4}\right\}$. We let $\widetilde{P} := T \backslash \left(\bigcup_{\theta \in \mathcal{Z}(\hat{\mu}, T)} B(\theta, \widetilde{\delta}) \right)$ denote the set of points that has a positive distance (at least $\widetilde{\delta}$) from the zero points of $\hat{\mu}(\xi)$ in T. It is clear that $\widetilde{P}$ is a compact set and $\alpha_4 := \inf_{\xi \in \widetilde{P}} |\hat{\mu}(\xi)| > 0$. We write

$$\widetilde{\alpha} := \min\{\alpha_1 \frac{\alpha_3}{2} (\frac{\alpha_2}{2})^{p+1}, \alpha_4\},$$

where α_1 comes from Lemma 4.

Given $\xi \in T$ and $I \in \Sigma_q^*$, write $\widetilde{\xi} := N^{-|I|}(\xi + F(I))$.

If $\widetilde{\xi} \in \widetilde{P}$, we take $\lambda_{\xi,I} = 0 \in \Lambda_I$. Then, we have

$$|\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| = |\hat{\mu}(\widetilde{\xi})| \geqslant \alpha_4 \geqslant \widetilde{\alpha}. \tag{22}$$

If $\widetilde{\xi} \notin \widetilde{P}$, there exists $\theta \in \mathcal{Z}(\hat{\mu}, T)$ such that $\widetilde{\xi} \in B(\theta, \widetilde{\delta})$. If there exists $J \in \Sigma^*$ such that

$$\theta + F_I(J) = 0,$$

we take $\lambda_{\xi,I} = F_I(J)$. Then, by a similar argument as (18), we have

$$|\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| \geq \alpha. \tag{23}$$

If there is no $J \in \Sigma^*$ such that

$$\theta + F_I(J) = 0,$$

by Lemma 3, there exist two integers $0 \leqslant w < \infty, 1 \leqslant v \leqslant p+1$ and a finite word $J := j_1 \cdots j_{w+v} \in \Sigma_q^*$ such that when $w = 0$, we have

$$0 < |m_D(N^{-l}(\theta + F_I(J|_l)))| < 1, \qquad 1 \leqslant l \leqslant v, \tag{24}$$

and $|\hat{\mu}(N^{-v}(\theta + F_I(J)))| > 0$; when $w > 0$, we have

$$m_D(N^{-l}(\theta + F_I(J|_l))) = 1, \qquad 1 \leqslant l \leqslant w, \tag{25}$$

$$0 < |m_D(N^{-l}(\theta + F_I(J|_l)))| < 1, \qquad w+1 \leqslant l \leqslant w+v, \tag{26}$$

and $|\hat{\mu}(N^{-w-v}(\theta + F_I(J)))| > 0$.

Take $\lambda_{\xi,I} := F_I(J)$. In the case $w = 0$, since $\widetilde{\xi} \in B(\theta, \widetilde{\delta})$, it is obvious that

$$N^{-l}(\widetilde{\xi} + F_I(J|_l)) \in B(N^{-l}(\theta + F_I(J|_l)), N^{-l}\widetilde{\delta}), \quad 1 \leqslant l \leqslant v. \tag{27}$$

Noting that $\theta \in \mathcal{Z}(\hat{\mu}, T) \cup (T \cap \mathbb{Z})$, by (24), we obtain

$$N^{-l}(\theta + F_I(J|_l)) \in V, \quad 1 \leqslant l \leqslant v.$$

Together with (20) and (27), the above inequality implies

$$|m_D(N^{-l}(\widetilde{\xi} + F_I(J|_l)))| > \frac{\alpha_2}{2}, \quad 1 \leqslant l \leqslant v. \tag{28}$$

Furthermore, since $N^{-v}(\theta + F_I(J)) \in V$ and $|\hat{\mu}(N^{-v}(\theta + F_I(J)))| > 0$, we have $N^{-v}(\theta + F_I(J)) \in W$ and $N^{-v}(\widetilde{\xi} + F_I(J)) \in B(N^{-v}(\theta + F_I(J)), N^{-v}\widetilde{\delta})$. From (21), it follows that

$$|\hat{\mu}(N^{-v}(\widetilde{\xi} + F_I(J|_l)))| \geqslant \frac{\alpha_3}{2}. \tag{29}$$

In combination with (28), this yields

$$\begin{aligned}
|\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| &= |\hat{\mu}(\widetilde{\xi} + \lambda_{\xi,I})| \\
&= \prod_{i=1}^{\infty} |m_D(N^{-i}(\widetilde{\xi} + F_I(J)))| \\
&= \prod_{i=1}^{v} |m_D(N^{-i}(\widetilde{\xi} + F_I(J))| |\hat{\mu}(N^{-v}(\widetilde{\xi} + F_I(J)))| \\
&\geqslant \frac{\alpha_3}{2} (\frac{\alpha_2}{2})^{p+1} \\
&\geqslant \widetilde{\alpha}.
\end{aligned} \tag{30}$$

In the case $w > 0$, we shall divide the product into three parts

$$\begin{aligned} &|\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| \\ =&\prod_{i=1}^{\infty}|m_D(N^{-i}(\widetilde{\xi} + F_I(J)))| \\ =&\prod_{i=1}^{w}|m_D(N^{-i}(\widetilde{\xi} + F_I(J)))| \prod_{i=w+1}^{w+v}|m_D(N^{-i}(\widetilde{\xi} + F_I(J)))||\hat{\mu}(N^{-w-v}(\widetilde{\xi} + F_I(J)))|. \end{aligned} \tag{31}$$

By (2) and (25), we have

$$N^{-l}(\theta + F_I(J|_l)) \in T \cap \mathbb{Z}, \qquad 1 \leqslant l \leqslant w. \tag{32}$$

Noting $\widetilde{\xi} \in B(\theta, \widetilde{\delta})$, we have

$$N^{-l}(\widetilde{\xi} + F_I(J|_l)) \in B(N^{-l}(\theta + F_I(J|_l)),\ N^{-l}\widetilde{\delta}), \qquad 1 \leqslant l \leqslant w.$$

Thus, by (10), we obtain

$$\prod_{l=1}^{w}|m_D(N^{-l}(\widetilde{\xi} + F_I(J|_l)))| \geqslant \alpha_1. \tag{33}$$

By (32), we have $N^{-w}(\theta + F_I(J|_w)) \in \mathcal{Z}(\hat{\mu}, T) \cup (T \cap \mathbb{Z})$. Then, by (26), we have

$$N^{-l}(\theta + F_I(J|_l)) \in V, \quad w + 1 \leqslant l \leqslant w + v$$

and

$$N^{-l}(\widetilde{\xi} + F_I(J|_l)) \in B(N^{-l}(\theta + F_I(J|_l)), N^{-l}\widetilde{\delta}), \quad w + 1 \leqslant l \leqslant w + v. \tag{34}$$

By (20) and (21), we obtain

$$|m_D(N^{-l}(\widetilde{\xi} + F_I(J|_l)))| > \frac{\alpha_2}{2}, \quad w + 1 \leqslant l \leqslant w + v, \tag{35}$$

and

$$|\hat{\mu}(N^{-w-v}(\widetilde{\xi} + F_I(J)))| \geqslant \frac{\alpha_3}{2}.$$

Together with (31), (33), and (35), the above inequality yields

$$|\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| \geq \alpha_1(\frac{\alpha_2}{2})^{p+1}\frac{\alpha_3}{2} \geq \widetilde{\alpha}. \tag{36}$$

In combination with (14), (18), (22), (23), (30), and (36), by Lemma 2, we obtain (μ, Λ) is a spectral pair, which is a contradiction to our hypothesis. We finish the proof of (i) $\Rightarrow$ (ii) in Theorem 2. $\square$

Finally, we shall prove Theorem 2 (ii) $\Rightarrow$ (iii).

Since T is compact, there exists $\xi^* \in T$ such that $Q_{\Lambda_I}(\xi^*) = 0$. Write

$$X := \{\xi \in T : \hat{\mu}(\xi) = 0 \text{ and } m_D(\xi) \neq 0\}.$$

It is clear that $0 \notin X$. Since $(\frac{1}{N}D, S_I)$ is a compatible pair, by Lemma 1, there exists an integer $j \in \Sigma_q$ with $m_D(\frac{1}{N}(\xi^* + F_I(j))) \neq 0$. Noting that

$$0 = Q_{\Lambda_I}(\xi^*) = \Sigma_{\lambda \in \Lambda_I}|\hat{\mu}(\xi^* + \lambda)|^2 \geq |m_D(\frac{1}{N}(\xi^* + F_I(j)))|^2|\hat{\mu}(\frac{1}{N}(\xi^* + F_I(j)))|^2,$$

we obtain $\hat{\mu}(\frac{1}{N}(\xi^* + F_I(j))) = 0$. By virtue of $\xi^* \in T$, we have $\frac{1}{N}(\xi^* + F_I(j)) \in T$. Hence, X is nonempty.

Next, we define a sequence of the subset of X by induction on n. Define $X_0 := \{\xi^*\}$ and

$$X_{n+1} := \{N^{-n-1}(\xi + F_I(J)) \in X : \ N^{-n}(\xi + F_I(J|_n)) \in X_n, \ J \in \Sigma_q^{n+1}\}, \quad n \geq 0.$$

We have the following conclusion.

Claim 2: $\#X_{n+1} \geqslant \#X_n$, $n \geqslant 0$.

Proof. When $n = 0$, by the definition of $Q_{\Lambda_I}(\xi^*)$, we have

$$0 = Q_{\Lambda_I}(\xi^*) = \sum_{j_1 \in \Sigma_q} |m_D(N^{-1}(\xi^* + F_I(j_1)))|^2 \cdot Q_{\Lambda_{Ij_1}}(N^{-1}(\xi^* + F_I(j_1))).$$

Noting that $(\frac{1}{N}D, S_I)$ is a compatible pair, Lemma 1(iii) implies that there exists at least one symbol $j_1 \in \Sigma_q$ such that $|m_D(N^{-1}(\xi^* + F_I(j_1)))| > 0$, which implies $Q_{\Lambda_{Ij_1}}(N^{-1}(\xi^* + F_I(j_1))) = 0$. Hence, we have $\hat{\mu}(N^{-1}(\xi^* + F_I(j_1))) = 0$. This leads to $\#X_1 \geqslant \#X_0$. Suppose Claim 2 holds for $n = k-1$. Then, X_k is nonempty. For any $y \in X_k$, there exists $\widetilde{J} \in \Sigma_q^k$ such that $y = N^{-k}(\xi^* + F_I(\widetilde{J}))$ and

$$\prod_{i=1}^{k} |m_D(N^{-i}(\xi^* + F_I(\widetilde{J}|_i)))| > 0.$$

By (1) and (4), we have

$$0 = Q_{\Lambda_I}(\xi^*) = \sum_{\widetilde{J} \in \Sigma_q^k} \prod_{i=1}^{k} |m_D(N^{-i}(\xi^* + F_I(\widetilde{J}|_i)))|^2 \cdot Q_{\Lambda_{I\widetilde{J}}}(N^{-k}(\xi^* + F_I(\widetilde{J}))).$$

Then, we obtain $Q_{\Lambda_{IJ}}(N^{-k}(\xi^* + F_I(\widetilde{J}))) = 0$. By a similar argument, we have

$$\begin{aligned} 0 =& Q_{\Lambda_{I\widetilde{J}}}(N^{-k}(\xi^* + F_I(\widetilde{J}))) \\ =& \sum_{j_{k+1} \in \Sigma_q} |m_D(N^{-k-1}(\xi^* + F_I(\widetilde{J}j_{k+1})))|^2 \cdot Q_{\Lambda_{I\widetilde{J}j_{k+1}}}(N^{-k-1}(\xi^* + F_I(\widetilde{J}j_{k+1}))). \end{aligned}$$

Noting that $(\frac{1}{N}D, S_{I\widetilde{J}})$ is a compatible pair, by Lemma 1(iii), there exists at least one symbol $j_{k+1} \in \Sigma_q$ such that

$$|m_D(N^{-k-1}(\xi^* + F_I(\widetilde{J}j_{k+1})))| > 0.$$

Hence, $Q_{\Lambda_{I\widetilde{J}j_{k+1}}}(N^{-k-1}(\xi^* + F_I(\widetilde{J}j_{k+1}))) = 0$, which implies $\hat{\mu}(N^{-k-1}(\xi^* + F_I(\widetilde{J}j_{k+1}))) = 0$. Thus, we obtain

$$N^{-k-1}(\xi^* + F_I(\widetilde{J}j_{k+1})) \in X_{k+1}.$$

If we consider $N^{-n-1}(\xi^* + F_I(\widetilde{J}j_{n+1}))$ as a "next generation" of $N^{-n}(\xi^* + F_I(\widetilde{J}))$ for $n \geq 1$, Proposition 2 implies that different points of X_k have different "next generations". Thus, we obtain $\#X_{k+1} \geqslant \#X_k$, which implies Claim 2 is true.

By noting the fact that X is a subset of the finite set $\mathcal{Z}(\hat{\mu}, T)$, there exists a positive integer $h \in \mathbb{N}$ such that

$$\#X_{h+m} = \#X_h, \quad m \geq 1. \tag{37}$$

From the above argument, it follows that for any $y = N^{-n}(\xi^* + F_I(j_1 \cdots j_n)) \in X_n$, if there exists a symbols $j_{n+1} \in \Sigma_q$ such that $|m_D(N^{-n-1}(\xi^* + F_I(j_1 \cdots j_n j_{n+1})))| > 0$, then y has a "next generation" $N^{-n-1}(\xi^* + F_I(j_1 \cdots j_n j_{n+1})) \in X_{n+1}$. Noting that $(\frac{1}{N}D, S_{Ij_1 \cdots j_n})$ is a compatible pair, by Lemma 1 (iii), we have

$$\Sigma_{j_{n+1} \in \Sigma_q} |m_D(N^{-1}(y + \mathcal{C}(IJj_{n+1})))|^2 = 1.$$

In combination with (37), we conclude that for any $n \geq h$, there exists only one symbol $j_{n+1} \in \Sigma_q$ such that $|m_D(N^{-1}(y + \mathcal{C}(IJj_{n+1}))| \neq 0$. In fact, $|m_D(N^{-1}(y + \mathcal{C}(IJj_{n+1}))| = 1$. Then, we obtain

$$N^{-n-1}(\xi^* + F_I(Jj_{n+1})) = N^{-1}(y + \mathcal{C}(IJj_{n+1}) \in \mathbb{Z}.$$

Continuing the process, we obtain a sequence of symbols $\{j_{h+l}\}_{l\geqslant 1} \subset \Sigma_q$, such that

$$N^{-h-l}(\xi^* + F_I(Jj_{h+1}\cdots j_{h+l})) \in \mathbb{Z}, \quad l \geqslant 1.$$

Define $\beta_1 := N^{-h}(\xi^* + F_I(J))$ and

$$\beta_l := N^{-h-l+1}(\xi^* + F_I(Jj_{h+1}\cdots j_{h+l-1})), \quad l \geqslant 2.$$

It is clear $\beta_l \in X_{h+l-1}$, which implies β_l is nonzero. Thus, the sequence of nonzero integers $\{\beta_l\}_{l\geqslant 1}$ and the increasing sequence of finite words $\{Jj_{h+1}\cdots j_{h+l}\}_{l\geqslant 1}$ with the prefix J fulfill the request. □

As a corollary of Lemma 2 and Theorem 2, we obtain another necessary and sufficient condition for Λ to be a spectrum of μ.

Proposition 5. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $\gcd(D) = 1$. Assume that a countable set Λ satisfies the conditions* (C1), (C2), *and* (C3)*. Then, (μ, Λ) is a spectral pair if and only if there exists a positive number $c > 0$ such that, for any ξ and $I \in \Sigma^*$, there is $\lambda_{\xi,I} \in \Lambda_I$ satisfying*

$$|\widehat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| \geq c.$$

Proof. The sufficiency follows from Lemma 2. We just prove the necessity here. Suppose that (μ, Λ) is a spectral pair. By Propositions 3 and 4, we obtain, for any $I \in \Sigma^*$,

$$Q_{\Lambda_I}(\xi) \equiv 1, \quad \xi \in \mathbb{R}.$$

By a similar argument in the proof of Theorem 2 (i) $\Rightarrow$ (ii), for any $\xi \in T$ and $I \in T$, there exists $\lambda_{\xi,I} \in \Lambda_I$ such that

$$|\widehat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| \geq c.$$

We finish the proof. □

4. An Example

In this section, we construct a self-similar measure and a set $\Lambda(N, \mathcal{B})$ with a tree structure. Neither the criterion of Łaba and Wang (Theorem 1) nor that of Strichartz ([24]) are applicable to this set $\Lambda(N, \mathcal{B})$. However, we show that there does not exist an infinite orbit $\{\beta_l\}_{l\geq 1} \subset \mathbb{Z} \setminus \{0\}$ associated with the dual IFS (see Theorem 3), which implies $\Lambda(N, \mathcal{B})$ is a spectrum by Theorem 2.

Example 1. *Let $N = 6$ and $D = \{0, 1, 2\}$. Write μ for the invariant measure associated with the IFS $\{\phi_1, \phi_2, \phi_3\}$ defined by*

$$\phi_1(x) = \frac{1}{6}x, \quad \phi_2(x) = \frac{1}{6}(x+1), \quad \phi_3(x) = \frac{1}{6}(x+2).$$

Let $B_1 = \{0, 8, 22\}$, $B_2 = \{0, 22, 38\}$, $B_3 = \{0, 8, 52\}$, and $B_4 = \{0, 38, 52\}$. By Lemma 1, a simple induction implies that $(\frac{1}{6}D, B_i)$ is a compatible pair for $1 \leq i \leq 4$. Noting

$$\frac{1}{6}(4+8) = 2,\ \frac{1}{6}(2+22) = 4,\ \frac{1}{6}(4+8) = 2,\ \frac{1}{6}(2+22) = 4,\ \cdots,$$

$$\frac{1}{6}(10+38) = 8,\ \frac{1}{6}(8+52) = 10,\ \frac{1}{6}(10+38) = 8,\ \frac{1}{6}(8+52) = 10,\ \cdots,$$

we see that both $\Lambda(6, B_1)$ and $\Lambda(6, B_4)$ have an infinite iterated nonzero integer sequence, where $\Lambda(N, S) := S + NS + N^2S + \cdots$ finite sum. Thus, by Theorem 1 or by Theorem 2, we conclude that both $\Lambda(6, B_1)$ and $\Lambda(6, B_4)$ are not a spectrum of μ. We consider the following set defined by $\{B_i : 1 \le i \le 4\}$.

$$\begin{aligned}\Lambda(N, \mathcal{B}) := B_1 + &\underbrace{NB_2 + N^2B_3}_{B_2 \text{ and } B_3 \text{ repeat 1 time}} + \\ N^3B_4 + &\underbrace{N^4B_3 + N^5B_2 + N^6B_3 + N^7B_2}_{B_3 \text{ and } B_2 \text{ repeat 2 times}} + \\ N^8B_1 + &\underbrace{N^9B_2 + N^{10}B_3 + N^{11}B_2 + N^{12}B_3 + N^{13}B_2 + N^{14}B_3 + N^{15}B_2 + N^{16}B_3}_{B_2 \text{ and } B_3 \text{ repeat } 2^2 \text{ times}} + \\ N^{17}B_4 + &\underbrace{N^{18}B_3 + N^{19}B_2 + \cdots + N^{32}B_3 + N^{33}B_2}_{B_3 \text{ and } B_2 \text{ repeat } 2^3 \text{ times}} + \cdots \quad \text{(finite sum)}.\end{aligned} \tag{38}$$

According to Remark 1, it is clear that Theorem 1 cannot work. We shall show $\Lambda(N, \mathcal{B})$ is a spectrum of μ by Theorem 2 in the following Theorem 3. Then, we show that Strichartz's criterion (Theorem 2.8 in [24]) is not appropriate by proving the following Theorem 4.

Let A_n denote the set of coefficients of $N^n (n \ge 0)$ in (38). Given two integers l and k with $l > k \ge 0$, we write

$$\Lambda_k^l := A_k + NA_{k+1} + N^2A_{k+2} + \cdots + N^{l-k-1}A_{l-1}. \tag{39}$$

We also write $\Lambda^k := \Lambda_0^k$ for simplicity. For three integers m, n, and k with $0 \le m < n < k$, we have

$$\begin{aligned}&\Lambda_m^n + N^{n-m}\Lambda_n^k \\ =&A_m + NA_{m+1} + \cdots + N^{n-m-1}A_{n-1} + N^{n-m}A_n + \cdots + N^{k-m-1}A_{k-1} \\ =&\Lambda_m^k.\end{aligned} \tag{40}$$

Theorem 3. *Given nonzero integer sequence $\{\beta_i\}_{i \ge 1}$, then, for any integer $M > 0$, there exists an integer $i \geqslant M$ such that*

$$\beta_{i+1} \ne N^{-1}(\beta_i + a_i),$$

for any $a_i \in A_i$.

Proof. Suppose that there exists a positive integer M such that, for any $i > M$, we have $\beta_{i+1} = 6^{-1}(\beta_i + a_i)$. Let T_0 be the self-similar set generated by the dual IFS $\{\frac{1}{6}(x+s) : s \in \bigcup_{j=1}^4 B_j\}$.

According to the definition of the attractor T_0, there exists a positive integer K such that, for any $i \ge K$, β_i belongs to a neighborhood of T_0, i.e.,

$$\beta_i \in (-1, \frac{53}{5}).$$

Recall a fact that $\bigcup_{i=0}^\infty A_i = \{0, 8, 22, 38, 52\}$. Then, $\beta_{K+1} = 6^{-1}(\beta_K + a_K)$ with $a_K \in \bigcup_{j=1}^4 B_j$ implies $\beta_K \in \{2, 4, 6, 8, 10\}$. By noting that $\beta_{K+2} = 6^{-1}(\beta_{K+1} + a_{K+1})$ with $a_{K+1} \in \bigcup_{j=1}^4 B_j$ implies $\beta_K \ne 6$, hence $\beta_K \in \{2, 4, 8, 10\}$. If $\beta_K = 2$, then

$$a_K = 22,\ a_{K+1} = 8,\ a_{K+2} = 22,\ a_{K+3} = 8,\ \cdots.$$

Hence, $\{8,22\} \cap A_i \neq \emptyset$ for all $i \geq K$, which contradicts that $\{8,22\} \cap B_4 = \emptyset$ and $B_4 = A_i$ for infinitely many i. Hence, $\beta_K \in \{4,8,10\}$.

By a similar argument for other cases, i.e., $\beta_K \in \{4,8,10\}$, we always obtain a contradiction. Then, we finish the proof. □

The following result shows that Strichartz's method (Theorem 2.8 in [24]) is not applicable to the above set $\Lambda(N,\mathcal{B})$.

Theorem 4. *We have*

$$\liminf_{n\to\infty} \inf_{\lambda\in\Lambda^n} |m_D(N^{-n}\lambda)| = 0.$$

Proof. Obviously, we need only to prove that there exists a subsequence $\{\lambda_{n_k}\}_{k\geq 1} \subset \Lambda^{n_k}$ such that $N^{-n_k}\lambda_{n_k}$ tends to a zero point of m_D as k tends to infinity. Let T_0 be the attractor of the $IFS\{\Phi_j(x) = \frac{1}{6}(x+j) : j \in \bigcup_{i=1}^4 B_i\}$. Thus, we have $T_0 \subset [0,\ \frac{52}{5}]$.

For $k \geq M$, we write $n_k = 2^{2k+2} + 2k + 1$, and we take

$$\beta_{n_k} = 38 + 52 \times 6 + 38 \times 6^2 + 52 \times 6^3 + \cdots + 38 \times 6^{2^{2k+1}} \in \Lambda^{n_k}_{2^{2k+1}+2k-1},$$

where the coefficients 38 and 52 appear alternately. By a simple deduction, we obtain

$$6^{-2^{2k+1}-2}(10 + \beta_{n_k}) = \frac{4}{3}. \tag{41}$$

Take arbitrarily $\alpha \in \Lambda^{2^{2k+1}+2k-1}$, and write

$$\lambda_{n_k} = \alpha + 6^{2^{2k+1}+2k-1}\beta_{n_k}.$$

By (40), we obtain

$$\lambda_{n_k} \in \Lambda^{n_k}.$$

According to the definition of T_0, we have

$$6^{-2^{2k+1}-2k+1}\alpha \in T_0,$$

which implies $|6^{-2^{2k+1}-2k+1}\alpha - 10| \leq \frac{52}{5}$. In combination with (41), we have

$$\begin{aligned}
&|6^{-n_k}\lambda_{n_k} - \frac{4}{3}| \\
=&|6^{-2^{2k+1}-2}(6^{-2^{2k+1}-2k+1}\alpha + \beta_{n_k}) - 6^{-2^{2k+1}-2}(10 + \beta_{n_k})| \\
=&|6^{-2^{2k+1}-2}(6^{-2^{2k+1}-2k+1}\alpha - 10)| \\
\leqslant& 6^{-2^{2k+1}-2} \times \frac{52}{5}.
\end{aligned}$$

Noting the fact that $m_D(\frac{4}{3}) = 0$, we finish the proof. □

5. Summary and Conclusions

In this paper, we introduced a tree structure with the language of symbolic space. The natural spectrum candidate of a self-similar measure associated with an IFS is a set with a special tree structure. We obtained three equivalent conclusions for Λ to be a spectrum of a self-similar measure. One of them implies that there exists an infinite orbit with an element of a nonzero integer associated with the dual IFS. An example involving a self-similar measure and a spectrum candidate $\Lambda(N,\mathcal{S}) = S_0 + NS_1 + N^2S_2 \cdots$ showed the tree structure expands essentially the field of spectrum candidates.

It is one of the most important problems to find all spectra of a spectral measure. We are not sure that every spectrum of a self-similar measure holds a tree structure. On the other hand, the self-similar $\mu_{N,D}$ measure has another description, $\mu_{N,D} = \delta_{\frac{1}{N}D} * \delta_{\frac{1}{N^2}D} * \cdots$. It is obvious to ask if Theorem 2 holds for the Moran-type self-similar measure. As mentioned in the Introduction, the version of Theorem 1 in higher-dimensional space has not been obtained completely. It is the next research direction to prove Theorem 2 the for self-affine measures.

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