FROM CLASSICAL TO QUANTUM: THE F^*-ALGEBRAIC APPROACH

JONATHAN GLEASON

Abstract. The purpose of this paper is to walk the reader through a mathematical development of physics, motivating everything along the way, sometimes with physical arguments, sometimes with mathematical ones, starting with Newtonian mechanics and ending with a modern axiomatization of quantum mechanics. To achieve this goal, we introduce the notion of an F^*-algebra and prove several results about these objects that enable us to axiomatize quantum mechanics using the language of F^*-algebras.

Contents

1. Introduction

As stated in the abstract, the purpose of this paper is to walk the reader through a mathematical development of the physics. Furthermore, we aim to do this as efficiently as possible, so that no more time is spent on a topic than is necessary to motivate the next. The hope is that it will give a clear, intuitive picture, with strong motivation throughout, of why physicists have chosen the path that they have in "fixing" previous physical theories. Because of the nature of this paper, it is inevitable that some topics will not feel fleshed out. I hope this doesn't put off too many readers, as such topics should be more thought of as mere "stepping stones" in the context of this paper, as opposed to a topic of interest in its own right.

A quick note on notation before we begin. For us, $\mathbb{N}$ will always contain 0. If we wish to omit 0, we shall use the symbol $\mathbb{Z}^+$ for the set of positive integers. Let V and W be normed linear spaces. Then, we shall denote by $\mathcal{L}[V, W]$ the set of

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all closed[1], densely-defined[2] linear operators from V to W. Furthermore, we shall denote by $\mathcal{B}[V,W]$ the set of all bounded linear operators from V to W.

2. Newtonian Mechanics

2.1. Space-time. The first notion in classical mechanics we want to make precise sense out of is *space-time*. In the same way that if one stands on the earth and looks in all directions without traveling a great distance, it might be reasonable to conclude (having never thought of something like a manifold before) that the earth is flat, if you stand in one place and look at space all around you, it would be reasonable to conclude that space if flat. Thus, it would be reasonable to take as an assumption that *space is* $\mathbb{R}^3$.

However, let's say that we perform some experiment, then travel around space, stop, and perform the same experiment again. We will find that our results of both experiments are identitcal, that is, we find that the world "looks the same" regardless of where we are and in what direction we are looking[3]. Thus, we don't want any point in space to be particularly "special". In $\mathbb{R}^3$, the origin plays a special role, and so it would not be natural to define space to just be $\mathbb{R}^3$. To get around this, we define the notion of *affine space*.

Definition 2.1 (Affine Space)**.** *Affine n-Space* is a nonempty set S equipped with a group action of $\mathbb{R}^n$ on S, written $(p,\mathbf{v})\mapsto p+\mathbf{v}=\mathbf{v}+p$, such that for all $p,q\in S$, there exists a unique $\mathbf{v}\in\mathbb{R}^n$ such that $p+\mathbf{v}=\mathbf{q}$.

The intuition here of course is that the elements of S are points in space and that, while it doesn't make sense to add two points together, it does make sense to *subtract* two points, the difference of course being thought of as the displacement between the two points or locations.

The observant reader probably noticed that we did not say "An affine n-space is. . ." in our definition. By *not* wording the definition like this, we mean to imply that affine n-space is somehow unique. This is indeed the case, but to say what we mean by unique, we must first define an affine map (the mathematically mature reader should be able to come up with these definitions and proofs on their own, and if they wish to skip ahead, they should do so after the following remark on notation).

Notation 2.2. Let S be affine n-space and let $p,q\in S$. Then, we shall denote the unique $\mathbf{v}\in\mathbb{R}^n$ such that $p+\mathbf{v}=q$ by $p-q$.

Of course, in mathematics, whenever we define some sort of mathematical object, we are also interested in maps between such objects that preserve the structure of the objects:

Definition 2.3 (Affine Map)**.** Let S and T be affine spaces of dimensions m and n respectively and let $f:S\to T$ be a function. Then, f is an *affine map* from S to T if and only if the map $\widetilde{f}:\mathbb{R}^m\to\mathbb{R}^n$ defined by $\widetilde{f}(p-q)=f(p)-f(q)$ is linear.

The reader should check the following:

[1] We need these operators to be closed so that $A^{**}=A$.

[2] We need these operators to be densely-defined so that the notion of an adjoint makes sense.

[3] These two ideas are usually referred to as *homogeneity* and *isotopy* of space(-time).

Fact 2.4. *Let S and T be affine spaces of dimensions m and n respectively and let $f : S \to T$ be an affine map. Then, for any $\mathbf{v} \in \mathbb{R}^m$, there exists $p, q \in S$ such that $p - q = \mathbf{v}$ and furthermore, $\widetilde{f}(\mathbf{v}) = f(p) - f(q)$ is independent of our choice of p and q.*

This fact essentially says that, when $\widetilde{f}$ exists, it is everywhere-defined and well-defined.

The reader should also check the following fact

Fact 2.5. *Define a category $\mathscr{A}$ whose objects are affine spaces and whose morphisms between two given affine spaces are the affine maps between the two affine spaces. Then, $\mathscr{A}$ is indeed a category.*

It then follows easily that

Fact 2.6. *Let S and T be affine spaces of dimensions m and n respectively. Then, S and T are isomorphic in $\mathscr{A}$ iff $m = n$.*

One can easily check that $\mathscr{A}$ carries a product, whose definition should be obvious to the reader (set-wise, the product is the Cartesian product and the group action is component-wise).

Furthermore, in Newtonian mechanics, we view space and time independently of one other. We can thus *define* space-time as follows:

Definition 2.7 ((Newtonian) Space-Time)**.** *Space* is S, *time* is T, and *space-time* is $S \times T$ where S is 3-dimensional affine space and T is 1-dimensional affine space.

Note that, once again, Fact 2.6 allows us to talk about 3-dimensional affine space instead of *a* 3-dimensional affine space.

Definition 2.8 (Event)**.** An *event* is an element of space-time.

Definition 2.9 (Time Difference)**.** Let p and q be events in space-time. Then, $p - q \in \mathbb{R}^3 \times \mathbb{R}$. The *time difference* between q and p, written $\Delta t(p - q)$, is $\pi_2(p - q)$, where $\pi_2 : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is the projection onto the 2^{nd}-coordinate.

Definition 2.10 (Simultaneity)**.** Let p and q be events in space-time. Then, we say that p and q are *simultaneous* if and only if $\Delta t(p - q) = 0$.

Furthermore, for events that are simultaneous, we would like to be able to talk about the distance they are apart.

Definition 2.11 (Distance)**.** Let p and q be simultaneous events in space-time. Then, the *distance* from q to be p, written $\Delta s(p - q)$, is $\|\pi_1(p - q)\|$, where $\pi_1 : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the projection onto the 1^{st}-coordinate and $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^3$.

The ability to talk about time difference between events and distance between simultaneous events now allows us to talk about the important notion of an inertial frame:

Definition 2.12 (Inertial Frame)**.** An *inertial frame* is a map from $S \times T$ to $\mathbb{R}^3 \times \mathbb{R}$ of the form $h \equiv f \times g$ where $f : S \to \mathbb{R}^3$ and $g : T \to \mathbb{R}$ are affine isomorphisms such that $\Delta t(p - q) = \pi_2 (h(p) - h(q))$ for all $p, q \in S \times T$ and such that $\Delta s(p - q) = \|\pi_1 (h(p) - h(q))\|$ for p and q simultaneous.

All this is saying is that h must be an affine isomorphism that preserves time intervals, and whenever two events are simultaneous, g must preserve the distance between them. In other words, a choice of inertial frame is essentially a choice of coordinate system in space-time, with the requirement that our inertial observer is not an idiot in the sense that he chooses coordinates that give incorrect results for measurements of time differences and spatial displacements.

2.2. **Mass and Force.** We all have this intuitive idea of what mass is: it's the amount of "stuff" that makes up an object. From our everyday experience, we expect that objects with a lot of "stuff" are going to be more resistant to movement than objects with a lot less "stuff". But unfortunately, this does not provide us with a good definition of mass, because defining mass in terms of its resistance to change in movement would give us a circular definition of force. We need to think of something else.

We then come up with an idea to properly define the notion of mass. First of all, we must choose a standard object that we shall arbitrarily declare to have one unit of mass.[4] Then, we pick an inertial reference frame in which this standard is at rest, and we take the object whose mass we wish to measure and send it on a collision course for the standard at a known speed v_0. The faster the standard travels after the impact, intuitively, the more "stuff" our object has. Conversely, the more *our* object slows down after the collision (the difference between the initial velocity and ending velocity), the less "stuff" we expect our object to have. It thus makes sense to *define* the mass of our object to be

$$\frac{v'}{v_0 - v} \text{ units of mass,}$$

where v' is the speed of the standard and v is the speed of our object *after* the collision. Note that, our intuition tells us that v should be less than v_0, and hence this quantity will always be positive. We note that it is an experimental fact that this definition is independent of inertial frame.[5]

Now that we have established the physical meaning of mass, we may define what it is we mean by a particle.

Definition 2.13 (Particle)**.** A *particle* is a pair $(\mathbf{x}, m)$, where $\mathbf{x}$ is a smooth[6] map from T to S and m is a positive real number.

Note here that, for this to make sense, I am using the fact that our definition of mass is independent of inertial frame, the intuition here of course being that $\mathbf{x}$ is the path in space that the particle traces out and m is the mass of the particle. For the purposes of this paper, we can consider classical mechanics to be the study of finitely many particles in space-time. For some purposes, one needs to consider "particles" which are not point particles. For example, the study of rigid body motion in which "particles" can have an orientation in space-time is a common subject in classical mechanics texts.

[4]There is actually a real-life object, known as the *international prototype kilogram*, that has been arbitrarily declared to have a mass of 1 kg. To be fair, this is probably not the best definition in the world we could have come up with, but for now, we're stuck with it.

[5]Remember, we are currently working in the realm of *classical mechanics*, not special relativity or otherwise, so statements I make such as these are made modulo special relativistic and quantum mechaniccal subtleties.

[6]Affine spaces have a natural smooth manifold structure induced by the action of $\mathbb{R}^n$.

Let us now consider n particles $(\mathbf{x}_1, m_1), \ldots, (\mathbf{x}_n, m_n)$ in space-time and choose any inertial frame. Then, we may think of each $\mathbf{x}_k$ as a function from $\mathbb{R}$ (thought of as time) to $\mathbb{R}^3$ (thought of as space). We will not distinguish between this function and the corresponding function from T to S, at least in terms of notation. Remember, for our purposes, the only things living in space-time are these n particles. Thus, if something is to affect the movement of one of these particles, it must be one of the other particles, or at least something produced by the existence of one of the other particles. We thus define:

Definition 2.14 (Force)**.** The *force on particle k* is a smooth function $\mathbf{F}_k : (TS)^n \times T \to TS$.[7]

Intuitively, for each of the n particles, the force $\mathbf{F}_k$ is dependent on both the particles' position and velocity, and hence, is a function of the n-fold product of TS. Furthermore, we allow $\mathbf{F}_k$ to also depend on time, which is the role of the $(n+1)^{\text{st}}$ coordinate of the domain.

The obvious question arises: hows does force affect the movement of particles? To answer this, let us perform a (thought) experiment. Let us attach a spring to a wall and connect an object of known mass m_1 to the end of the spring and pull the spring a fixed difference away from the wall. The mass will accelerate, say with an acceleration of a_1. Now, let us take another object of mass, say, m_2. If we do the exact same thing, the "pull" of the spring should be the same because, after all, it's not as if it knows what is at the other end of the spring, and we will record another acceleration, a_2. *It is an experimental fact that $m_1 a_1 = m_2 a_2$.* This thought experiment only dealt with things up to absolute value[8]; nevertheless, this experiment can be modified to experimentally verify the general statement that is Newton's Second Law:

Axiom 2.15 (Newton's Second Law)**.** Let $(\mathbf{x}_1, m_1), \ldots, (\mathbf{x}_n, m_n)$ be n particles in space-time and pick any inertial frame. Then, for $1 \leq k \leq n$,

$$\mathbf{F}_k\left((\mathbf{x}_1(t), \mathbf{v}_1(t)), \ldots, (\mathbf{x}_n(t), \mathbf{v}_n(t)), t\right) = m_k\left(\mathbf{x}_k(t), \mathbf{a}_k(t)\right)$$

holds for all $t \in \mathbb{R}$, where $\mathbf{v}_k = \dot{\mathbf{x}}_k$ and $\mathbf{a}_k = \dot{\mathbf{v}}_k$.

A quick note on the notation here. Each pair $(\mathbf{x}_i(t), \mathbf{v}_i(t))$ and $(\mathbf{x}_k(t), \mathbf{a}_k(t)) \in TS$; however, after picking an inertial frame, as already mentioned, we can think of each $\mathbf{x}_i$, $\mathbf{v}_i$, and $\mathbf{a}_i$ as a smooth function from $\mathbb{R}$ into $\mathbb{R}^3$, and with this idea in mind, Newton's Second Law may be written as

$$\mathbf{F}_k\left(\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t), \mathbf{v}_1(t), \ldots, \mathbf{v}_n(t), t\right) = m_k \mathbf{a}_k(t),$$

which is perhaps more familiar and a little less tedious.

Partly to simplify notation and partly so we don't have to continually worrying about the fact that we are dealing with n particles we define

$$\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}^{3n},$$

$$\mathbf{F} = (\mathbf{F}_1, \ldots, \mathbf{F}_n) \in \mathbb{R}^{3n},$$

[7]Here, TS is the tangent bundle of S.

[8]That is to say, our thought experiment proved nothing about the direction of the acceleration of the force, although Newton's Second Law does make an assertion about the relation of these two directions (they're the same).

and

$$m = \begin{bmatrix} m_1 I_3 & 0 & \cdots & 0 \\ 0 & m_2 I_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n I_3 \end{bmatrix},$$

where I_3 is the 3×3 identity matrix. We also similarly define $\mathbf{v} = \dot{\mathbf{x}}$ and $\mathbf{a} = \dot{\mathbf{v}}$. Then, Newton's Second Law may be written as

$$\mathbf{F} = m\mathbf{a},$$

which is even more familiar and a lot cleaner. Using this notation, we shall call $\mathbf{F}$ a force acting on n particles with mass m.

2.3. **States and Observables.** The study of classical mechanics is not the study of how to determine what the forces on particles are (in principle), but rather, to determine, for a *given* force, what happens to our classical system. So, if we ignore the problem of how to determine $\mathbf{F}$, all of classical mechanics essentially reduces to how to solve the above second-order ordinary differential equation (ODE) for each $\mathbf{x}_k$. However, from the theory of ODEs, if for some $t_0 \in \mathbb{R}$, we know $\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0), \mathbf{v}_1(t_0), \ldots, \mathbf{v}_n(t_0)$, then we can determine $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ *for all* t.[9] This information thus essentially encodes, at least in principle, everything we would ever want to know about the system, which hence motivates our definition of a *state*:

Definition 2.16 ((Newtonian) System)**.** A *Newtonian system* of n particles is a collection of n particles along with the forces acting on each.

Definition 2.17 ((Newtonian) State)**.** The *Newtonian state* of a given n-particle Newtonian system is a collection of n-elements of TS along with one element of T.

The intuition here being that the elements of TS encode the initial position and velocities and the element of T encodes the time at which the n particles had these positions and velocities. As just mentioned, if we know the state of the system, picking an inertial frame reduces the problem of solving for the particles' trajectories for all time to solving a second-order ODE, which is now an initial-value problem by virtue of knowing the state of the system, to which a unique solution exists for all time by the theory of ODEs.

Now, it is an experimental fact that we can never measure something with *infinite* precision; however, there are such things that we can, in principle, measure to an arbitrarily precise degree. We call such things *observables.*

We would now like to come up with a mathematically precise way to characterize these observables. A first natural requirement is that observables depend on the state of the system, that is, observables better be functions of $(TS)^n \times T$, for a system of n particles. Secondly, we better require that these functions be real-valued. Thirdly, we must require that there is some way to make the error, when we measure an observable in the laboratory, arbitrarily small. Let us assume (by virtue of experimental fact, in the classical realm of course), that we can always

[9]The mathematical theorem only guarantees existence in an interval containing t_0, but we shall simply declare cases where we do not have existence for all time as unphysical. In any case, these problems do not matter for the purpose of this paper.

measure position, velocity, and time arbitrarily precisely.[10] Now, say somebody gives me a function on $(TS)^n \times T$, call it f, and would like me to measure f with error less than some $\varepsilon > 0$. Now, I know that I can make the error in the positions, velocities, and time arbitrarily small, so I think, if there is some maximum error in the positions, velocities, and time, call it δ, so that when I plug in my measured values for the positions, velocities, and time, my experimental value of f will be within ε of the true value of f, then f will be observable. But of course, this is just the definition of a continuous function! Thus, the natural definition for an observable in classical mechanics can be stated as follows:

Definition 2.18 (Newtonian Observables)**.** The *Newtonian observables* of a system of n particles are exactly the continuous functions from $(TS)^n \times T$ to $\mathbb{R}$.

To simplify notation, we shall simply write $M \equiv (TS)^n \times T$. Furthermore, we shall simply denote the set of all observables on M as $\mathcal{O} = C(M, \mathbb{R})$.

Theorem 2.19 (Properties of Classical Observables)**.** *The set of observables $\mathcal{O}$ of a* classical system *is exactly the set of self-adjoint elements of a separable, commutative, unital F^*-algebra $\mathcal{A}$.*[11]

Proof. Take $\mathcal{A} = C(M, \mathbb{C})$. Equipping $\mathcal{A}$ with the usual addition, scalar multiplication, multiplication, and complex conjugation as an involution turns $\mathcal{A}$ into a *-algebra. Now, let $\{K_n |\, n \in \mathbb{N}\}$ be a sequence of compact sets such that $K_n \subseteq K_{n+1}$ and $\bigcup_{n\in\mathbb{N}} K_n = M$. Then, we may define the seminorm p_n by $p_n(f) = \|f|_{K_n}\|$, that is, p_n is the usual supremum norm of f restricted to the compact set K_n. It is easy to check that this structure turns $\mathcal{A}$ into an F^*-algebra of which $\mathcal{O}$ is exactly the set of self-adjoint elements. □

This is big time. Mathematically, it's pretty trivial; however, this theorem will serve as our guide for axiomatizing quantum mechanics. Eventually, we will take the above theorem (with a slight modification) as an *axiom* of the observables in quantum mechanics. A quick remark on notation before we continue: we will be using the notation $\mathcal{A} = C(M, \mathbb{C})$ equipped with the structure noted in the above proof that turns $\mathcal{A}$ into an F^*-algebra.

Now we wish to do something similar with the states of a classical system. That is to say, we would like to examine the mathematical description of states as given above and arrive at a result that we can hopefully take as an axiom for our theory of quantum mechanics. There is a natural way of viewing states in classical mechanics as linear functionals on $\mathcal{O}$.

Definition 2.20. Let $x \in M$ be a state. Then, we define $\widehat{x} : \mathcal{A} \to \mathbb{C}$ such that, for $f \in \mathcal{A}$,

$$\widehat{x}(f) = f(x).$$

With this definition, it is easy to see that each state x induces a positive, multiplicative, unital linear functional $\widehat{x}$ on $\mathcal{A}$, but not just any positive, multiplicative, unital linear functional. Linear functionals of the form $\widehat{x}$ for $x \in M$ have the special property that, for any inverse sequence $\{\mathcal{A}_n |\, n \in \mathbb{N}\}$ of C^*-algebras converging to $\mathcal{A}$ (in the sense of an inverse limit), we have that $\widehat{x} = \widehat{\pi_n}(\psi)$ where π_n is the map

[10] Of course, we mean to imply that the measurements of position and velocity are simultaneous, which, in classical mechanics, is perfectly acceptable.

[11] See the Appendix for the definition of an F^*-algebra.

from $\mathcal{A}$ into $\mathcal{A}_n$ that makes $\mathcal{A}$ the inverse limit of the sequence $\{\mathcal{A}_n | n \in \mathbb{N}\}$ and $\widehat{-}$ is the contravariant character functor. We will say that a linear functional of this form is a *restricted* linear functional.[12]

Theorem 2.21 (Properties of Classical States)**.** *The map that sends $x \in M$ to $\widehat{x}$ in $\mathcal{A}^*$ is a homeomorphism onto the set of positive, multiplicative, unital, restricted linear functionals of $\mathcal{A}$.*

Proof. Step 1: Note that every linear functional of the form $\widehat{x}$ is positive, multiplicative, and unital.
Let $x \in M$. We have already mentioned that it is easy to show that $\widehat{x}$ is a positive, multiplicative, unital linear functional on $\widehat{A}$.

Step 2: Show that every linear functional of the form $\widehat{x}$ is positive, multiplicative, unital, and restricted.
To show that $\widehat{x}$ is restricted, let $\{\mathcal{A}_n | n \in \mathbb{N}\}$ be an inverse sequence of C^*-algebras such that $\mathcal{A}$ is the inverse limit of this sequence with maps $\pi_n : \mathcal{A} \to \mathcal{A}_n$. Because the image of each π_n is dense and $\mathcal{A}$ is commutative, it follows that each $\mathcal{A}_n$ must be commutative. Similarly, because $\mathcal{A}$ is unital, it follows that each $\mathcal{A}_n$ must be unital. Then, by the Commutative Gelfand-Naimark Theorem, it follows that $\mathcal{A}_n$ is isomorphic to $C(K_n)$ for some compact Hausdorff space K_n. Furthermore, because $\mathcal{A}$ is separable, it follows that each $C(K_n)$ is separable. The map from $C(K_{n+1})$ into $C(K_n)$ induces a map from the set of all positive, multiplicative, unital linear functionals on $C(K_n)$ into the set of all positive, multiplicative, unital linear functionals on $C(K_{n+1})$, which is injective because the map from $C(K_{n+1})$ into $C(K_n)$ has dense image. Thus, by Lemma 5.21, this continuous injection gives us a continuous injection from K_n into K_{n+1}. Similarly, we also get continuous injections from each K_n into M. Note that each of these maps must be a homeomorphisms onto its image and so we identify each K_n as a subspace of M. It follows from the universal property that $M = \bigcup_{n\in\mathbb{N}} K_n$. Thus, $x \in K_n$ for some $n \in \mathbb{N}$, in which case $\widehat{x} = \widehat{\pi_n}(\widehat{x})$, where on the right hand side we regard $\widehat{x}$ as a linear functional on $C(K_n)$. Thus, $\widehat{x}$ is restricited.

Step 3: Show that each positive, multiplicative, unital, restrictive linear functional is of the form $\widehat{x}$ for some $x \in M$.
Let ψ be a positive, multiplicative, unital, restrictive linear functional on $\mathcal{A}$. By Theorem 5.19, $\mathcal{A}$ is the inverse limit of some sequence of commutative, unital, separable C^*-algebras. By the argument given in the previous step, this sequence is of the form $\{C(K_n) | n \in \mathbb{N}\}$ for $\{K_n | n \in \mathbb{N}\}$ a sequence of increasing compact sets such that $\bigcup_{n\in\mathbb{N}} K_n = M$. Then, because ψ is restricited, $\psi = \widehat{\pi_n}(\psi_n)$ for some positive, multiplicative, unital linear functional ψ_n on $C(K_n)$. However, by the previous lemma, $\psi_n = \widehat{x}$ for some $x \in K_n$, and hence $\psi = \widehat{x}$ for some $x \in M$.

Step 4: Show that this map is injective.
M is normal, so continuous functions separate distinct points on M, so that the map that sends x to $\widehat{x}$ is injective.

Step 5: Show that this map is continuous.
Let $\{x_n | n \in \mathbb{N}\}$ be a sequence converging to $x \in M$. Then, for $f \in \mathcal{A}$, the sequence $\{f(x_n) | n \in \mathbb{N}\}$ converges to $f(x)$, so by the definition of the weak-* topology, the sequence $\{\widehat{x_n} | n \in \mathbb{N}\}$ converges to $\widehat{x}$, and hence the map $x \mapsto \widehat{x}$ is continuous.

[12]It turns out that every multiplicative, unital linear functional on $\mathcal{A}$ is restricted; however, for quantum mechanical reasons, we will eventually want to remove the assumption of multiplicative, in which case there will be such linear functionals that are not restricted.

Step 6: Show that the inverse map is continuous.
Let I be a directed set and let $\{\widehat{x_i} | \, i \in I\}$ be a net of linear functionals converging to another linear functional $\widehat{x}$.[13] It follows that for every $f \in \mathcal{A}$, $\{f(x_i) | \, i \in I\}$ converges to $f(x)$. However, for this to occur, it must be the case that $\{x_i | \, i \in I\}$ converges to x, and hence the inverse map is continuous. Thus, the map that sends $x \in M$ to $\widehat{x}$ in $\mathcal{A}^*$ is a homeomorphism onto the set of positive, multiplicative, unital, restrictive linear functionals of $\mathcal{A}$. □

At this point, we have fully characterized both the observables and states in classical mechanics. The characterization of the observables is given in Theorem 2.19 and the characterization of the states is given in Theorem 2.21. The idea now is to examine what's wrong with Theorems 2.19 and 2.21, and to figure out in what way we can modify them so that they are consistent with the way in which our world actually works (up to quantum mechanical considerations).

3. From Classical to Quantum

Before we attempt at "fixing" our notions of states and observables for classical mechanics, we first want to gain a more enlightening view of states when viewed as linear functionals. A nice theorem, due to Riesz and Markov[14], actually characterizes these states very nicely:

Theorem 3.1 (Riesz-Markov Theorem). *Let X be a locally compact, Hausdorff space, and let ψ be a positive, unital linear functional on $C(X, \mathbb{R})$. Then, there exists a unique Borel probability measure μ_ψ on X such that, for all $f \in C(X, \mathbb{R})$,*

$$\psi(f) = \int_X f d\mu_\psi.$$

When viewed in the light of the Riesz-Markov Theorem, it makes sense to view $\psi(f)$ as *the expected value of the observable f in the state ψ*. Physically, if we measure f many times in the laboratory, and our particle is in the state ψ, the our results *should* average to the value $\psi(f)$. With this intuition in mind, it makes sense to define the variance of an observable with respect to a state:

Definition 3.2 (Variance). Let $x \in M$ be a state and let $f \in \mathcal{O}$. Then, the *variance of f with respect to x* is defined as

$$\sigma_x(f)^2 \equiv \widehat{x}\left[(f - \widehat{x}(f))^2\right].$$

The reader may check (it's trivial) that, $\sigma_x(f) = 0$ for all $f \in \mathcal{O}$. Keeping our experimental knowledge of quantum mechanics in mind, we know that this is *not the case for all states.* A simple counterexample is a particle an infinite square well. We would like to develop our theory so that the ground "state" of this particle is to be considered a state in the mathematical theory. Unfortunately, with the ability of hindsight, we know that the variance of the position in this state is *nonzero*, and so if it is to be included in our notion of a state, we must modify the classical definition of a state to include it.

[13] Note that we have already shown that the image consists entirely of linear functionals of this form. Furthermore, it is easy to check that the limit of a net of positive, multiplicative, unital linear functionals must also be positive, multiplicative, and unital, and hence must also be of this form.

[14] See [11], pg. 130.

To include such states, we must now throw away our notion that our states are points living in a smooth manifold. However, it is natural, and still mathematically possible, to take the set of all states of a *quantum* system to be the set of all positive, multiplicative, unital, restricted linear functionals on the algebra of observables (just as in the classical case). We still have to come back to this and make this formal, however, because we have not yet defined the notion of an observable for a quantum system.

Now for the observables. It is *an experimental fact* that

$$\sigma_\psi(p)\sigma_\psi(x) \geq \frac{\hbar}{2} \tag{3.3}$$

for any state ψ, where $p = mv$ is the momentum of the particle and m is the mass of the particle.[15] We would like such a result to be derivable from our mathematical theory, and the following derivation suggests that we should take our algebra of observables to be *noncommutative*, in contrast to the classical case.

For the following argument, we shall assume all the properties of the observables stated in Theorem 2.19 except for commutativity. Let $A, B \in \mathcal{O}$ and fix some state ψ. Without loss of generality, we may assume that $\psi(A) = \psi(B) = 0$ (because we could just as well take the observables $A - \psi(A)$ and $B - \psi(B)$). Thus,

$$\sigma_\psi(A)^2\sigma_\psi(B)^2 = \psi(A^2)\psi(B^2).$$

Now, $(\alpha A + i\beta B)^* = \alpha A - i\beta B$ for $\alpha, \beta \in \mathbb{R}$ (here, we have used the fact that A and B are self-adjoint), so, by positivity of states, we have that

$$\begin{aligned}\psi\left((\alpha A - i\beta B)(\alpha A + i\beta B)\right) &= \psi(\alpha^2A^2 + i\alpha\beta AB - i\alpha\beta BA + \beta^2B^2) \\ &= \psi(A)^2\alpha^2 + \psi\left(i[A, B]\right)\alpha\beta + \psi(B^2)\beta^2 \geq 0,\end{aligned}$$

where $[A, B] \equiv AB - BA$ is the *commutator* of A and B. Defining

$$M \equiv \begin{bmatrix} \psi(A^2) & \frac{1}{2}\psi\left(i[A, B]\right) \\ \frac{1}{2}\psi\left(i[A, B]\right) & \psi(B^2) \end{bmatrix} \text{ and } v \equiv \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

we see that the above inequality becomes

$$v^TMv \geq 0.$$

Thus, M is positive-definite, and hence

$$\det[M] = \psi(A^2)\psi(B^2) = \frac{1}{4}\psi\left(i[A, B]\right)^2 \geq 0,$$

and hence

$$\sigma_\psi(A)\sigma_\psi(B) \geq \frac{1}{2}\left|\psi\left([A, B]\right)\right|.$$

We immediately see that the Equation (3.3) is derivable from the above equation if $[x, p] = \alpha\hbar$ where $\alpha \in \mathbb{C}$ has norm 1. Of course, we must also have that (because all observables must be self-adjoint)

$$[x, p]^* = (xp - px)^* = px - xp = -[x, p],$$

so that $\alpha^* = -\alpha$, so that $\alpha = \pm i$. In the end, it makes no difference whether we take $\alpha = i$ or $\alpha - i$, so we might as well take $\alpha = i$. Thus, we see that if our theory takes the observables to be a *noncommutative algebra*, in particular, with the relation $[x, p] = \mathbf{i}\hbar$, then Equation (3.3) will be derivable in this theory. This suggests modifying Theorem 2.19 only slightly, removing the requirement that the

[15] We refer the reader to any standard textbook on quantum mechanics, e.g., [13].

algebra be commutative, and taking this as a *definition* of observables in quantum mechanics. We now make this formal:

Axiom 3.4 (Quantum Observables)**.** The set of *observables* $\mathcal{O}$ *of a quantum system* is exactly the set of all self-adjoint elements of a separable, (in general, noncommutative) unital F^*-algebra $\mathcal{A}$.

The reader should compare Axiom 3.4 with Theorem 2.19. Note how little we are changing between the classical and the quantum.

Now that we have precisely defined what the observables are in quantum mechanics, the definition we would have liked to take for states before as the set of all positive, multiplicative, unital, restricted linear functionals on the algebra of observables makes sense. However, this is not quite the definition we want, as, with the variance defined as above, if our states are taken to be multiplicative, the variance will *always* be 0, which is exactly the problem we had in the classical case. Evidently, in the quantum world, our states are not in general going to be multiplicative, which yields the following definition, which itself is just a slight modification of Theorem 2.21):

Axiom 3.5 (Quantum States)**.** The set of *states* $\mathcal{S}$ *of a quantum system* is exactly the set of all positive, unital, restricted linear functionals ψ on $\mathcal{A}$.

So what have we accomplished so far? We first took a very natural, intuitive formulation of classical mechanics, Newtonian mechanics, and defined two important entities: the states, which are objects that tell us everything we would ever want to know about the system, and observables, a very intuitive motivation for which is given shortly after the definition of a classical state. We really wish to stress how natural all this was. Look around you: we see that space is essentially 3-dimensional Euclidean space with no preferred origin. Time is a naturally flowing, 1-dimensional entity that also has no preferred moment at which we should call $t = 0$. Objects in our universe have a natural notion of something we call mass, which is intuitively how much "stuff" the object has, and can be defined precisely in terms of collisions. Furthermore, we have Newton's Second Law, which if we ignore for the moment how forces actually arise, essentially says the more you "push" an object the more its speed will change. All of this gives rise to natural definitions of states and observables, from which we can *prove* Theorems 2.19 and 2.21. We then took this mathematical characterization of states and observables contained in Theorems 2.19 and 2.21, and tried to figure out why these characterizations are incompatible with what we know about quantum mechanics. We eventually determined that to make these characterizations of states and observables compatible with quantum mechanics (with the benefit of hindsight of course), we should modify them slightly in the manner presented in Axioms 3.4 and 3.5.

Unfortunately, however, most students of quantum mechanics will not find either of these definitions familiar at all. But, if we invoke the magic of Theorem 5.20, we see that *the set of observables in quantum mechanics is exactly the set of self-adjoint elements of a separable, unital F^*-algebra which is a subset of the set of all closed, densely-defined linear operators on a separable, complex Hilbert space, and furthermore that to each quantum state (a positive, unital, restricted linear* *functional on $\mathcal{A}$) corresponds a unique positive operator of trace* 1, which is of course the usual formulation of the definition of observables in quantum mechanics.[16]

We would like to point out the similarity and difference between this F^*-algebraic formulation of quantum mechanics and the C^*-algebraic formulation of quantum mechanics. The ideas behind both formulations are in fact exactly the same: characterize observables and states in classical mechanics, determine why these formulations don't hold true in the quantum world, and modify them slightly so that they do hold true in the quantum world. The difference is that in the C^*-algebraic formulation one must *assume* that all the observables are bounded, which is emphatically not the case (for example, the kinetic energy $E = 1/2mv^2$ is not going to be bounded in general). Second of all, when invoking the Gelfand-Naimark Theorem, one obtains that the observables are exactly the self-adjoint operators of $\mathcal{B}[H]$ for some Hilbert space H, which is, once again, emphatically not the case (for example, the usual position and momentum operators are not going to be bounded). In fact, we prove in the next section that there is *no* C^*-algebra that has elements x and p that satisfy the canonical commutation relations. This motivates the introduction of F^*-algebras to account for the unboundedness we would like to include, and in fact, proceeding the same way as with the usual C^*-algebraic formulation of quantum mechanics replacing along the way C^*-algebras with F^*-algebras, one obtains the formulation of quantum mechanics we have just derived.

In George W. Mackey's *Mathematical Foundations of Quantum Mechanics*, immediately after stating what he believes to be an unmotivated axiom that invokes in some way the usual Hilbert space formulation of quantum mechanics, the author states the following:

> Ideally one would like to have a list of physically plausible assumptions from which one could deduce [this axiom]. Short of this one would like a list from which one could deduce a set of possibilities for the structure of [the pure states], all but one of which could be shown to be inconsistent with suitably planned experiments. At the moment such lists are not available and we are far from being forced to accept [this axiom] as logically inevitable.

The problem he is referring to is a problem that has bothered me ever since I first learned quantum mechanics, and while I do not think the argument we have just given is as strong an argument as Mackey is referring to, in my opinion, it is damn well close, and indeed, close enough that this author will finally be able to sleep at night.

4. The Canonical Commutation Relations

Before we conclude this paper, we first want to show that this abstract formulation of quantum mechanics can actually encode the canonical commutation relations, a crucial feature that was missing from the C^*-algebraic formulation. You recall that we argued in the previous section that if we are to have (3.3) derivable in our theory, then the canonical commutation relations better be satisfied,

[16] Perhaps the reader thinks that the states should correspond to normalized vectors in the Hilbert space. This definition is too narrow and in fact only encompasses the *pure* states. However, the projection operator corresponding to the one dimensional subspace spanned by such a vector is a positive operator of trace 1, and in general, we need to include more than just these projection operators to take into account more general *mixed* states.

that is, we better have that $[x, p] = \mathbf{i}\hbar$. The following theorem shows that this can be done in our abstract formulation given above.

Theorem 4.1. *There exists a separable, unital F^*-algebra with two self-adjoint elements X, P that satisfy $[X, P] = i\hbar$.*

Proof. STEP 1: CONSTRUCT THE UNDERLYING ASSOCIATIVE ALGEBRA.
Let $D \subseteq L^2(\mathbb{R})$ be the collection of functions f such that the function obtained from f by any finite series of applications of the operations that are either multiplication by x or taking the derivative is a function in $L^2(\mathbb{R})$. This is clearly a subspace of $L^2(\mathbb{R})$, however, we wish to furthermore show that D is dense in $L^2(\mathbb{R})$. Let H_n be the n^{th} Hermite polynomial and define

$$\phi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} H_n(x) e^{-x^2/2}.$$

It is well-known[17] that this forms an orthonormal basis for $L^2(\mathbb{R})$. It is trivial to check that each of these is an element of D, and hence D is dense in $L^2(\mathbb{R})$. Let $\mathcal{A}$ be the associative algebra generated by the operators X and P defined by $[X(f)](x) = xf(x)$ and $P(f) = -i\hbar\partial f$ both with domains D.

STEP 2: SHOW THAT $[X, P] = i\hbar$.
Let $f \in D$. Then,

$$\begin{aligned}[][X, P](f)](x) &= -i\hbar\left[x\partial_x f(x) - \partial_x\left(xf(x)\right)\right] = i\hbar\left[f(x) + x\partial_x f(x) - x\partial_x f(x)\right] \\ &= i\hbar f(x).\end{aligned}$$

Thus, $[X, P] = i\hbar$.

STEP 3: TURN $\mathcal{A}$ INTO A *-ALGEBRA.
Note that, D was defined so that every operator in $\mathcal{B}$ may have the same domain D. Equip this associative algebra with a *formal* involution defined such that X and P are both self-adjoint (with respect to this involution, not as operators). Thus, for example, $(3iXP - 5P)^* = -3iPX - 5P$. This turns $\mathcal{A}$ into a *-algebra.

STEP 4: TURN $\mathcal{A}$ INTO A LOCALLY CONVEX TOPOLOGICAL VECTOR SPACE.
Now, let D_n be the subspace[18] spanned by the functions $\phi_0, \ldots, \phi_n$ and define $p_n(A) = \|A|_{D_n}\|$. Each p_n is clearly a seminorm.

STEP 5: CONSTRUCT THE F^*-ALGEBRA.
We show that the collection $\{p_n | \, n \in \mathbb{N}\}$ is separating. Suppose that $p_n(A) = 0$ for all $n \in \mathbb{N}$. Then, in particular, $A(\phi_n) = 0$ for all $n \in \mathbb{N}$, and hence it must be the case that $A = 0$. Thus, by Proposition 5.7, $\mathcal{A}$ is metrizable, so let $\overline{\mathcal{A}}$ be its completion. By density, we can extend addition, scalar multiplication, multiplication, and involution to turn $\overline{\mathcal{A}}$ into a *-algebra. Via the completion process, the extension of the collection $\{p_n | \, n \in \mathbb{N}\}$ to all of $\overline{\mathcal{A}}$ turns $\mathcal{A}$ into a Fréchet space. However, it is easily checked that each seminorm p_n satisfies $p_n(AB) \leq p_n(A)p_n(B)$, $p_n(A^*A) = p_n(A)^2$. and $p_n(A) \leq p_{n+1}(A)$ for $A \in \mathcal{A}$, and hence the corresponding extension to $\overline{\mathcal{A}}$ must satisfy the same properties. Thus, these extensions turn $\overline{\mathcal{A}}$ into an F^*-algebra. Furthermore, the elements $X, P \in \overline{\mathcal{A}}$ are both self-adjoint and satisfy $[X, P] = i\hbar$. Finally, it is easy to check that finite linear combinations of elements of the form $|\phi_m\rangle\langle\phi_n|$[19] with rational coefficients are dense in $\mathcal{A}$, so that $\mathcal{A}$ is separable. $\square$

[17] See [8], pg. 360.

[18] Note that each of these will be a subspace of D.

[19] That is, the unique linear operator on D that sends ϕ_n to ϕ_m.

Furthermore, we now show that this emphatically *cannot* be done with a mere C^*-algebra.

Theorem 4.2. *There exists no C^*-algebra with elements X, P that satisfy $[X, P] = i\hbar$.*

Proof. We proceed by contradiction: let $\mathcal{A}$ be some C^*-algebra with two elements X, P that satisfy $[X, P] = i\hbar$. Given this, it is easy to prove inductively that $XP^n = ni\hbar P^{n-1} + P^n X$. We then see that, for $a \in \mathbb{R}$,

$$\begin{aligned} Xe^{iaP} &= \sum_{n\in\mathbb{N}} XP^n \frac{(ia)^n}{n!} = (i\hbar)(ia) \sum_{n\in\mathbb{Z}^+} \frac{(iaP)^{n-1}}{(n-1)!} + \sum_{n\in\mathbb{N}} \frac{(iaP)^n}{n!} X \\ &= -ae^{iaP} + e^{iaP} X. \end{aligned}$$

Thus, $\left[X, e^{iaP}\right] = -ae^{iaP}$. It follows that

$$|a| = \left\| \left[X, e^{iaP}\right] \right\| \leq 2 \left\| X \right\|.$$

But this can't possibly hold for a arbitrary: a contradiction. Thus, there exists no such C^*-algebra. □

5. Appendix: Locally Convex Spaces, Fréchet Spaces, and F^*-algebras

We review a couple of the main facts relevant to the theory of Fréchet spaces, a type of very nice topological vector space that generalizes the notion of a Banach space, and from there we proceed to define and classify all F^*-algebras, a fundamental result, that, to the author's best knowledge, is an original one, that is crucial to the transition from classical mechanics to quantum mechanics. In general, the term *topological vector space* means exactly what you think it would mean, that is, it is a vector space over a topological field such that the addition and scalar multiplication maps are continuous. While in general, it makes sense to talk about such objects, for our purposes, we only care about the case when the field we are working over is a subfield of the complex numbers.

We first introduce some terminology:

Definition 5.1 (Absorbing Set)**.** Let V be a topological vector space over a subfield of the complex numbers and let $A \subseteq V$. Then, we say that A is *absorbing* if and only if $\bigcup_{a>0} aA = V$.

Definition 5.2 (Balanced Set)**.** Let V be a topological vector space over a subfield of the complex numbers and let $B \subseteq V$. Then, we say that B is *balanced* if and only if $\alpha B \subseteq B$ for all $\alpha \in F$ with $|\alpha| \leq 1$.

Definition 5.3 (Bounded Set)**.** Let V be a topological vector space over a subfield F of the complex numbers and let $B \subseteq V$. Then, we say that B is *bounded* iff for every open neighborhood U of 0, there is some $M \geq 0$ such that $B \subseteq aU$ whenever $a \geq M$.

Definition 5.4 (Separating)**.** Let V be a topological vector space and let $\mathcal{P}$ be a collection of seminorms on V. Then, we say that $\mathcal{P}$ is *separating* if and only if whenever $p(v) = 0$ for all $p \in \mathcal{P}$ it follows that $v = 0$.

Definition 5.5 (Locally Convex Topological Vector Space)**.** Let V be a topological vector space over a subfield of the complex numbers. Then, we say that V is *locally convex* if and only if there exists a base for V's topology consisting entirely of convex sets.

Definition 5.6 (Fréchet Space)**.** Let V be a locally convex topological vector space. Then, we say that V is a *Fréchet space* if and only if V is completely metrizable with a translation invariant metric.

We now present an important theorem that tells us a convenient way to construct locally convex topological vector spaces.

Proposition 5.7. *Let V be a vector space over a subfield of the complex numbers, let $\mathcal{P}$ be a collection of seminorms on V, and for each $p \in \mathcal{P}$, $\varepsilon > 0$, and $v_0 \in V$ define $U_{p,\varepsilon v_0} = \{v \in V \,|\, p(v - v_0) < \varepsilon\}$. Then, V equipped with the topology generated by the collection*

$$\{U_{p,\varepsilon,v_0} \,|\, p \in \mathcal{P}, \varepsilon > 0, v_0 \in V\}$$

is a locally convex topological vector space with the following properties:

1. *The collection of all finite intersections of elements of the form U_{p,ε,v_0} for $p \in \mathcal{P}$, $\varepsilon > 0$ rational, and $v_0 \in V$ forms a base for the topology.*
2. *For a fixed v_0, the collection of all finite intersections of elements of the form U_{p,ε,v_0} for $p \in \mathcal{P}$ and $\varepsilon > 0$ rational forms a local base at v_0.*
3. *Each $p \in \mathcal{P}$ is continuous. In fact, this topology is the initial topology of the collection $\mathcal{P}$.*
4. *A set $B \subseteq V$ is bounded iff every $p \in \mathcal{P}$ is bounded on E.*
5. *The topology is Tychonoff if and only if $\mathcal{P}$ is separating.*
6. *If $\mathcal{P} = \{p_n \,|\, n \in \mathbb{N}\}$ is countable and separating, then the topology is metrizable with translation invariant metric d satisfying*
   (a) *$d(v, w) = \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(v-w)}{1+p_n(v-w)}$.*
   (b) *The open balls centered at 0 are balanced.*
   (c) *d makes V into a Fréchet space if and only if whenever $\{v_n \,|\, n \in \mathbb{N}\}$ is a sequence that is Cauchy with respect to each p_n, it follows that $\{v_n \,|\, n \in \mathbb{N}\}$ converges to some $v \in V$ with respect to each p_n.*

Proof. Step 1: Introduce notation.
Write $\mathcal{S} = \{U_{p,\varepsilon,v_0} \,|\, p \in \mathcal{P}, \varepsilon > 0, v_0 \in V\}$, let τ be the topology generated by $\mathcal{S}$, let $\mathcal{B}$ be the collection of all finite intersections of elements of $\mathcal{S}$, and for a fixed $v_0 \in V$, let $\mathcal{B}_{v_0}$ be the collection of all finite intersections of elements of the form U_{p,ε,v_0} for $p \in \mathcal{P}$ and $\varepsilon > 0$ rational.

Step 2: Verify property (1).
As the elements in the collection mentioned in property (1) are trivially open, clearly cover V, and are closed under intersection they form a base for the topology.

Step 3: Prove that $\mathcal{B}_{v_0}$ does indeed form a local base at v_0.
Let U be an arbitrary open neighborhood of v_0. Then, we can find $p_1, \ldots, p_n \in \mathcal{P}$, $\varepsilon_1, \ldots, \varepsilon_n > 0$, and $v_1, \ldots, v_n \in V$ such that

$$v_0 \in U_{p_1,\varepsilon_1,v_1} \cap \cdots \cap U_{p_n,\varepsilon_n,v_n} \subseteq U.$$

Let ε_k' be any positive rational less than $\varepsilon_k - p_k(v_k - v_0)$. Note that this is always possible as each $\varepsilon_k - p_k(v_k - v_0)$ must be positive. Furthermore, it follows from the triangle inequality that $U_{p_k,\varepsilon'_k,v_0} \subseteq U_{p_k,\varepsilon_k,v_k}$, and hence

$$v_0 \in U_{p_1,\varepsilon'_1,v_0} \cap \cdots \cap U_{p_n,\varepsilon'_n,v_0} \subseteq U_{p_1,\varepsilon_1,v_1} \cap \cdots \cap U_{p_n,\varepsilon_n,v_n} \subseteq U,$$

and hence $\mathcal{B}_{v_0}$ forms a local base at v_0.

STEP 4: PROVE THAT ADDITION IS CONTINUOUS.
Let $v_0, w_0 \in V$ and let U be an open neighborhood of $v_0 + w_0$ in V. Then, because $\mathcal{B}_{v_0+w_0}$ is a local base at $v_0 + w_0$, we can find $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$U_{p_1,\varepsilon_1,v_0+w_0} \cap \cdots \cap U_{p_n,\varepsilon_n,v_0+w_0} \subseteq U.$$

Define $V' = U_{p_1,\varepsilon_1/2,v_0} \cap \cdots \cap U_{p_n,\varepsilon_n/2,v_0}$ and $W' = U_{p_1,\varepsilon_1/2,w_0} \cap \cdots \cap U_{p_n,\varepsilon_n/2,w_0}$. It then follows by the triangle inequality that

$$V' + W' \subseteq U_{p_1,\varepsilon_1,v_0} \cap \cdots \cap U_{p_n,\varepsilon_n,v_0} \subseteq U,$$

and hence the addition map is continuous.

STEP 5: PROVE THAT SCALAR MULTIPLICATION IS CONTINUOUS.
Let $v_0 \in V$, let $\alpha_0 \in F$, where V is over the field F, and let $\alpha_0 v_0 + U$ be an open neighborhood of $\alpha_0 v_0$ in V. Then, because $\mathcal{B}_0$ is a local base at 0, we can find $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$U_{p_1,\varepsilon_1,0} \cap \cdots \cap U_{p_n,\varepsilon_n,0} \subseteq U.$$

Define $U' = U_{p_1,\varepsilon_1/2,0} \cap \cdots \cap U_{p_n,\varepsilon_n/2,0}$. Then, there is some $a > 0$ such that $v_0 \in aU'$. Define $b = \frac{a}{1+|\alpha|a}$. Then, if $v \in v_0 + bU'$ and $|\alpha - \alpha_0| < 1/a$, it follows that

$$\alpha v - \alpha_0 v_0 = \alpha(v - v_0) + (\alpha - \alpha_0)v_0$$

is an element of

$$|\alpha|bU' + |\alpha - \alpha_0|aU' \subseteq U' + U' \subseteq U,$$

and hence

$$\alpha v \in \alpha_0 v_0 + U.$$

It follows that scalar multiplication is continuous, and hence that V is a topological vector space.

STEP 6: SHOW THAT V IS A LOCALLY CONVEX TOPOLOGICAL VECTOR SPACE.
Let $p_1, \ldots, p_n \in \mathcal{P}$, let $\varepsilon_1, \ldots, \varepsilon_n > 0$, and let $v_1, \ldots, v_n \in V$. Then, we wish to show that

$$U \equiv U_{p_1,\varepsilon_1,v_1} \cap \cdots \cap U_{p_n,\varepsilon_n,v_n}$$

is convex, so let $v \in U$ and let $t \in (0,1)$. Then,

$$\begin{aligned} p_k\left(((1-t)v + tv) - v_k\right) &= p_k\left((1-t)(v - v_k) + t(v - v_k)\right) \\ &\leq (1-t)p_k(v - v_k) + tp_k(v - v_k) < (1-t)\varepsilon_k + t\varepsilon_k = \varepsilon_k, \end{aligned}$$

and hence U is convex. Thus, V is a locally convex topological vector space.

STEP 7: VERIFY (3).
Let $0 \leq a < b$ and let $p \in \mathcal{P}$. Then, $p^{-1}((a,b)) = \{v \in V \,|\, a < p(v) < b\} = U_{p,b,0} \backslash U_{p,a,0}$, which is open, and hence p is continuous.
Conversely, if τ' is another topology on V that makes each $p \in \mathcal{P}$ continuous, then, by considering the preimage of $[0, \varepsilon)$ under the map that sends v to $p(v - v_0)$ for a fixed $v_0 \in V$, we see that each U_{p,ε,v_0} must be open, so that this topology is in fact the initial topology of the collection $\mathcal{P}$.

STEP 8: VERIFY (4).
Let $B \subseteq V$.

($\Rightarrow$) Suppose that B is bounded. Let $p \in \mathcal{P}$. Then, $U_{p,1,0}$ is a neighborhood of 0, so there is some $M \geq 0$ such that $B \subseteq aU_{p,1,0}$ for all $a \geq M$. Thus, for $v \in B$, $p(v) < a$ for all $a \geq M$, and hence $p(v) \leq M$. Thus, p is bounded on B.
($\Leftarrow$) Suppose that every $p \in \mathcal{P}$ is bounded on B. Let U be a neighborhood of the origin. Then, there are $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $U' \equiv U_{p_1,\varepsilon_1,0} \cap \cdots \cap U_{p_n,\varepsilon_n,0} \subseteq U$. Let $M \geq \max\{1/\varepsilon_1, \ldots, 1/\varepsilon_n\}$ such that that $p_1|_B, \ldots, p_n|_B \leq M$. Then, whenever $a \geq M$, it follows that

$$B \subseteq aU' \subseteq aU.$$

Thus, B is bounded.

Step 9: Verify (5).
($\Rightarrow$) Suppose that the topology is Tychonoff. Let $v \in V$ and suppose that $p(v) = 0$ for all $p \in \mathcal{P}$. Then, this v would be contained in every neighborhood of the origin. However, as the topology is Tychonoff, then in particular, it is Hausdorff, so that this implies that $v = 0$.
($\Leftarrow$) Suppose that $\mathcal{P}$ is separating. Let $v, w \in V$ be distinct. Suppose that every neighborhood of v contains w. Then in particular, $p(v - w) < \varepsilon$ for every $\varepsilon > 0$, and hence $p(v - w) = 0$, and hence $v = w$: a contradiction. Thus, the topology on V is T_0. However, $(V, +)$ is a topological group, and hence from the theory of topological groups, we know that the topology is Tychonoff.

Step 10: Verify (6).
Suppose that $\mathcal{P} = \{p_n | \, n \in \mathbb{N}\}$ is countable and separating. Define

$$d(v, w) = \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(v - w)}{1 + p_n(v - w)}.$$

Clearly, d is nonnegative and $d(v, v) = 0$. Furthermore, because $\mathcal{P}$ is separating, whenever $d(v, w) = 0$, it follows that $v = w$. Thus, d is positive-definite. d is trivially symmetric, translation-invariant, and satisfies the triangle inequality because each p_n does. Thus, d is a translation invariant metric on V.
Since each p_n is continuous and the series defining d converges absolutely, it follows that d is continuous from $V \times V$ into $\mathbb{R}$. It follows that the set of all open balls centered at the origin are open in the original topology, and hence the metric topology is coarser than the original topology. On the other hand, let U be an open set in the original topology and let $v_0 \in U$. Then, we can find $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $U' \equiv U_{p_1,\varepsilon_1,v_0} \cap \cdots \cap U_{p_n,\varepsilon_n,v_0} \subseteq U$. On the other hand, if

$$d(v, v_0) = \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(v - v_0)}{1 + p_n(v - v_0)} < \varepsilon,$$

then it must be the case that $p_k(v - v_0) < \frac{2^n \varepsilon}{1 - 2^n \varepsilon}$ for each $k \in \mathbb{N}$. Thus, we can choose ε sufficiently small so that whenever $d(v, v_0) < \varepsilon$, it follows that $p_k(v - v_0) < \varepsilon_k$ for $1 \leq k \leq n$. For such an ε, $v_0 \in B(\varepsilon, v_0) \subseteq U' \subseteq U$, so that U is open in the metric topology. Thus, the metric topology is the same as the original topology.
It is trivial to check properties (b) and (c). □

This previous result gives us a way of constructing a locally convex topology given a collection of seminorms on a vector space. Conversely, we may ask whether the topology on any locally convex topological vector space comes from such a family of seminorms, and indeed the answer is yes; however, before we begin the proof, we first need a series of small lemmas:

Lemma 5.8. *Let V be a topological vector space over a subfield of the complex numbers and let B be a balanced subset of V that contains the origin. Then, B^o is balanced.*

Proof. For $0 < |a| \leq 1$, the map that sends v to av is a homeomorphism, and hence

$$aB^o = (aB)^o \subseteq aB \subseteq B$$

because B is balanced. However, aB^o is open, and hence $aB^o \subseteq B^o$. Trivially, however, this inclusion also holds in the case $a = 0$ because $0 \in B$, so that B^o is balanced. $\square$

Lemma 5.9. *Let V be a topological vector space and let U be a convex neighborhood of 0. Then, there is a convex, balanced neighborhood U' of 0 such that $U' \subseteq U$.*

Proof. By continuity of scalar multiplication, there is some neighborhood V of 0 and some $\delta > 0$ such that $aV \subseteq U$ whenever $|a| < \delta$. Define $W = \bigcup_{|a|<\delta} aV$. Then, W is a neighborhood of the origin contained in U which is easily checked to be balanced.

Now, (forget the original definition of V and) redefine $V = \bigcap_{|a|=1} aU$. For $a \in \mathbb{C}$ with $|a| = 1$, because W is balanced, we have that $a^{-1}W = W \subseteq U$, so that $W \subseteq aU$, so that $W \subseteq V$. Thus, V^o is a neighborhood of 0. In fact, being the intersection of convex sets, V is convex, and hence so if V^o.

Now, let $0 \leq r \leq 1$ and let $|b| = 1$. Then,

$$rbV = \bigcap_{|a|=1} rbaU = \bigcap_{|a|=1} raU.$$

However, aU is a convex set containing the origin, and hence $raU \subseteq aU$. Thus, $rbV \subseteq V$. However, any complex number of norm less than or equal to 1 can be represented of the form rb, so that V is balanced. But then, by Lemma 5.8, V^o is balanced. Thus, V^o is a convex, balanced neighborhood of the origin contained in U. $\square$

Lemma 5.10. *Let V be a topological vector space over a subfield of the complex numbers and let $A \subseteq V$ be a neighborhood of the origin. Then, A is absorbing.*

Proof. Let F be the field that V is over, let $v \in V$, and define $f_v : (a) = av$. f is continuous, and so $f^{-1}(A)$ is an open neighborhood of 0 (because $0 \in A$). For all $M \geq 0$ sufficiently large, we have that $1/Mv \in A$, and hence $v \in MA$ for M sufficiently large. Thus, A is absorbing. $\square$

Lemma 5.11. *Let V be a topological vector space over a subfield of the complex numbers, let $A \subseteq V$, and define $\mu_A : V \to [0, \infty)$ by*

$$\mu_A(v) = \inf \left\{ a > 0 | \, a^{-1}v \in A \right\}.$$

Then, if A is convex, balanced, and absorbing, μ_A is a seminorm.

Proof. Note that, because A is absorbing, it must be the case that $0 \in A$. Trivially, μ_A is nonnegative. Furthermore, because $0 \in A$, it follows that $\mu_A(0) = 0$.

For each $v \in V$, define

$$M_A(v) = \left\{ a > 0 | \, a^{-1}v \in A \right\}.$$

Because A is convex and contains 0, it follows that $a \in M_A(v)$ and $b > a$, then $b \in M_A(v)$. That is, $M_A(v)$ is either of the form $(\mu_A(v), \infty)$ or $[\mu_A(v), \infty)$.

Let $b > 0$ nonzero and let $a \in M_A(bv)$. Then, $\frac{b}{a}v = \frac{1}{a/b}v \in A$. Thus, $a/b \in M_A(v)$, so that $a \in bM_A(v)$. Thus, $M_A(bv) \subseteq bM_A(v)$. The other inclusion is similar, so that we have $M_A(bv) = bM_A(v)$. For $b < 0$, write $b = -c$ for $c > 0$ and use the fact that $-A \subseteq A$ because A is balanced. Thus, $M_A(bv) = |b|M_A(v)$ for all $b \in \mathbb{R}$. Thus, $\mu_A(bv) = |b|\mu_A(v)$.
Now, let a and b be arbitrary positive real numbers such that $\mu_A(v) < a$ and $\mu_A(w) < b$, and define $c = a + b$. Then, $a^{-1}v \in A$ and $b^{-1}w \in A$. Because A is convex, we have that

$$c^{-1}(v + w) = \frac{a}{c}a^{-1}v + \frac{b}{c}b^{-1}w \in A.$$

Thus, $\mu_A(v+w) \leq c = a+b$. As a and b were arbitrary, we have that $\mu_A(v+w) \leq \mu_A(v) + \mu_A(w)$. It follows that μ_A is a seminorm. □

Proposition 5.12. *Let V be a locally convex topological vector space over a subfield of the complex numbers. Then, there exists a collection $\mathcal{P}$ of seminorms on V that induce the original topology on V.*

Proof. Because V is locally convex, we can find a local base at 0 consisting of convex sets. By Lemma 5.9, we can furthermore find a local base at 0 consisting of balanced, convex sets. Write this collection as $\mathcal{A}$. By Lemma 5.10, each element of $\mathcal{A}$ is also absorbing, and hence by Lemma 5.11, μ_A is a seminorm on V for each $A \in \mathcal{A}$, and so $\mathcal{P} \equiv \{\mu_A | \, A \in \mathcal{A}\}$ induces a locally convex topology on V, as given by Proposition 5.7. We wish to show that this topology is the original topology on V.
We first show that each μ_A is continuous, so let $\varepsilon > 0$. Then, whenever $v - w \in \varepsilon A$, it follows that

$$|\mu_A(v) - \mu_A(w)| \leq \mu_A(v - w) < \varepsilon,$$

so that μ_A is continuous. Thus, the original topology is finer than the seminorm topology (because the seminorm topology is the coarsest topology which makes each μ_A continuous, via Proposition 5.7.(3). Conversely, it suffices to show that each $A \in \mathcal{A}$ is open in the seminorm topology, however, this is trivial as $\mu_A^{-1}([0, 1)) = A$ because A is open and balanced. Thus, the seminorm topology is the same as the original topology. □

In conclusion, a collection of seminorms defines a locally convex topology on a topological vector space, and conversely, the topology on a locally convex topological vector space arises from the topology induced by a collection of seminorms. Furthermore, this topology is T_0, which in the context of topological groups, is equivalent to being Tychonoff, if and only if the collection of seminorms is separating. Furthermore, this topology is metrizable if and only if the collection of seminorms is separating *and* countable, and is complete with respect to any metric inducing this topology if and only if either hypothesis of Proposition 5.7.(6).(c) is satisfied.

We are now in a position to define a Fréchet algebra:

Definition 5.13 (Fréchet Algebra)**.** Let $\mathcal{A}$ be a Fréchet space with topology induced by the seminorms $\{p_n | \, n \in \mathbb{N}\}$[20] that is also an associative algebra. Then, we

[20]Note here that we are using both the results of Proposition 5.12 and Proposition 5.7. We need Proposition 5.12 to deduce that the topology on $\mathcal{A}$ arises from a collection of seminorms and we need Proposition 5.7 to deduce that this collection is countable.

say that $\mathcal{A}$ is a *Fréchet algebra* if and only if $p_n(AB) \leq p_n(A)p_n(B)$ for all $n \in \mathbb{N}$ and $A, B \in \mathcal{A}$ and $p_m \leq p_n$ for $m \leq n$.

Definition 5.14 (Fréchet *-algebra)**.** Let $\mathcal{A}$ be a Fréchet algebra with a map $^* : V \to V$. Then, we say that $\mathcal{A}$ is a *Fréchet *-algebra* if and only if $\mathcal{A}$ is over $\mathbb{C}$ and for all $A, B \in \mathcal{A}$ and $a \in \mathbb{C}$

(1) $(A + B)^* = A^* + B^*$.
(2) $(aA)^* = a^*A^*$.
(3) $(AB)^* = B^*A^*$.
(4) $(A^*)^* = A$.

Definition 5.15 (F^*-algebra)**.** Let $\mathcal{A}$ be a Fréchet *-algebra with topology induced by the seminorms $\{p_n | \, n \in \mathbb{N}\}$. Then, we say that $\mathcal{A}$ is an F^*-algebra if and only if $p_n(A^*A) = p_n(A)^2$ for all $A \in \mathcal{A}$.

The reader should take note that, in the same way that a Fréchet space is a generalization of a Banach space, an $F-^*$-algebra is a generalization of a $C-^*$-algebra.

Before we proceed, we prove a small lemma that will be useful to us later.

Lemma 5.16. *Let $\mathcal{A}$ be an F^*-algebra with collection of seminorms $\{p_n | \, n \in \mathbb{N}\}$. Then, $p_n(A^*) = p_n(A)$ for all $A \in \mathcal{A}$ and $n \in \mathbb{N}$.*

Proof. We have that

$$p_n(A)^2 = p_n(A^*A) \leq p_n(A^*)p_n(A),$$

and hence $p_n(A) \leq p_n(A)^*$. Replacing A with A^* yields the reverse inequality, and hence $p_n(A) = p_n(A^*)$. □

There is a deep theorem[21] essentially classifying all C^*-algebras:

Theorem 5.17 (Gelfand-Naimark Theorem)**.** *Let $\mathcal{A}$ be a separable, unital C^*-algebra. Then, there exists a separable complex Hilbert space H such that:*

(1) There exists an isometric *-isomorphism π from $\mathcal{A}$ into a closed subalgebra of $\mathcal{B}[H]$.
(2) ψ is a positive, unital linear functional on $\mathcal{A}$ if and only if there exists a positive trace-class operator of trace 1 Ψ such that $\psi(A) = \mathrm{tr}\,[\Psi\pi(A)]$.

For a complete proof of this result, see [6, Theorem 6.4]. However, in quantum mechanics, bounded operators are not enough, and we would like to extend such a result to all operators on a Hilbert space.

There is a little subtlety, here, however. First of all, a general unbounded operator on a Hilbert space need not even have an adjoint. To remedy this problem, however, we only consider *densely-defined* operators. Every densely-defiend operator has an adjoint; however, the adjoint need not be densely defined. Furthermore, iven the usual definition of the adjoint for unbounded operators, $D(A^{**})$ will not in general be equal to $D(A)$. To remedy this problem, we only consider *closed* operators, as, for densely-defined operators, being closed is equivalent to $A = A^{**}$. Thus, $\mathcal{L}[H]$ will refer to the set of closed, densely-defined linear operators.

[21] This is actually a slight modification of said theorem that gives a one-to-one correspondence between positive, unital linear functionals and positive trace-class operators of trace 1.

There is a problem with this however. For $A, B \in \mathcal{L}[H]$ closed, $A+B$ (defined on $D(A) \cap D(B)$) need not even be closable.[22] Evidently, there is no nice way of turning $\mathcal{L}[H]$ into a vector space, much less an F^*-algebra. Nevertheless, certain nice subsets of $\mathcal{L}[H]$ evidently do form an F^*-algebra, and furthermore, we show (see Theorem 5.20) that essentially every F^*-algebra is of this form. In these cases, the algebraic structure is what you would expect and the Fréchet space structure is given by choosing a countable orthonormal basis $\{e_n | \, n \in \mathbb{N}\}$ for the Hilbert space, defining $H_n = \operatorname{span}\{e_0, \ldots, e_n\}$, and defining $p_n(A) = \left\| A|_{D(A) \cap H_n} \right\}$.

Returning our thought back to classical mechanics for the moment, we remember that the observables in Newtonian mechanics were exactly the self-adjoint elements of a separable, unital F^*-algebra that was a inverse limit of a sequence of separable, unital C^*-algebras,[23] and indeed, we wish to show that, in a sense that can be made precise, *every* F^*-algebra that is the inverse limit of a sequence of C^*-algebras is a closed subalgebra of some $\mathcal{L}[H]$. But first, however, we must prove that the inverse limit of a sequence of C^*-algebras is in fact an F^*-algebra:

Theorem 5.18. *Let $\{\mathcal{A}_m | \, m \in \mathbb{N}\}$ be a sequence of separable, unital C^*-algebras with morphisms[24] that have dense image $f_{m,n} : \mathcal{A}_n \to \mathcal{A}_m$ for $m \leq n$ such that*

(1) $f_{m,m}$ is the identity on $\mathcal{A}_m$.

(2) $f_{i,j} \circ f_{j,k} = f_{i,k}$ for all $i \leq j \leq k$.

Then, there exists a separable, unital F^*-algebra $\mathcal{A}$, that is unique up to isomorphism, with morphisms that have dense image $\pi_m : \mathcal{A} \to \mathcal{A}_m$ such that $\pi_m = f_{m,n} \circ \pi_n$ for $m \leq n$. Furthermore, $\mathcal{A}$ is universal in the sense that, for any other separable, unital F^*-algebra $\mathcal{B}$ with morphisms that have dense image $\rho_m : \mathcal{B} \to \mathcal{A}_m$ such that $\rho_m = f_{m,n} \circ \rho_n$ for $m \leq n$, there exists a unique morphism $f : \mathcal{B} \to \mathcal{A}$ such that the following diagram commutes

$$\begin{tikzcd}
 & \mathcal{B} \arrow[d, dashed, "f"] \arrow[ddl, dotted, "\rho_m"'] \arrow[ddr, dotted, "\rho_n"] & \\
 & \mathcal{A} \arrow[dl, "\pi_m"] \arrow[dr, "\pi_n"'] & \\
\mathcal{A}_m \arrow[rr, "f_{m,n}"] & & \mathcal{A}_n
\end{tikzcd}$$

for $m \leq n$.

Proof. Step 1: Define $\mathcal{A}$.
Define

$$\mathcal{A} = \left\{ A \in \prod_{n \in \mathbb{N}} \mathcal{A}_n | \, f_{m,n}(A_n) = A_m \right\}.$$

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[22]The following counterexample is due to Robert Israel given to me on `math.stackexchange.com`. On ℓ^2, define A and B such that $[A(x)]_n = -[B(x)]_n = n^2$ for $n > 1$ and $[A(x)]_1 = \sum_{n \in \mathbb{Z}^+} n x_n$ and $[B(x)]_1 = 0$ with $D(A) = D(B) = \{x \in \ell^2 | \sum_{n \in \mathbb{Z}^+} n^4 |x_n|^2 < \infty\}$.

[23]The C^*-algebras we are referring to are $C(K_n)$, where $\{K_n | \, n \in \mathbb{N}\}$ was the sequence of increasing compact subsets whose union was all of M, the space of states.

[24]Morphisms in the category of F^*-algebras are continuous *-homomorphisms, that is, continuous linear maps that preserve multiplication and involution, and send the identity to the identity (for unital algebras).

Addition, multiplication, and involution are performed on $\mathcal{A}$ component-wise and define seminorms on $\mathcal{A}$ by $p_n(v) = \|A_n\|$[25].

STEP 2: SHOW THAT $\mathcal{A}$ IS A LOCALLY CONVEX TOPOLOGICAL VECTOR SPACE. By virtue of Proposition 5.7, we just need to verify that each p_n is a seminorm, which follows trivially from the fact that $\|\cdot\|_n$ is a norm on $\mathcal{A}_n$.

STEP 3: SHOW THAT $\mathcal{A}$ IS A FRÉCHET SPACE.
Suppose that $p_n(A) = 0$ for all $n \in \mathbb{N}$. Then, each $A_n = 0$, and hence $A = 0$. Thus, the collection $\{p_n | \, n \in \mathbb{N}\}$ is separating on $\mathcal{A}$. Now, suppose that $\{A^n | \, n \in \mathbb{N}\}$ is a sequence in $\mathcal{A}$ that is Cauchy with respect to each p_m. It follows that the sequence $\{A^n_m | \, n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{A}_m$, and hence we can define $A_m = \lim_{n\to\infty} A^n_m$. By continuity of $f_{i,j}$, we have that

$$f_{i,j}(A_j) = \lim_{n\to\infty} f_{i,j}(A^n_j) = \lim_{n\to\infty} A^n_i = A_i.$$

Thus, the element defined by $A = (A_0, A_1, \ldots, A_n, \ldots) \in \mathcal{A}$, and furthermore, the sequence $\{A^n | \, n \in \mathbb{N}\}$ converges to A with respect to each p_m, and hence $\mathcal{A}$ is a Fréchet space, once again, by virtue of Proposition 5.7.

STEP 4: SHOW THAT $\mathcal{A}$ IS A SEPARABLE, UNITAL F^*-ALGEBRA.
It is easy to verify that $\mathcal{A}$ is a *-algebra. Furthermore, $p_n(AB) \le p_n(A)p_n(B)$ and $p_n(A^*A) = p_n(A)^2$ because the respective properties hold for each $\|\cdot\|_n$. Furthermore, it is clear that the element $(1, 1, \ldots)$ is a multiplicative identity, and hence $\mathcal{A}$ is a unital F^*-algebra. $\prod_{n\in\mathbb{N}} \mathcal{A}_n$ is a product of separable spaces, and hence is separable. Countable products of metrizable spaces are metrizable, so this product is metrizable. Countable products of separable spaces are separable, so this product is also separable, and hence second-countable, as separability is equivalent to second-countability for metrizable spaces. Thus, the subspace $\mathcal{A}$ is second-countable, and hence separable.

STEP 5: DEFINE π_m AND PROVE THEY SATISFY THE DESIRED PROPERTY.
Define $\pi_m : \mathcal{A} \to \mathcal{A}_m$ such that $\pi_m(A) = A_m$. Then, for $m \le n$,

$$f_{m,n}\left(\pi_n(A)\right) = f_{m,n}(A_n) = A_m,$$

where we have applied the defining property of $\mathcal{A}$ in the last equality. Furthermore, it is easy to see that each π_m has dense image because each $f_{n+1,n}$ does.

STEP 6: SHOW THAT $\mathcal{A}$ IS UNIVERSAL WITH RESPECT TO THESE PROPERTIES.
Let $\mathcal{B}$ be another F^*-algebra with maps $\rho_m : \mathcal{B} \to \mathcal{A}_m$ satisfying $\rho_m = f_{m,n} \circ \rho_n$ for $m \le n$. Define $f : \mathcal{B} \to \mathcal{A}$ by $f(B) = (\rho_0(B), \rho_1(B), \ldots, \rho_n(B), \ldots)$. This is clearly a morphism in the category of F^*-algebras that, by construction, makes the desired diagram commute. Furthermore, by composing π_m with f, we see that any such map f must satisfy $\pi_m\left(f(B)\right) = \rho_m(B)$, so that the morphism f is unique.

STEP 7: SHOW THAT $\mathcal{A}$ IS UNIQUE UP TO ISOMORPHISM.
Let $\mathcal{A}'$ be any other such F^*-algebra. Then, by taking $\mathcal{B} = \mathcal{A}'$, there is a unique morphism $f : \mathcal{A}' \to \mathcal{A}$ that makes a certain diagram commute. Then, switching the roles of $\mathcal{A}$ and $\mathcal{A}'$, we see that there must be a unique morphism $f' : \mathcal{A} \to \mathcal{A}'$ that makes another certain diagram commute. It follows that the map $f \circ f'$ is the unique map from $\mathcal{A}$ to itself that, if drawn into the diagram as a loop from $\mathcal{A}$ to itself, would yield a commuting diagram. But the identity is such a map, and

[25] When there is no ambiguity, we shall not distinguish between the norms on each $\mathcal{A}_n$, however, when there is ambiguity, we shall write $\|\cdot\|_n$ for the norm on $\mathcal{A}_n$.

hence $f \circ f'$ is the identity. Similarly, $f' \circ f$ is the identity, and so $\mathcal{A}$ is isomorphic to $\mathcal{A}'$. $\square$

We have just proved that the inverse limit of a sequence[26] of separable, unital C^*-algebras is an F^*-algebra. Conversely, we may ask the question "Is every separable, unital F^*-algebra the inverse limit of a sequence of separable, unital C^*-algebras?", and indeed, the answer is affirmative:

Theorem 5.19. *Let $\mathcal{A}$ be a separable, unital F^*-algebra. Then, there exists a sequence $\{\mathcal{A}_m | \, m \in \mathbb{N}\}$ of separable, unital C^*-algebras such that $\mathcal{A}$ is the inverse limit of this sequence.*

Proof. STEP 1: CONSTRUCT THE SPACES BY WHICH WE WISH TO QUOTIENT.
Let $\{p_n | \, n \in \mathbb{N}\}$ be the collection of seminorms on $\mathcal{A}$ that make it into an F^*-algebra and define $\mathcal{V}_n = p_n^{-1}(0)$. We wish to show that each $\mathcal{V}_n$ is a two-sided ideal that is closed under involution.

STEP 2: SHOW THAT THESE SPACES DO INDEED FORM TWO-SIDED IDEALS THAT ARE CLOSED UNDER INVOLUTION.
It is trivial to check using the fact that p_n is a seminorm that $\mathcal{V}_n$ is a vector space over the field F, where F is the field that $\mathcal{A}$ is over. Furthermore, if $A \in \mathcal{V}_n$ and $B \in \mathcal{A}$, then

$$0 \le p_n(AB) \le p_n(A)p_n(B) = 0p_n(B) = 0,$$

so that $AB \in \mathcal{V}_n$. Similarly, $BA \in \mathcal{V}_n$, so that $\mathcal{V}_n$ is a two-sided ideal. Furthermore, if $A \in \mathcal{V}_n$, we have that

$$p_n(A^*) = p_n(A) = 0,$$

so that $A^* \in \mathcal{V}_n$, where we have applied Lemma 5.16. Thus, each $\mathcal{V}_n$ is a two-sided ideal closed under involution. Thus, we may define the associative *-algebra $\widetilde{\mathcal{A}_n} \equiv \mathcal{A}/\mathcal{V}_n$.

STEP 3: TURN EACH $\widetilde{\mathcal{A}_n}$ INTO NORMED UNITAL *-ALGEBRA.
First of all, note that we *do not* equip $\widetilde{\mathcal{A}_n}$ with the quotient topology. Instead, we give it the norm topology of the norm we are about to define. For $A \in \mathcal{A}$, define

$$\|A + \mathcal{V}_n\| = p_n(A).$$

To show that this is well-defined, let $B \in \mathcal{A}$ be such that $A - B \in \mathcal{V}_n$. Then,

$$|p_n(A) - p_n(B)| \le p_n(A - B) = 0,$$

so that $p_n(A) = p_n(B)$. This clearly defines a seminorm on $\widetilde{\mathcal{A}_n}$ as p_n itself is a seminorm. To show that it is in fact a norm, suppose that $\|A + \mathcal{V}_n\| = 0$. Then, $p_n(A) = 0$, and hence $A \in \mathcal{V}_n$. This norm thus turns $\widetilde{\mathcal{A}_n}$ into a normed unital *-algebra. We furthermore note that

$$\|(A + \mathcal{V}_n)(B + \mathcal{V}_n)\| \le \|A + \mathcal{V}_n\| \, \|B + \mathcal{V}_n\|$$

and

$$\|(A^* + \mathcal{V}_n)(A + \mathcal{V}_n)\| = \|A + \mathcal{V}_n\|^2$$

because p_n satisfies these properties. Unfortunately, however, $\widetilde{\mathcal{A}_n}$ is not necessarily going to be complete, so let $\mathcal{A}_n$ be its completion.

[26] For our purposes, the inverse limit of a sequence of C^*-algebras is the unique F^*-algebra that satisfies the universal property of the previous theorem. In particular, we are assuming that these morphisms have dense image, which to the best of my knowledge, is not a universal assumption.

Step 4: Turn each $\mathcal{A}_n$ into a separable, unital C^*-algebra.
By density of $\widetilde{\mathcal{A}_n}$ in $\mathcal{A}_n$, it is easy to extend definitions of addition, scalar multiplication, multiplication, and involution to $\mathcal{A}_n$ that turn it into a complete unital *-algebra. Furthermore, because the norm on $\widetilde{\mathcal{A}_n}$ satisfies these properties, the norm on $\mathcal{A}_n$ must satisfies $\|AB\| \leq \|A\| \|B\|$ and $\|A^*A\| = \|A\|^2$. Thus, $\mathcal{A}_n$ is a C^*-algebra. Furthermore, each $\widetilde{\mathcal{A}_n}$ is separable because $\mathcal{A}$ is, and in turn each $\mathcal{A}_n$ is separable because each $\widetilde{\mathcal{A}_n}$ is.

Step 5: Show that $\mathcal{A}$ is the inverse limit of this sequence.
Let $\pi_m : \mathcal{A} \to \mathcal{A}_m$ be the composite of the injection from $\widetilde{\mathcal{A}_m}$ into its completion $\mathcal{A}_m$ with the projection map π_m from $\mathcal{A}$ into $\widetilde{\mathcal{A}_m}$. This map is certainly a *-homomorphism and it is easy to check that it is also continuous. Furthermore, because $p_m \leq p_n$ for $m \leq n$, we have that $\mathcal{V}_n \subseteq \mathcal{V}_m$, and hence the map $\pi_{m,n}$ defined by

$$\pi_{m,n}(A + \mathcal{V}_n) = \pi_m(A)$$

is well-defined. Once again, it is easy to check that $\pi_{m,n}$ is a continuous *-homomorphism from $\widetilde{\mathcal{A}_n}$ into $\widetilde{\mathcal{A}_m}$, and hence we obtain a continuous *-homomorphism from $\mathcal{A}_n$ into $\mathcal{A}_m$: call it $f_{m,n}$. It is then easy to check that these maps make $\mathcal{A}$ satisfy the necessary universal stated in Theorem 5.18, and hence $\mathcal{A}$ is the inverse limit of the sequence of separable, unital C^*-algebras $\{\mathcal{A}_m | \, m \in \mathbb{N}\}$. □

The following theorem allows us to classify all F^*-algebras in the same vein that the Gelfand-Naimark Theorem allows us to classify all C^*-algebras:

Theorem 5.20 (Gelfand-Naimark Theorem for F^*-Algebras)**.** *Let $\mathcal{A}$ be a separable, unital F^*-algebra. Then, there exists a separable complex Hilbert space H such that:*

(1) There is an isomorphism π from $\mathcal{A}$ to a separable, unital F^*-algebra which is a subset of $\mathcal{L}[H]$.
(2) For every positive, unital, restricted linear functional ψ on $\mathcal{A}$, there exists a unique positive trace-class operator of trace 1 Ψ such that $\psi(A) = \operatorname{tr}[\Psi\pi(A)]$.

Proof. Step 1: Define the F^*-algebra that is a subset of $\mathcal{L}[H]$.
By the previous theorem, there exists a sequence of separable, unital C^* algebras $\{\mathcal{A}_m | \, m \in \mathbb{N}\}$ such that $\mathcal{A}$ is the inverse limit of this sequence. By the Gelfand-Naimark Theorem, there exist separable complex Hilbert spaces H_m such that $\mathcal{A}_m$ is isomorphic to a closed subalgebra of $\mathcal{B}[H_m]$. Let ϕ_m be the isomorphism from $\mathcal{A}_m$ into $\mathcal{B}[H_m]$. Write $\mathcal{B}_m = \phi_m(\mathcal{A}_m)$. Let $f_{m,n}$ be the morphism from $\mathcal{A}_n$ to $\mathcal{A}_m$ whose existence is guaranteed by the previous theorem and define $g_{m,n} : \mathcal{B}_n \to \mathcal{B}_m$ such that $g_{m,n} = \phi_m \circ f_{m,n} \circ \phi_n^{-1}$. Define the Hilbert space $H = \bigoplus_{n\in\mathbb{N}} H_n$ and define

$$\mathcal{B} = \left\{ A \in \mathcal{L}[H] | \, A = \bigoplus_{n\in\mathbb{N}} A_n, A_n \in \mathcal{B}_n, g_{m,n}(A_n) = A_m \right\},$$

where the domain of each $A \in \mathcal{B}$ is defined to be the set of all $v = \bigoplus_{n\in\mathbb{N}} v_n \in H$ such that $A(v) = \bigoplus_{n\in\mathbb{N}} A_n(v_n) \in H$ where $A = \bigoplus_{n\in\mathbb{N}} A_n$, i.e., such that $\sum_{n\in\mathbb{N}} \|A_n(v_n)\|^2 < \infty$.

Step 2: Prove that $\mathcal{B}$ satisfies the desired properties.
If $\{e_n^m | \, n \in \mathbb{N}\}$ is an orthonormal basis for H_m, then $\bigcup_{m,n\in\mathbb{N}} e_n^m$ is an orthonormal

basis for H, and hence H is separable. Moreover, as the domain of $A \in \mathcal{B}$ contains each element in this basis, the domain of each $A \in \mathcal{B}$ is dense in $\mathcal{B}$. We now show that each $A \in \mathcal{B}$ is closed. Let $\{v^m |\, m \in \mathbb{N}\}$ be a sequence in $D(A)$ that converges to $v \in H$ and such that $A(v^m)$ converges to $w \in H$. We may write $v^m = \bigoplus_{n\in\mathbb{N}} v_n^m$ and $A = \bigoplus_{n\in\mathbb{N}} A_n$ so that $A(v^m) = \bigoplus_{n\in\mathbb{N}} A_n(v_n^m)$. Becuase $\{v^m |\, m \in \mathbb{N}\}$ converges, it follows that for each $n \in \mathbb{N}$, the sequence $\{v_n^m |\, m \in \mathbb{N}\}$ converges, say to v_n, so that $v = \bigoplus_{n\in\mathbb{N}} v_n$. Furthermore, because the sequence $\{A(v^m)|\, m \in \mathbb{N}\}$ converges, it follows that for each $n \in \mathbb{N}$, the sequence $\{A_n(v_n^m)|\, m \in \mathbb{N}\}$ converges *uniformly in* m, say to w_n. Thus, because each A_n is closed, it follows that $v_n \in D(A_n)$ and $w_n = A_n(v_n)$, so that $A(v) = w$. Now, let $\varepsilon > 0$ and choose M sufficiently large so that $\|w_n\|^2 \le \frac{\varepsilon}{2^n} + \left\|A_n(v_n^M)\right\|^2$, where we can pick an M that works for all n because the convergence is uniform in m. Then,

$$\sum_{n\in\mathbb{N}} \|A_n(v_n)\|^2 = \sum_{n\in\mathbb{N}} \|w_n\|^2 \le 2\varepsilon + \sum_{n\in\mathbb{N}} \left\|A_n(v_n^M)\right\|^2 = 2\varepsilon + \left\|A(v^M)\right\|^2 < \infty.$$

Thus, $v \in D(A)$ and $A(v) = w$, and hence A is closed. Furthermore, define $\rho_n : \mathcal{B} \to \mathcal{B}_n$ by $\rho_n(A) = A_n$ if $A = \bigoplus_{n\in\mathbb{N}} A_n$. It is then easy to see that $\mathcal{B}$ is the inverse limit of the sequence $\{\mathcal{B}_m |\, m \in \mathbb{N}\}$. But via the isomorphisms from $\mathcal{A}_m$ into $\mathcal{B}_m$, this means that $\mathcal{B}$ is also the inverse limit of the sequence $\{\mathcal{A}_m |\, m \in \mathbb{N}\}$, and hence, by uniqueness, $\mathcal{A}$ is isomorphic to $\mathcal{B}$, that is, $\mathcal{A}$ is isomorphic to a separable, unital F^*-algebra which is a subset of $\mathcal{L}[H]$.

Step 3: Prove (2).

Let π be the isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Let ψ be a positive, unital, restricited linear functional on $\mathcal{A}$. Then, $\psi = \widehat{f_n}(\psi_n)$ for some positive, unital linear functional ψ_n on $\mathcal{A}_n$, where $f_n : \mathcal{A} \to \mathcal{A}_n$ is the given morphism that makes $\mathcal{A}$ the inverse limit of $\{\mathcal{A}_n |\, n \in \mathbb{N}\}$. Then, by the Gelfand-Naimark Theorem for C^*-algebras, there exists a positive trace-class operator of trace 1 Ψ_n on H_n such that $\psi_n(A) = \operatorname{tr}[\Psi_n \phi_n(A)]$. Define Ψ on H to be the unique operator that sends v to $\Psi_n(v)$ for $v \in H_n$ and sends v to 0 otherwise. It follows that Ψ defines an operator on H that is positive and of trace-class with trace 1 that furthermore satisfies the property that $\psi(A) = \operatorname{tr}[\Psi\pi(A)]$. Ψ is uniquely determined on H_n because Ψ_n is unique (by Theorem 5.17) and must be uniquely 0 on H_m for $m \neq n$ because it must be positive and of trace 1. Thus, Ψ is unique. □

We end this appendix with a miscellaneous result that is required in the proof of Theorem 2.21.

Lemma 5.21. *Let X be a compact Hausdorff space such that $C(X)$ is separable. Then, the map that sends $x \in X$ to $\widehat{x} \in C(X)^*$ is a homeomorphism onto the set of all positive, multiplicative, unital linear functionals.*[27]

Proof. Step 1: Note that each linear functional of this form is positive, multiplicative, and unital.

First of all, we note that it is easy to check that each $\widehat{x}$ is a positive, multiplicative, unital linear functional.

Step 2: Show that each positive, multiplicative, unital linear functional is of this form.

To show that each such linear functional is of this form, let $\psi \in C(X)^*$ be positive, multiplicative, and unital. For each $x \in X$, let $\{K_{x,n} |\, n \in \mathbb{N}\}$ be a decreasing

[27]We always equip the dual space with the weak-* topology.

sequence of compact sets such that $\bigcap_{n\in\mathbb{N}} K_{x,n} = \{x\}$. We may construct such a sequence because X is compact and normal. By Urysohn's Lemma, there exists a function $f_{x,n} \in C(X)$ of norm 1 such that $f_{x,n}|_{K_n} = 1$ and $f_{x,n}|_{K_{n-1}^C} = 0$. For each $x \in X$, consider the sequence defined by $a_{x,n} = \psi(f_{x,n})$. Because ψ is bounded, it follows that the sequence $\{a_{x,n}|\, n \in \mathbb{N}\}$ is bounded. We proceed by contradiction: suppose that every such sequence converged to 0. Then, as finite linear combinations of functions of the form $\{f_{x,n}|\, x \in X, n \in \mathbb{N}\}$ are dense in $C(X)$, then it would follows that $\psi(1) = 0$: a contradiction. Thus, there must be some $x_0 \in X$ such that the sequence $\{a_{x_0,n}|\, n \in \mathbb{N}\}$ does not converge to 0. Furthermore, for $x \neq x_0$ and large enough n, $f_{x,n}f_{y,n} = 0$, so that

$$0 = \psi(0) = \psi(f_{x,n}f_{x_0,n}) = a_{x,n}a_{x_0,n}.$$

As $\{a_{x_0,n}|\, n \in \mathbb{N}\}$ does not converge to 0, it follows that $\{a_{x,n}|\, n \in \mathbb{N}\}$ must. As every subsequence $\{a_{x_0,n}|\, n \in \mathbb{N}\}$ is bounded, every subsequence has in turn some convergent subsequence $\{a_{x_0,m_n}|\, n \in \mathbb{N}\}$. However, we also know that $f_{x_0,m_{n+1}} \leq f_{x_0,m_n}^2$, and hence, taking limits, $a_{x_0} \leq a_{x_0}^2$, where $a_{x_0} \equiv \lim \psi(f_{x_0,n})$. As we already know that $a_{x_0} \neq 0$, it follows that $a_{x_0} \geq 1$. Then, by Proposition 4.12 of [6][28], we have that $a_{x_0} \leq 1$, and hence $a_{x_0} = 1$. Thus, every subsequence of $\{a_{x_0,n}|\, n \in \mathbb{N}\}$ has in turn a subsequence that converges to 1, and hence this sequence converges to 1. In conclusion,

$$\lim \psi(f_{x,n}) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases}.$$

Now, let $f \in C(X)$ be arbitrary. Then, as X has a natural uniform structure[29], f is uniformly continuous, and hence $\{(f - f(x_0))\, f_{x_0,n}|\, n \in \mathbb{N}\}$ must converge to 0, and hence $\{\psi((f - f(x_0))\, f_{x_0,n})\,|\, n \in \mathbb{N}\}$ must converge to 0, and hence $\psi(f) = f(x_0)$. Hence, $\psi = \widehat{x_0}$.

Step 3: Note that this map is injective.
As X is normal, we can separate any two distinct points by a function, and hence the map that sends x to $\widehat{x}$ must be injective.

Step 4: Show that this map is continuous.
Let I be a directed set and let $\{x_i|\, i \in I\}$ be a net in X converging to $x \in X$ and let $f \in C(X)$. Then, because f is continuous, $\{f(x_i)|\, i \in I\}$ converges to $f(x)$, and hence, by definition of the weak-* topology, $\{\widehat{x_i}|\, i \in I\}$ converges to $\widehat{x}$. Thus, the map $x \mapsto \widehat{x}$ is continuous.

Step 5: Show that the set of all positive, multiplicative, unital linear functionals is Hausdorff.
By Proposition 4.12 of [6] again, the set of all positive, multiplicative linear, unital linear functionals on $C(X)$ is contained in the unit ball of $C(X)^*$, which itself is metrizable, and hence the set of all positive, multiplicative, unital linear functionaos is Hausdorff.

Step 6: Deduce that this map is a homeomorphism.
This means that we have that the map that sends x to $\widehat{x}$ is an injective, continuous map from a compact space into a Hausdorff space, and hence is a homeomorphism onto its image. Thus, the map that sends $x \in X$ to $\widehat{x} \in C(X)^*$ is a homeomorphism onto the set of all positive, multiplicative, unital linear functionals. □

[28]All this tells us is that $\|\psi\| = 1$.

[29]See [7], pg. 106.

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