On Brace Products and the Structure of Fibrations with Section

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Abstract

We relate the brace products of a fibration with section to higher order differentials in its Serre spectral sequence. As an application, we determine the homology of some free iterated loop spaces of spheres. Our main result is related to a classical theorem of G. Whitehead on free loop fibrations and we use Whitehead's result to give a simple geometric proof of a result of Havlicek on spaces of rational maps into complex projective space.

§1 Introduction

Let $\zeta : F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration with a section $B \xrightarrow{s} E$. One of our main results in this note shows that the differentials on the spherical classes in the Serre spectral sequence for ζ are determined entirely by "brace products"in ζ.

Brace products for a fibration with section were originally defined by James ([J]). Given $\alpha \in \pi_p(B)$ and $\beta \in \pi_q(F)$, one can take the Whitehead product $[s_*(\alpha), i_*(\beta)]$ in $\pi_{p+q-1}(E)$. Since $\pi_*([s_*(\alpha), i_*(\beta)]) = 0$, one deduces from the long exact sequence in homotopy associated to ζ that $[s_*(\alpha), i_*(\beta)]$ must lift to a unique class

$$\{\alpha, \beta\} \in \pi_{p+q-1}(F);$$

the so called *brace product* of α and β. Note that this class depends on the choice of section. The brace product operation gives then a pairing

$$\{,\} : \pi_p(B) \times \pi_q(F) \longrightarrow \pi_{p+q-1}(F).$$

Let $h : \pi_*(X) \longrightarrow H_*(X; \mathbf{Z})$ denote the Hurewicz homomorphism. Our main result can now be stated

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Theorem 1.1: *Let $F \to E \to B$ be a fibration with section, and assume B is simply connected. Then in the Serre spectral sequence (with any untwisted coefficients), the following diagram commutes*

$$\begin{array}{ccc}
\pi_p(B)\otimes\pi_q(F) & \xrightarrow{\{,\}} & \pi_{p+q-1}(F) \\
\downarrow & & \downarrow h \\
H_p(B, H_q(F)) & & H_{p+q-1}(F) \\
\downarrow \cong & & \downarrow \cong \\
E^2_{p,q} & \xrightarrow{d^p} & E^2_{0,p+q-1}
\end{array}$$

Remarks: Some interpretations are in order. First the map $\pi_p(B)\otimes\pi_q(F)\longrightarrow H_p(B,H_q(F))$ is of course the composite

$$\pi_p(B)\otimes\pi_q(F)\xrightarrow{h\otimes h}H_p(B)\otimes H_q(F)\xrightarrow{\nu}H_q(F,H_p(B)),$$

where ν is a universal coefficient homomorphism. Secondly the differential d^p is really a map $E^p_{p,q}\longrightarrow E^p_{0,p+q-1}$, but the point is that any class in $E^2_{p,q}$ coming from $\pi_p(B)\otimes\pi_q(F)$ actually lives until $E^p_{p,q}$. Finally, even though the brace product does depend on the choice of section s, commutativity of the above diagram does not.

A particularly interesting application of this theorem occurs for the evaluation fibration

$$F\xrightarrow{i}E\xrightarrow{ev}S^n,$$

where $E = \mathrm{Map}(S^k,S^n) := \Lambda^k S^n$ is the free loop space on S^n and $F=\Omega^kS^n$ the subspace of basepoint preserving (or *based*) maps. The Serre spectral sequence for this fibration is discussed in detail in §4 for $1\leq k<n$. General arguments show that the spectral sequence collapses at E^2 with mod-2 coefficients. When n is odd, the same collapse occurs with mod-p coefficients. The case n even is therefore most interesting to study and we obtain our next main result as a direct consequence of 1.1.

Let $x\in H_{2n}(S^{2n})$ be the orientation class and $e\in H_{2n-k}(\Omega^kS^{2n})$ be the infinite cyclic generator representing the class of the inclusion $S^{2n-k}\longrightarrow\Omega^kS^{2n}$ which is adjoint to the identity map of S^{2n}. Finally let $a\in H_{4n-k-1}(\Omega^kS^{2n})$ be the torsion free generator (see §4).

Proposition 1.2: *Assume $1\leq k<n$ and n is even. Then in the homology Serre spectral sequence (with integral coefficients) for the fibration $\Omega^kS^n\xrightarrow{i}\Lambda^kS^n\xrightarrow{ev}S^n$;*

$$d^n_{n,n-k}(x\cdot e)=2a.$$

A more extensive study of the differentials in the evaluation fibration with relation to homology operations in the fiber will appear in a sequel ([K2]). We point out for now that the differential in

1.2 is already sufficient to determine $H^*(\Lambda^k S^n;\mathbf{Q})$ and $H^*(\Lambda^2 S^n;\mathbf{A})$ (for any untwisted coefficients) entirely.

Corollary 1.3: *Suppose $1\le k<n$ and n is even, then the Poincaré series for $H^*(\Lambda^k S^n;\mathbf{Q})$ is given as follows*

$$\begin{cases} 1+(x^n+x^{n-k})/(1-x^{n-k-1}) & ,\ k \text{ is odd} \\ (1+x^{3n-k-1})/(1-x^{n-k}) & ,\ k \text{ is even.} \end{cases}$$

Corollary 1.4 (F. Cohen): *Suppose $n>2$ even and p odd. Then in the cohomology Serre spectral sequence for $\Lambda^2 S^n$, the mod-p differentials are generated by $d_n(x\cdot e)=x_0$, where $H^*(\Omega^2 S^n;\mathbf{Z}_p)$ is a tensor product of a divided power algebra on generators e, y_i, and an exterior algebra on generators x_i, $dim(x_i)=2(n-1)p^i-1=dim(y_i)+1$, $i\ge 0$.*

The proof of theorem 1.1 relies on a beautiful theorem of G. Whitehead [W] relating the boundary homomorphism in the homotopy long exact sequence of a free loop fibration on a space X to Whitehead products in X. More precisely, let X be a finite CW complex (based at x_0) and consider the evaluation fibration

$$\Omega^n X \xrightarrow{\ i\ } \Lambda^n X \xrightarrow{\ ev\ } X$$

where again $\Lambda^n(X):=\mathrm{Map}(S^n,X)$. We let $\Lambda^n_f(X)$ denote the component containing a given map f. We can assume $f(S)=x_0\in X$ (where S is the south pole of S^n) and hence denote by $\Omega^n_f(X)$ the subspace of $\Lambda^n_f(X)$ of based maps sending S to x_0.

Theorem 1.5: [W] *The homotopy boundary $\partial:\pi_p(X)\to\pi_{p-1}(\Omega^n_f(X))\cong\pi_{p+n-1}(X)$ in the long exact sequence in homotopy associated to*

$$\Omega^n_f(X)\longrightarrow\Lambda^n_f(X)\xrightarrow{\ ev\ }X$$

is given (up to sign) by the Whitehead product: $\partial\alpha=[\alpha,f],\alpha\in\pi_p(X)$.

We give a short application of this theorem to the theory of rational maps ([K3]). Let $\mathbf{P}^1$ be the Riemann sphere, and write $\mathrm{Hol}_k(\mathbf{P}^1)$ for the space of holomorphic maps from $\mathbf{P}^1$ into itself of degree k. Denote by $\mathrm{Rat}_k(\mathbf{P}^1)$ the subspace of basepoint preserving holomorphic maps (that fix the north pole say).

Corollary 1.6 (Havlicek): *The (cohomology) Serre SS for the fibration*

$$Rat_k(\mathbf{P}^1)\longrightarrow Hol_k(\mathbf{P}^1)\longrightarrow\mathbf{P}^1$$

has only one non-zero differential $d_2(x)=2k\iota$, where $x\in H^1(Rat_k(\mathbf{P}^1))$ and $\iota\in H^2(\mathbf{P}^1)$ are the generators. It follows in particular that the spectral sequence collapses with mod-p coefficients whenever $p=2$ or p divides k.

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§2 Brace Products: Examples and Properties

Notation and Conventions: We often (but not always) identify a map $f : S^p \longrightarrow X$ with its homotopy class $[f] \in \pi_p(X)$. We do so when there is no risk of confusion and to ease notation. We also write *ad* for the adjoint isomorphism

$$ad : \pi_{i+k}(X) \xrightarrow{\cong} \pi_i(\Omega^k X)$$

(the index k is to be determined from the context).

In the introduction we defined Brace products for a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ with a section $B \xrightarrow{s} E$. Brace products are related to Whitehead products by the commutative diagram

2.1
$$\begin{array}{ccc} \pi_p(B) \otimes \pi_q(F) & \xrightarrow{\{,\}} & \pi_{p+q-1}(F) \\ \downarrow{\scriptstyle s\otimes i} & & \downarrow{\scriptstyle i} \\ \pi_p(E) \otimes \pi_q(E) & \xrightarrow{[,]} & \pi_{p+q-1}(E) \end{array}$$

The next three examples compute the brace product pairing for some classes of fibrations with section.

Example 2.2: Let E be a sphere bundle over $B = S^n$ with fiber $F = S^k$ and group $O(k+1)$;

$$S^k \xrightarrow{i} E \xrightarrow{\pi} S^n.$$

This fibration is classified (up to homotopy) by a clutching function $\mu : S^{n-1} \longrightarrow O(k+1)$. If E has a section then the group of the bundle reduces to $O(k)$ (because the associated vector bundle does). The map μ factors (up to homotopy) through $S^{n-1} \longrightarrow O(k) \hookrightarrow O(k+1)$, giving a class $\alpha \in \pi_{n-1}O(k)$. Let J be the Hopf-Whitehead construction

$$J : \pi_{n-1}(O(k)) \longrightarrow \pi_{n+k-1}(S^k).$$

Finally let $\iota_n \in \pi_n(E)$ be the class of $s : S^n \longrightarrow E$ and $\iota_k \in \pi_k(E)$ be the class of the fiber. Then (up to sign)

Proposition 2.3: $\{\iota_n, \iota_k\} = J\alpha$ *in* $\pi_{n+k-1}(S^k)$

Proof: We will make use of some crucial intermediate results we prove in §3. Start with the map $\alpha : S^{n-1} \longrightarrow O(k)$. One can think of $O(k)$ as transformations of the closed unit disc D^k. It follows that α adjoins to a map $S^{n-1} \times D^k \longrightarrow D^k$ and by pinching the boundary of D^k we get the following commutative diagram

$$\begin{array}{ccc} S^{n-1} \times D^k & \xrightarrow{\alpha} & D^k \\ \downarrow & & \downarrow \\ S^{n-1} \wedge S^k & \xrightarrow{J} & S^k \end{array}$$

Since $S^{n-1} \wedge S^n = S^{n+k-1}$ the bottom map can indeed be identified with (and actually is) the J homomorphism. Consider the composite

$$\phi : S^{n-1} \wedge S^k \xrightarrow{J} S^k \xrightarrow{i} E$$

We write its adjoint as a map $g : S^{n-1} \longrightarrow \Omega^k E$. Notice that the image of g lies in the component containing the fiber inclusion $i : S^k \longrightarrow E$ and hence maps into $\Omega^k_{\iota_k} E$. Moreover notice that g when extended to $\Lambda^k_{\iota_k} E$ is trivial (lemma 3.4) and hence the map g factors through the fiber ΩE of the inclusion $\Omega^k E \hookrightarrow \Lambda^k E$ as follows

2.4
$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{ad(s)} & \Omega E & & \\ \downarrow = & & \downarrow & & \\ S^{n-1} & \xrightarrow{g} & \Omega^k E & \longrightarrow & \Lambda^k E \end{array}$$

The adjoint of the top map is $s : S^n \longrightarrow E$ and the class of this map we denote by ι_n. According to lemma 3.2 we must have that

$$\phi = [\iota_k, \iota_n] \in \pi_{n+k-1}(E)$$

Both ϕ and the Whitehead product map lift to S^k. Since the lift of ϕ is J and the lift of $[\iota_k, \iota_n]$ is $\{\iota_k, \iota_n\}$, the proof is complete. ∎

NOTE: In this case $\{\iota_n, \iota_k\}$ is independent of the choice of section (see 3.1 and 3.2).

EXAMPLE 2.5: Let ζ be a vector bundle over $B = S^n$ with fiber $F = \mathbf{R}^k$ and group $O(k)$. Consider the open unit disc bundle associated to ζ and compactify it fiberwise by adding a point at infinity. One then gets a bundle $S^k \to E \to S^n$ which has a canonical cross section (sending each point in S^n to the point at infinity in the fiber). Thus one computes $\{\iota_n, \iota_k\}$ as indicated in 2.3.

EXAMPLE 2.6: It is known that the fiber of the inclusion $X \vee X \longrightarrow X \times X$ is $\Sigma(\Omega X \wedge \Omega X)$ (a theorem of Ganea). Taking $X = \mathbf{P} = \mathbf{P}^\infty$ the infinite complex projective space, we find that there is a fibration

2.8
$$S^3 \longrightarrow \mathbf{P} \vee \mathbf{P} \longrightarrow \mathbf{P} \times \mathbf{P}$$

and hence after looping we obtain a fibration $\Omega S^3 \longrightarrow \Omega(\mathbf{P} \vee \mathbf{P}) \longrightarrow S^1 \times S^1$ with a section given by the composite

$$S^1 \times S^1 \longrightarrow \Omega\mathbf{P} \times \Omega\mathbf{P} \xrightarrow{=} \Omega(\mathbf{P} \vee *) \times \Omega(* \vee \mathbf{P}) \hookrightarrow \Omega(\mathbf{P} \vee \mathbf{P}) \times \Omega(\mathbf{P} \vee \mathbf{P}) \xrightarrow{*} \Omega(\mathbf{P} \vee \mathbf{P}).$$

It turns out that 2.8 has an interesting brace product given as follows. Denote by a_1 (resp. a_2) the generator of the second homotopy group for the first (resp. second) copy of $\mathbf{P}$ in $\mathbf{P} \vee \mathbf{P}$. The fiber S^3 maps to $\mathbf{P} \vee \mathbf{P}$ via the Whitehead product $[a_1, a_2]$. Taking $G : S^3 \to \mathbf{P} \vee \mathbf{P}$ to be the class of the fiber, we then have that $[a_1, G] = [a_1, [a_1, a_2]] \in \pi_4(\mathbf{P} \vee \mathbf{P})$ and this corresponds to the class $\{a, G\}$ via the isomorphism $\pi_4(\mathbf{P} \vee \mathbf{P}) \cong \pi_4(S^3)$. It is shown in [K1] that this triple Whitehead product is non-trivial and hence

Lemma 2.8 [K1]: $\{a, G\}$ is the generator of $\pi_4(S^3) = \mathbf{Z}_2$.

Example 2.9 (Saaidia): Suppose $F \longrightarrow E \longrightarrow B$ is a fibration with section, and F is a G-space with a G-invariant basepoint. Consider the classifying bundle $F \longrightarrow EG \times_G F \longrightarrow BG$. This fibration also admits a section and its brace products are identified with the so-called "secondary Eilenberg invariant" of the fibration E (cf. [Sa]). These invariants are fundamental in the study of the homotopy type of the space of sections of E.

Brace Products and Samelson Products

The commutator map at the level of loop spaces (better known as the Samelson product) is related to the Whitehead product as follows. First write S for the commutator

$$\begin{array}{ccc} \Omega(X) \wedge \Omega(X) & \xrightarrow{S} & \Omega(X) \\ (a,b) & \mapsto & aba^{-1}b^{-1} \end{array}$$

Then the following commutes (up to sign)

2.10
$$\begin{array}{ccc} \pi_p(\Omega X) \times \pi_q(\Omega X) & \xrightarrow{S} & \pi_{p+q}(\Omega X) \\ \Big\downarrow{\scriptstyle ad \times ad} & & \Big\downarrow{\scriptstyle ad} \\ \pi_{p+1}(X) \times \pi_{q+1}(X) & \xrightarrow{[\ ,\ ]} & \pi_{p+q+1}(X). \end{array}$$

where ad is the adjoint isomorphism. This fact (originally due to H. Samelson) can be combined with 2.1 to show that

Lemma 2.11: *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega B \wedge \Omega F & \xrightarrow{\{,\}} & \Omega F \\ \Big\downarrow{\scriptstyle \Omega s \wedge \Omega i} & & \Big\downarrow{\scriptstyle \Omega i} \\ \Omega E \wedge \Omega E & \xrightarrow{S} & \Omega E \end{array}$$

where the upper map (which we also denote by a brace) induces James' brace product at the level of homotopy groups).

Proof: The composite

$$\Omega B \wedge \Omega F \xrightarrow{S \circ (\Omega s \wedge \Omega i)} \Omega E$$

is trivial when projected into ΩB (because $\Omega p \circ \Omega i$ is trivial.) It then lifts to ΩF as desired. This lift is unique up to homotopy since any two maps differ by a map

$$\Omega B \wedge \Omega F \longrightarrow \Omega^2 B \xrightarrow{\partial} \Omega F$$

and that this "boundary" map ∂ is null-homotopic since $\Omega E \simeq \Omega F \times \Omega B$ (see below). The rest of the claim follows from 2.10. ∎

Brace products as obstructions

As pointed out in [J], brace products form an obstruction to retracting the total space E into the fiber F. They also represent obstructions to the triviality of certain pull-back fibrations in the postnikov tower for B (see [Sa]). In what follows we exhibit yet another obstruction expressed in terms of these brace products.

Let $F \xrightarrow{i} E \to B$ be a fibration and consider the loop fibration

2.12 $$\Omega F \longrightarrow \Omega E \longrightarrow \Omega B$$

Suppose that 2.12 has a section s'. Being principal, it splits and we have a trivialization

$$\Omega i * s' : \Omega F \times \Omega B \xrightarrow{\simeq} \Omega E$$

induced by loop sum in ΩE. This trivialization however is not necessarily an H-space map and its failure to be such is measured by the commutator $(\Omega i)s'(\Omega i)^{-1}(s')^{-1}$. We illustrate this by an example

Example 2.13: Consider the Hopf fibering $S^1 \to S^3 \to S^2$ which can be looped to a fibering

$$\Omega S^3 \longrightarrow \Omega S^2 \longrightarrow S^1$$

This has an obvious section and it follows from above that $S^1 \times \Omega S^3 \xrightarrow{\simeq} \Omega S^2$. Notice that the left hand side is abelian (since S^3 is a topological group). It turns out however that ΩS^2 is not abelian. To see this consider the map $S^1 \longrightarrow \Omega S^2$ and take its self commutator in ΩS^2. This commutator in homotopy is adjoint (by the result of Samelson) to the Whitehead product $[\iota_2, \iota_2] = 2\eta \in \pi_3(S^2)$ where η is the class of the hopf map. This shows that indeed ΩS^2 is not abelian and that the splitting above is not an H space splitting.

Lemma 2.14: *Let $F \to E \to B$ be a fibration with section s. If the brace products in this fibration vanish identically, then*

$$\theta = \Omega s * \Omega i : \Omega B \times \Omega F \xrightarrow{\simeq} \Omega E$$

is an H-space splitting.

Proof: Here one first notes that the induced loop fibration has a section $s' = \Omega s$. We need only check that the following diagram homotopy commutes

$$\begin{array}{ccccc}
(\Omega B \times \Omega F)^2 & \xrightarrow{1\times\chi\times 1} & (\Omega B)^2 \times (\Omega F)^2 & \xrightarrow{*\times *} & \Omega B \times \Omega F \\
\downarrow{\scriptstyle \Omega s \times \Omega i} & & & & \downarrow{\scriptstyle \theta} \\
\Omega E \times \Omega E & & \xrightarrow{\quad\quad *\quad\quad} & & \Omega E
\end{array}$$

where $1 \times \chi \times$ is the shuffle map $(x, a, b, y) \mapsto (x, b, a, y)$. Since the images of Ωs and Ωi commute in ΩE (this follows from 2.11 and 2.10 and from the fact that the brace products vanish), the claim follows immediately. ∎

§3 Whitehead's theorem and the Proof of Theorem 1.1

In this section we prove theorems 1.5 and 1.1 in the introduction. Denote by D^n the closed unit disc in $\mathbf{R}^n$ and by $\partial D^n = S^{n-1}$ its boundary. If $D^n = D^p\times D^q$, we can then write $S^{n-1} = \partial D^n = D^p\times\partial D^q\cup\partial D^p\times D^q$ (where the union is over $\partial D^p\times\partial D^q$). We also recall that an element in $\pi_q(\Lambda^p_\alpha X,\Omega^p_\alpha X)$ is represented by a map $D^p\times D^q\rightarrow X$ which sends $\partial D^p\times D^q$ to basepoint. We now have the following pivotal lemma

Lemma 3.1 (Whitehead): *Start with a map*

$$\phi : S^{p-1}\wedge S^q\longrightarrow X$$

and adjoin it to get $g : S^{p-1}\longrightarrow\Omega^q_\alpha X$ *(where* $\Omega^q_\alpha X$ *is some component of* $\Omega^p X$ *containing a representative map* α*). Suppose that* g *extends to a map* $D^p\longrightarrow\Lambda^q_\alpha X$ *and hence gives rise to an element* $\beta\in\pi_p(\Lambda^q_\alpha X,\Omega^q_\alpha X)\cong\pi_p(X)$*. Then*

$$\phi = [\alpha,\beta]\in\pi_{p+q-1}(X)$$

Sketch of Proof: First we note that the statement of the lemma is independent of the choice of the extension $D^p\longrightarrow\Lambda^q X$. Observe for now that the map ϕ can be represented by a map

$$\phi : (D^p\times\partial D^q)\cup(\partial D^p\times D^q)\longrightarrow X$$

sending $D^p\times\partial D^q$ to basepoint x_0 in X. On the other hand, the Whitehead product $[\alpha,\beta]$ is represented by a map

$$S^{p+q-1} = (D^p\times\partial D^q)\cup(\partial D^p\times D^q)\longrightarrow X$$

which sends $\partial D^q\cup\partial D^p$ to point, maps $(D^p,\partial D^p)\longrightarrow X$ via α and maps $(D^q,\partial D^q)\longrightarrow X$ via β. Consider the following homotopy from S^{p+q-1} to itself

$$F(x,y) = \begin{cases} (x,2y), & |x|=1,\ |y|\le\frac{1}{2},\\ ([2-2|y|]x)\,, y/|y|), & |x|=1,\ |y|\ge\frac{1}{2},\\ (0,y), & |x|\le 1,\ |y|=1\end{cases}$$

This has the effect of deforming the top half of D^q (i.e $|y|\ge\frac{1}{2}$) to the boundary and shrinking D^p to point. By composing with F, we see that the map representing $[\alpha,\beta]$ is homotopic to the map $\phi : (D^p\times\partial D^q)\cup(\partial D^p\times D^q)\longrightarrow X$ sending D^q via α and D^p to point. ∎

An alternative formulation of this lemma that is better suited to us is as follows.

Lemma 3.2: *Let* E *be a space and think of* ΩE *as the fiber of* $\Omega^q E\longrightarrow\Lambda^q E$*. Given a composite*

$$\phi : S^{p-1}\xrightarrow{\beta}\Omega E\longrightarrow\Omega^q_\alpha E$$

then necessarily $ad\phi = [\alpha, ad\beta]\in\pi_{p+q-1}E$.

Proof: That $\phi : S^{p-1}\longrightarrow\Omega^q_\alpha E$ factors through ΩE is exactly the same as having an extension diagram

$$\begin{array}{ccc} S^{p-1} & \xrightarrow{\phi} & \Omega^q_\alpha E\\ \downarrow & & \downarrow\\ D^p & \longrightarrow & \Lambda^q_\alpha E\end{array}$$

such that the element of $\pi_p(\Lambda^q_\alpha E, \Omega^q_\alpha E) \cong \pi_p(E)$ that this diagram defines is the class of $ad\beta$. It follows from 3.1 that $ad\phi = [\alpha, ad\beta]$. ∎

Theorem 3.3: [W] *The homotopy boundary* $\partial : \pi_p(X) \to \pi_{p-1}(\Omega^n_f(X)) = \pi_{p+n-1}(X)$ *in the long exact sequence in homotopy associated to*

$$\Omega^q_f(X) \longrightarrow \Lambda^q_f(X) \xrightarrow{ev} X$$

is given (up to sign) by the Whitehead product as follows. Let $\alpha \in \pi_p(X)$, *then*

$$\partial\alpha = ad[\alpha, f] \in \pi_{p-1}(\Omega^q_f X).$$

Proof: Recall that given a fibration $F \longrightarrow E \longrightarrow B$, it extends to the left $\Omega B \longrightarrow F$ and the boundary homomorphism is given by the induced map in homotopy

$$\partial : \pi_p(B) = \pi_{p-1}(\Omega B) \longrightarrow \pi_{p-1}(F)$$

This means that if we take $\alpha \in \pi_p(B)$, then the following commutes

$$\begin{array}{ccc} S^{p-1} & \xrightarrow{ad(\alpha)} & \Omega B \\ \downarrow = & & \downarrow \\ S^{p-1} & \xrightarrow{\partial\alpha} & F \end{array}$$

Letting $B = X$, $F = \Omega^f_q X$ and $E = \Lambda^q_f(X)$, we deduce from lemma 3.2 that $ad^{-1}(\partial\alpha) = [\alpha, f]$ and the claim follows. ∎

We need one more lemma before we can proceed with the proof of 1.1. Let $\zeta : F \longrightarrow E \longrightarrow S^n$ be a fibration with section s, and let $\mu : S^{n-1} \longrightarrow Aut(F)$ be the cluching function. Here $Aut(F)$ consists of based homotopy equivalences and we denote by $\text{Map}^*(F, E)$ the space of based maps from F into E. There are inclusions $Aut(F) \hookrightarrow \text{Map}^*(F, E) \hookrightarrow \text{Map}(F, E)$ and we assert that

Lemma 3.4: *There is an extension diagram*

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\mu} Aut(F) \hookrightarrow & Map^*(F, E) \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & Map(F, E) \end{array}$$

such that the element $\beta \in \pi_n(Map(F, E), Map^*(F, E)) \cong \pi_n(E)$ *that this defines corresponds to the class of* $s : S^n \longrightarrow E$

Proof: We have the following sequence of fibrations $F \longrightarrow E \longrightarrow S^n \longrightarrow BAut(F)$ and the last map classifies the fibration ζ. By looping and letting $S^{n-1} \to \Omega S^n$ be the adjoint to the identity map, we get the following diagram

$$\begin{array}{ccccc} & & \Omega E & & \\ & & \downarrow & & \\ S^{n-1} & \longrightarrow & \Omega S^n & \longrightarrow & Aut(F) \end{array}$$

The lower composite, which we label θ, can be identified with the clutching map μ. If one has a section $\Omega s : \Omega S^n \longrightarrow \Omega E$, then θ factors through ΩE which is the fiber of $\mathrm{Map}^*(F, E) \longrightarrow \mathrm{Map}(F, E)$. The lemma follows. ∎

Theorem 3.5: *There is a commutative diagram*

$$\begin{array}{ccc}
\pi_p(B) \otimes \pi_q(F) & \xrightarrow{\{,\}} & \pi_{p+q-1}(F) \\
\downarrow h\otimes h & & \downarrow h \\
H_p(B, H_q(F)) & & H_{p+q-1}(F) \\
\downarrow \cong & & \downarrow \cong \\
E^2_{p,q} & \xrightarrow{d^p} & E^2_{0,p+q-1}
\end{array}$$

Remark 3.6: We first explain why 3.5 is independent of the choice of section. Suppose $F \to E \to B$ is as above and assume it has two distinct sections s_1 and s_2. Let $\alpha \in \pi_p(B)$ and $\beta \in \pi_q(F)$. The brace products associated to s_1 and s_2 are given by $\{\alpha, \beta\}_1$ and $\{\alpha, \beta\}_2$ (respectively). Notice that $s_1(\alpha) - s_2(\alpha)$ projects to zero in $\pi_*(B)$ and hence must lift to a class $\alpha_F \in \pi_p(F)$. The difference element

$$\{\alpha, \beta\}_1 - \{\alpha, \beta\}_2 = [s_1(\alpha) - s_2(\alpha), i(\beta)] = [\alpha_F, \beta] \in \pi_{p+q-1}(F)$$

is a Whitehead product in F and hence must necessarily map to zero by the Hurewicz homomorphism. This obviously shows that the composite $h \circ \{,\}$ in the top half of the diagram in 3.5 is independent of the choice of section as asserted.

Proof of 3.5: Let $\alpha : S^p \to B$ represent a class in $\pi_p(B)$. Consider the pullback diagram

$$\begin{array}{ccc}
F & \longrightarrow & F \\
\downarrow & & \downarrow \\
E' & \longrightarrow & E \\
\downarrow & & \downarrow \\
S^p & \xrightarrow{\alpha} & B
\end{array}$$

By naturality of the Serre spectral sequence it suffices to prove the theorem for the pull back fibration $F \longrightarrow E' \longrightarrow S^p$. In other words we must prove that the following diagram commutes:

$$\begin{array}{ccc}
\pi_p(S^p) \otimes \pi_q(F) & \xrightarrow{\{,\}} & \pi_{p+q-1}(F) \\
\downarrow h\otimes h & & \downarrow h \\
H_p(S^p, H_q(F)) & & H_{p+q-1}(F) \\
\downarrow \cong & & \downarrow \cong \\
E^2_{p,q} & & E^2_{0,p+q-1} \\
\downarrow \cong & & \downarrow \cong \\
E^p_{p,q} & \xrightarrow{d^p} & E^p_{0,p+q-1}
\end{array}$$

Now associated to $F \to E' \to S^p$ is a Wang sequence

$$\cdots \longrightarrow H_i(F) \longrightarrow H_i(E') \longrightarrow H_{i-p}(F) \xrightarrow{\tau_*} H_{i-1}(F) \longrightarrow \cdots$$

where τ_* is determined in terms of the clutching function of the bundle. Recall that this clutching function is given by a map

$$\mu : S^{p-1} \times F \longrightarrow F$$

whose homotopy class determines the bundle (up to fiber homotopy). Identifying $H_{i-p}(F)$ with $E^2_{p,i-p}$ and $H_{i-1}(F)$ with $E^2_{0,i-1}$ it isn't hard to see that $\tau_* = d^p : E^2_{p,i-p} \to E^2_{0,i-1}$ (see [S], p:482).

Choose a basepoint $p \in F$. Given $\beta : S^q \longrightarrow F$ representing a spherical class (of the same name) in $H_q(F)$, then τ_* can be made explicit as follows. We first have an isomorphism $H_q(F) \cong H_{p+q}(S^p \wedge F)$ and the class β is represented under this isomorphism by a map $S^p \wedge S^q \to S^p \wedge F$. Writing $D^{p+q} = D^p \times D^q$ and $\partial D^{p+q} = (D^p \times \partial D^q) \cup (\partial D^p \times D^q)$ as before, we can represent β as a map of pairs

$$(D^{p+q}, \partial D^{p+q}) \longrightarrow (D^p \times F, D^p \times p \cup \partial D^p \times F).$$

The map on the second component is the boundary map ∂ and it can be prolonged into F

3.6 $$\tau : \partial D^{p+q} \xrightarrow{\partial} D^p \times p \cup \partial D^p \times F \longrightarrow F$$

by collapsing $D^p \times p$ to $p \in F$ and sending $\partial D^p \times F = S^{p-1} \times F$ to F via the clutching function μ. (This is possible since $\mu(\partial D^p \times p) = p \in F$.) The composite in 3.6 is a map $S^{p+q-1} \longrightarrow F$ whose Hurewicz image gives a class in $H_{p+q-1}(F)$. This class is exactly $\tau_*(\beta) = d_p(\beta)$.

Note at this point that the map τ gives rise by restriction to a map

$$\begin{array}{l} \partial D^p \times D^q \\ \quad\downarrow \\ S^{p-1} \wedge S^q \longrightarrow S^{p-1} \wedge F \xrightarrow{\mu} F \xrightarrow{i} E \end{array}$$

The horizontal composite adjoins to a map $\theta : S^{p-1} \longrightarrow \Omega^q E$ and the component it lies in contains the map $\beta : S^q \longrightarrow F \longrightarrow E$. By precomposing and using lemma 3.4, one gets the following extension diagram

$$\begin{array}{ccccc} S^{p-1} & \longrightarrow & Aut(F) & \longrightarrow & \Omega^q E \\ \downarrow & & \downarrow & & \downarrow \\ D^p & \longrightarrow & \mathrm{Map}(F, E) & \longrightarrow & \Lambda^q E \end{array}$$

and the homotopy class this defines is given by (lemma 3.4)

$$s(S^p) \in \pi_p(E) \cong \pi_p(\Lambda^q_\beta E, \Omega^q_\beta E).$$

One can now apply lemma 3.1 directly to obtain

$$i \circ \tau = [s(\alpha), i(\beta)] \ \text{ in } \pi_{p+q-1}(E).$$

Both maps lift to F; the LHS lifts to τ and the RHS lifts to $\{\alpha, \beta\} : S^{p+q-1} \to F$. Notice that in homology, the Hurewicz images of $i_* \circ \tau_*$ and $[s(\alpha), i(\beta)]_*$ are zero in $H_{p+q-1}(E)$ (in the first case because of the Wang exact sequence and in the second because of a known property of Whitehead products). It follows by the Wang exact sequence again that the class in the image of $h\circ\{\alpha,\beta\}_*$ in $H_{p+q-1}(F)$ is also in the image of τ_* and by the arguments above it must follow that it is exactly $\tau_*(\beta)$. The proposition follows. ∎

§4 Free Loop Spaces of Spheres $\Lambda^k S^n, 1 \le k < n$

Let $\Lambda^k S^n = \mathrm{Map}(S^k, S^n)$ denote the space of free maps from S^k to S^n. When $k < n$, this space is connected and moreover the evaluation fibration

4.1 $$\Omega^k S^n \xrightarrow{i} \Lambda^k S^n \xrightarrow{ev} S^n.$$

admits a section. We use our previous results to determine $H^*(\Lambda^k S^n)$ for many values of $1 \le k \le n$, and for field coefficients $\mathbf{F} = \mathbf{Q}, \mathbf{Z}_p$. (The general case is discussed in [K2].)

Remark 4.2: When $n = 1, 3$ or 7, $\Lambda^k S^n$ is an H-space and so the existence of a section yields a space level splitting for these values of n. Generally and for n odd, the localised sphere $S^n_{(p)}$ at an odd prime becomes an H-space and hence so is $\Lambda^k(S^n)$. We therefore have a space level splitting for odd n and after inverting 2. The Serre spectral sequence for 4.1 collapses for odd spheres with $\mathbf{Z}_p$ coefficients (p odd).

First of all, observe that our main theorem 1.1 combined with the result of Hansen described in the appendix yields the diagram

4.3 $$\begin{array}{ccc} \pi_n S^n \otimes \pi_{i+k}(S^n) & \xrightarrow{[,]} & \pi_{n+i+k-1}(S^n) \\ \downarrow{\scriptstyle 1\otimes ad} & & \downarrow{\scriptstyle ad} \\ \pi_n S^n \otimes \pi_i(\Omega^k S^n) & \xrightarrow{\{,\}} & \pi_{n+i-1}(\Omega^k S^n) \\ \downarrow{\scriptstyle h\otimes h} & & \downarrow{\scriptstyle h} \\ H_n(S^n)\otimes H_i(\Omega^k S^n) & \xrightarrow{d_i^{n,i}} & H_{n+i-1}(\Omega^k S^n) \end{array}$$

where here we have identified $H_n(S^n)\otimes H_i(\Omega^k S^n)$ with $E^2_{n,i}$ and $H_{n+i-1}(\Omega^k S^n)$ with $E^2_{0,n+i-1}$. The situation is most interesting when $i = n-k$. The top map in this case is given by the Whitehead square $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$ and the Hurewicz map $h : \pi_{n-k}(\Omega^k S^n) \to H_{n-k}(\Omega^k S^n)$ is an isomorphism.

Remark 4.4 (The Whitehead Square):
- When n is odd, the Whitehead square is two torsion; i.e. $2[\iota_n, \iota_n] = 0$ (this follows from the commutation relation $[\alpha,\beta] = (-1)^{|a||b|}[\beta,\alpha]$). Naturally this Whitehead square is zero if and only if S^n is an H-space and this corresponds to the values of $n = 1, 3, 7$.
- When $n = 2q$, $[\iota_{2q}, \iota_{2q}]$ generates an infinite cyclic group in

$$\pi_{4q-1}(S^{2q}) \cong \mathbf{Z} \oplus \text{torsion}.$$

This follows from the fact that there is a homomorphism $H : S^{4q-1} \to S^{2q}$ (the Hopf invariant) for which $H([\iota_{2q}, \iota_{2q}]) = 2$. The copy of $\mathbf{Z}$ in $\pi_{4q-1}(S^{2q})$ is then identified with the even integers by H. This Whitehead square is twice another class τ if and only if τ has Hopf invariant one in which case $q = 1, 2$ or 4.

Lemma 4.5: *Assume $1 \leq k < n$. Then under the composite*

$$\theta : \pi_{2n-1}S^n \xrightarrow{ad} \pi_{2n-k-1}(\Omega^k S^n) \xrightarrow{h} H_{2n-k-1}(\Omega^k S^n),$$

the Whitehead square maps as follows

$$\theta([\iota_n, \iota_n]) = \begin{cases} 0 & n \text{ is odd,} \\ 2x & n \text{ is even} \end{cases}$$

(here x is the infinite cyclic element in $H_{2n-k-1}(\Omega^k S^n; \mathbf{Z})$, n even.)

Proof: Since $[\iota_n, \iota_n]$ is two torsion for n odd, the claim in this case follows mod-p, p odd and also rationally. For the case n even, we refer to [C2] for details. One however need convince himself that θ maps indeed to an infinite cyclic element mod-p (similary mod-$\mathbf{Q}$). To see this, we use the fact that $[\iota_n, \iota_n]$ generates an infinite cyclic summand in $\pi_{2n-1}(S^n)$ and hence by adjointing it generates a torsion free generator $\beta \in \pi_{2n-k-1}\Omega^k S^n$. On the other hand, we know by [S] that loops on an even sphere split after localizing at any odd prime p;

4.6 $$\Omega^k S^n \simeq_{(p)} \Omega^{k-1} S^{n-1} \times \Omega^k S^{2n-1}$$

Under this equivalence, the generator β maps to the generator of $\pi_{2n-k-1}\Omega^k S^{2n-1} \cong \mathbf{Z}_p$ (remember the spaces are localized). It follows that by composing with the mod-p Hurewicz map $h : \pi_{2n-k-1}\Omega^k S^{2n-1} \longrightarrow H_{2n-k-1}\Omega^k S^{2n-1} \cong \mathbf{Z}_p$ we get the non-trivial generator. The composite

$$\pi_{2n-1}(S^n) \xrightarrow{ad} \pi_{2n-k-1}(\Omega^k S^n) \xrightarrow{pr} \pi_{2n-k-1}\Omega^k S^{2n-1} \xrightarrow{h} H_{2n-k-1}\Omega^k S^{2n-1} \hookrightarrow H_{2n-k-1}(\Omega^k S^n)$$

maps $[\iota_n, \iota_n]$ to an infinite cyclic generator as desired. ∎

We can now prove proposition 1.2 of the introduction.

Proposition 4.7: *Assume $1 \leq k < n$ and n is even. Then in the Serre spectral sequence for the fibration $\Omega^k S^n \xrightarrow{i} \Lambda^k S^n \xrightarrow{ev} S^n$, the differential $d^n_{n,n-k}$ is given by multiplication by 2 on the torsion free generator of $H_{2n-k-1}(\Omega^k S^n)$. In particular, $d^n_{n,n-k}$ is an isomorphism with rational coefficients.*

Proof: The differential $d^n_{n,n-k}$ is determined according to diagram 4.3 by the image of the Whitehead square under the map θ described in 4.5. The claim now follows from lemma 4.5. ∎

4.1 Rational and Mod-2 Calculations

The mod-2 cohomology of $\Lambda^k S^n$, $k < n$ is completely determined according to the following lemma

Lemma 4.9: *The Serre SS for* $\Omega^k S^n \longrightarrow \Lambda^k S^n \longrightarrow S^n$ *collapses with mod-2 coefficients whenever* $k < n$.

Proof: The following short and quite lovely argument was provided to us by F. Cohen [C]. Consider the suspension $\Omega^n E : \Omega^n S^{n+q} \to \Omega^{n+1} S^{n+q+1}$ and the following induced map of fibrations

$$\begin{array}{ccc} \Omega^n S^{n+q} & \xrightarrow{\Omega^n E} & \Omega^{n+1} S^{n+q+1} \\ \downarrow & & \downarrow \\ \Lambda^n(S^{n+q}) & \xrightarrow{\Lambda^n(E)} & \Lambda^n \Omega S^{n+q+1} \\ \downarrow & & \downarrow \\ S^{n+q} & \xrightarrow{E} & \Omega(S^{n+q+1}). \end{array}$$

Since ΩS^{n+q+1} is an H-space, then so is $\Lambda^n \Omega S^{n+q+1}$ and consequently we have a splitting

$$\Lambda^n \Omega S^{n+q+1} \simeq \Omega S^{n+q+1} \times \Omega^{n+1} S^{n+q+1}.$$

It is known (cf. [C3], pp. 228-231) that the map $\Omega^i E$ is injective in mod-2 homology (for all i) and hence in the diagram above both fiber and base inject in $\mathbf{Z}_2$-homology. The Lemma follows. ∎

We now use proposition 4.7 to calculate $H^*(\Lambda^k S^n)$ with rational coefficients. We also give a complete answer mod-p (p odd) for the case of a two fold loop space. We make use throughout of the following standard fact. Consider the path-loop fibration $\Omega^k S^n \longrightarrow P \longrightarrow \Omega^{k-1} S^n$ for $k < n$. Then

4.8 $$H^*(\Omega^k S^n) = Tor^{H^*(\Omega^{k-1} S^n)}(\mathbf{F}, \mathbf{F})$$

This follows because the Eilenberg-Moore spectral sequence collapses at the E^2 term (cf. [CM]).

Proposition 4.10: *Let* $1 \leq k < n$ *and suppose* n *even. then the Poincaré series for* $H^*(\Lambda^k S^{2n}; \mathbf{Q})$ *is given as follows*

$$\begin{cases} 1 + (x^n + x^{n-k})/(1 - x^{n-k-1}) & , k \text{ is odd} \\ (1 + x^{3n-k-1})/(1 - x^{n-k}) & , k \text{ is even.} \end{cases}$$

Proof: When n is even, one has $H^*(\Omega S^n) = \Lambda(e_{n-1}) \otimes \mathbf{Q}(a_{2n-2})$, where $\Lambda(e_{n-1})$ is an exterior algebra on an $n-1$ dimensional generator. It is easy to see (see §4) that

$$Tor^{\Lambda(e_{n-1})}(\mathbf{Q}, \mathbf{Q}) = \mathbf{Q}(e_{n-2}), \quad \text{and} \quad Tor^{\mathbf{Q}(a_{2n-2})}(\mathbf{Q}, \mathbf{Q}) = \Lambda(a_{2n-3})$$

Iterating these constructions yields

$$H^*(\Omega^k S^n; \mathbf{Q}) = \begin{cases} \mathbf{Q}(e) \otimes \Lambda(a), & k \text{ even} \\ \Lambda(e) \otimes \mathbf{Q}(a), & k \text{ odd} \end{cases}$$

where $\deg(e) = n - k$ and $\deg(a) = 2n - k - 1$. In the Serre spectral sequence for 4.1 with $\mathbf{Q}$ coefficients, the class a hits $e\iota$ and this differential generates all other differentials. When k is odd, one has (up to a unit)

$$d(a^k) = e\iota a^{k-1}, \quad d(ea^k) = e^2 \iota a^{k-1} = 0$$

and the classes that survive are ea^k and xa^k, $k = 1, 2, \ldots$. The first claim follows. When k is even, $H^*(\Lambda^k S^n; \mathbf{Q}) \cong \mathbf{Q}(e)[a\iota, 1]$ and that leads to the second assertion. ∎

REMARK: The Poincaré series for ΛS^n; $(1 + x^n + x^{n-k} - x^{n-k-1})/(1 - x^{n-k-1})$, is well-known and is given for instance in [Ro].

§4.2 Second fold (free) loop spaces

We here show that the Serre SS for the fibration $\Omega^2(S^{2q+2}) \longrightarrow \Lambda^2 S^{2q+2} \longrightarrow S^{2q+2}$ has only one non-zero differential at the E^{2q+2} term. This recovers an unpublished calculation of F. Cohen (we work below with mod-p coefficients.)

Recall first the description of $\Omega^2 S^{2q+2}$ over the mod-p Steenrod algebra (see [C3] or [R] for a general discussion). We have that

$$\Omega^2 S^{2q+2} \simeq \Omega S^{2q+1} \times \Omega^2 S^{4q+3}$$

(this follows from 4.6), and that $H^*(\Omega^2 S^{4q+3})$ is given by

$$H^*(\Omega^2 S^{4q+3}) = \Lambda(x_0, x_1, \ldots) \otimes \Gamma(y_1, y_2, \ldots)$$

where $|x_i| = 2(2q + 1)p^i - 1$ and $|y_i| = 2(2q + 1)p^i - 2$. The action of the Steenrod algebra is given by

$$\beta(y_i) = x_i, \quad \text{and} \quad \mathcal{P}^1(y_i^p) = y_{i+1}.$$

EXAMPLE: We work out below the details for $\Omega^2 S^4$ (the general case being identical). Start with the Hopf fibration $S^3 \to S^7 \to S^4$ and observe that it has to split after looping $\Omega S^4 \simeq S^3 \times \Omega S^7$ (see §2). This gives an algebra isomorphism

$$H^*(\Omega S^4) \cong H^*(S^3) \otimes H^*(\Omega S^7) \cong \Lambda(\epsilon) \otimes \Gamma(a).$$

Here $\Gamma(a)$ is a divided power algebra on a six dimensional generator while $\Lambda(\epsilon)$ is the exterior algebra on a three dimensional generator. According to 4.8

$$H^*(\Omega^2 S^4, \mathbf{Z}_p) \cong Tor^{\Lambda(\epsilon)}(\mathbf{Z}_p, \mathbf{Z}_p) \otimes Tor^{\Gamma(a) \otimes \mathbf{Z}_p}(\mathbf{Z}_p, \mathbf{Z}_p)$$

It is easy to see that a minimal resolution for $\Lambda(\epsilon)$ is generated by $|\epsilon|, |\epsilon|\epsilon|, \ldots$ (sub-resolution of the Bar construction) and these elements generate a divided power algebra (where the algebra structure comes from the shuffle product in the Bar construction), so one has

4.11 $$Tor^{\Lambda(\epsilon)}(\mathbf{Z}_p, \mathbf{Z}_p) \cong \Gamma(|\epsilon|).$$

On the other hand (and mod-p) one can verify that $\Gamma(a)$ splits as a product of truncated polynomial algebras (a result of H. Cartan)

$$\Gamma(a) \otimes \mathbf{Z}_p \cong P_T(a, p) \otimes \cdots \otimes P_T(\gamma_{p^i}, p) \otimes \cdots$$

where $P_T(a, p)$ denotes the truncated polynomial algebra $\mathbf{Z}_p(a)/a^p = 0$. It can be easily checked using the Bar construction (see [K1]) that

$$Tor^{P_T(a,p)}(\mathbf{Z}_p, \mathbf{Z}_p) = \Lambda(|a|) \otimes \Gamma(|a^{p-1}|a|)$$

This then combines with 4.11 to yield the following description of $H^*(\Omega^2 S^4, \mathbf{Z}_p)$ as an algebra.

Lemma 4.12: *$H^*(\Omega^2 S^4, \mathbf{Z}_p) \cong \Gamma(\epsilon) \otimes \Lambda(x_0, x_1, \ldots,) \otimes \Gamma(y_1, y_2, \ldots)$ where e is of degree 2, the x_i's are of degree $6p^i - 1$ and the y_i's of degree $6p^i - 2$.*

Here of course $e = |\epsilon|$, $x_i = |\gamma_{p^{i+1}}|$ and $y_i = |\gamma_{p^i}^{p-1}|\gamma_p|$. It isn't hard to see that as a module over the Hopf algebra, $H^*(\Omega^2 S^4, \mathbf{Z}_p)$ is generated by e and the x_i (here $\beta x_i = y_i$ where β is the mod-p Bockstein).

Theorem 4.13: *In the cohomology Serre spectral sequence for $\Omega^2 S^{2q+2} \to \Lambda^2 S^{2q+2} \to S^{2q+2}$ we have that*

$$d_{4q+1} x_0 = e \cdot \iota,$$

where e is the generator of $H^{2q}(\Omega S^{2q+1})$ in the fiber and ι is the generator of $H^{2q+2}(S^{2q+2})$ in the base.

Proof: The differential d^{4q+1} is described by 4.7 and is non-trivial. The differentials vanish on the y's by dimensions argument. It follows that there are no non-zero differentials on the x_i's, $i \geq 1$ since $dx_i = d(\beta y_i) = \beta d y_i = 0$. The claim follows. ∎

§5 The Space of Rational Maps of the Riemann Sphere

We here give a short proof of a theorem of Havlicek on the structure of the space $Hol(\mathbf{P}^1, \mathbf{P}^n)$ of holomorphic maps (unbased) from the Riemann sphere into complex projective space. We start with a lemma

Lemma 5.1: *Let $F \to E \to B$ be a fibration over a simply connected space B. Assume all elements of $H_n(B; \mathbf{A})$ are transgressive. Then the transgression is given according to the diagram*

$$\begin{array}{ccc} \pi_n(B) & \xrightarrow{\partial} & \pi_{n-1}(F) \\ \downarrow h & & \downarrow h \\ H_n(B) & \xrightarrow{\tau} & E^{0,n-1} \end{array}$$

where here $E^{0,n-1}$ is thought of as a quotient of $H_{n-1}(F)$.

Proof: The following diagram commutes

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \pi_i(F) & \longrightarrow & \pi_i(E) & \longrightarrow & \pi_i(E,F) \cong \pi_i(B) & \xrightarrow{\partial} & \pi_{i-1}(F) & \longrightarrow \cdots \\ & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & \\ \cdots \longrightarrow & H_i(F) & \longrightarrow & H_i(E) & \longrightarrow & H_i(E,F) & \xrightarrow{\partial} & H_{i-1}(F) & \longrightarrow \cdots \\ & & & & & \downarrow p_* & & \downarrow & \\ & & & & & H_i(B) & \xrightarrow{\tau} & E^{0,i-1} & \end{array}$$

and the lemma follows. ∎

Lemma 5.2: *The d_2 differential for the (cohomology) Serre spectral sequence associated to $\Omega^2_k S^2 \to \Lambda^2_k S^2 \to S^2$ is of the form*

$$d_2(x) = 2k\iota$$

where x is the generator of $H^1(\Omega^2 S^3) \cong \mathbf{Z}$. The SS collapses at E^2 with mod-2 coefficients and at E^3 mod-p, p odd.

PROOF: Let $k : S^2 \longrightarrow S^2$ denote multiplication by k. From the previous lemma we deduce that the transgression on $\iota \in H_2(S^2)$ is given as the image under the Hurewicz homomorphism of the Whitehead product

$$[k, \iota] = k[\iota, \iota] = 2k \in \pi_3(S^2) \cong \pi_1(\Omega^2_k S^2) \cong \mathbf{Z}$$

Note that all components of $\Omega^2 S^2$ are equivalent to $\Omega^2_0 S^2 \simeq \Omega^2 S^3$. Since $\pi_1 \Omega^2_0 S^2$ is abelian, the Hurewicz map is an isomorphism and one has that $\tau(\iota_2) = d_2(\iota_2) = 2kx$ where τ is the transgression. Dualizing in cohomology we see that $\tau(x) = 2k\iota_2$. We now have that

$$H^*(\Omega^2 S^3; \mathbf{Z}_2) \cong \Lambda(q_1, \ldots, q_{2^{i+1}-1}, \ldots)$$

where the right hand side denotes an exterior algebra on (odd) generators q_i of dimensions $2^{i+1} - 1$ ($q_1 = x$). A quick inspection of the mod-2 quadrant for the serre spectral sequence shows that the only possible differential is the one going from q_1 to ι_2 in the base. But this is multiplication by $2k$ and hence is zero mod-2. The first claim of the lemma follows.

The very same argument applies for the case $p > 2$. In this case

$$H^*(\Omega^2 S^3; \mathbf{Z}_2) \cong \Lambda(q_1, \ldots, q_{2p^i-1}, \ldots) \otimes P_T(\beta q_1, \ldots, \beta q_i, \ldots)$$

where now the q_i have degree $2p^j - 1$ and the additional term is given by a truncated polynomial algebra (that is a polynomial algebra on the indicated generators with $(\beta q_i)^p = 0$.) Since the differentials on $q_i, i > 1$ vanish, they must also vanish on βq_i and the second claim follows. ∎

We now turn to the space of meromorphic functions on $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ or $\mathrm{Hol}(\mathbf{P}^1)$. This space can be regarded as as a subspace of $\Omega^2 S^2$ and hence is also graded by degree. Inside $\mathrm{Hol}(\mathbf{P}^1)$ we let $\mathrm{Rat}(\mathbf{P}^1)$ be the subspace of maps sending the north pole ∞ to 1. An element of $\mathrm{Rat}(\mathbf{P}^1)$, say of degree k, is identified with the quotient $\frac{p}{q} = \frac{z^k + a_{k-1}z^{k-1} + \cdots + a_0}{z^k + b_{k-1}z^{k-1} + \cdots + b_0}$ where p and q have no roots in common. It is easy to see for example that $\mathrm{Hol}_1(\mathbf{P}^1)$ corresponds to $PSL(2, \mathbf{C})$, the automorphism group of $\mathbf{P}^1$ (which is up to homotopy RP^3), and that $\mathrm{Rat}_1(\mathbf{P}^1) = \mathbf{C} \times \mathbf{C}^* \simeq S^1$.

Consider now the map of fibrations

5.3
$$\begin{array}{ccc} \mathrm{Rat}_k(\mathbf{P}^1) & \hookrightarrow & \Omega^2_k S^2 \\ \downarrow & & \downarrow \\ \mathrm{Hol}_k(\mathbf{P}^1) & \hookrightarrow & \Lambda^2_k S^2 \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{=} & S^2. \end{array}$$

According to Segal [S], the top inclusion is an isomorphism in homology group up to dimension k. In [C^2M^2] and also in [K1] it is shown that $H_*(\mathrm{Rat}_k(\mathbf{P}^1))$ actually injects in $H_*(\Omega^2_0 S^2)$ and it does so in the following nice way. Recall that $\Omega^2_0 S^2 \simeq \Omega^2 S^3 = \Omega^2\Sigma^2 S^1$ and hence it stably splits as an infinite wedge $\bigvee_{j\geq 1} D_j$ where the summands D_j are given in terms of configuration spaces with labels. It now turns out that (stably)

$$\mathrm{Rat}_k(\mathbf{P}^1) \simeq_s \bigvee_{j=1}^{k} D_j.$$

It is known that $H_1(\mathrm{Rat}_1(\mathbf{P}^1;\mathbf{Z})) \cong \pi_1(\mathrm{Rat}_1(\mathbf{P}^1)) \cong \mathbf{Z}$. Since the map of the fibers in 5.3 is an injection, it is direct to see by comparison of spectral sequences that the transgression on the orientation class of the base (in the left hand fibration in 5.3) is also given by multiplication by $2k$, and that all other differentials are trivial. This then shows that

Corollary 5.5 (Havlicek): *The Serre SS for the fibration*

$$Rat_k(\mathbf{P}^1) \longrightarrow Hol_k(\mathbf{P}^1) \longrightarrow \mathbf{P}^1$$

has only one non-zero differentials $d_2(x) = 2k\iota$. It follows in particular that the spectral sequence collapses with mod-p coefficients whenever $p = 2$ of p divides k.

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