LIMIT CLOSURES OF CLASSES OF COMMUTATIVE RINGS

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Abstract. We study and, in a number of cases, classify completely the limit closures in the category of commutative rings of subcategories of integral domains.

1. Introduction.

In a paper to appear [Barr et al. (2015)], we have studied the following general question: Given a complete (respectively, cocomplete) category $\mathscr{C}$ and a full subcategory $\mathscr{A}$, what is the smallest limit closed (respectively, colimit closed) subcategory of $\mathscr{C}$ that contains $\mathscr{A}$?

This paper studies the question for several categories of integral domains as subcategories of commutative rings, which leads to interesting problems. Section 2 contains some general results. Section 3 gives conditions that the limit closure be the ring of global sections of a sheaf with stalks that are domains in the limit closure of $\mathscr{A}$. The base of the sheaf is the spectrum of all prime ideals, with a topology between the domain topology and the patch topology, as defined in 2.2.20. This is analogous to the known fact that commutative von Neumann regular rings are characterized as the global sections of a sheaf of fields. Section 4 characterizes rings that are in the limit closure of the subcategory of all domains, thereby clarifying an earlier paper on this subject, see [Kennison 1976]. Section 5 gives a simple necessary and sufficient condition for a ring to be in the limit closure of the subcategory of domains that are integrally closed (in their field of fractions). The same condition characterizes rings that are in the limit closure of GCD domains (ones in which every pair of elements has a greatest common divisor) and also rings in the limit closure of Bézout domains (ones in which every finitely generated ideal is principal). Section 6 defines perfect domains and characterizes their limit closure. Section 7 explores the limit closure of UFDs finding two necessary conditions for a ring to be in this limit closure, but no sufficient conditions. We show that the limit closure of UFDs is not closed under ultraproducts and therefore cannot be characterized by first-order conditions, see 2.4. We do show that every quadratic extension of the ring $\mathbf{Z}$ of integers is in the limit closure of UFDs. The final section summarizes the results of the paper and mentions some open problems.

In this paper, $\mathscr{C}$ is usually the category $\mathcal{CR}$ of commutative rings and $\mathscr{A}$ is a full subcategory of integral domains. A commutative ring is called **semiprime** or **reduced**

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if it has no non-zero nilpotents.

2. Domain induced subcategories.

2.1. PRELIMINARIES. The rest of this paper is concerned with subcategories of commutative rings generated by a full subcategory of domains. It is well known (and easily proved) that the set of nilpotents in a commutative ring is just the intersection of all the prime ideals, so that a ring is semiprime if and only if the intersection all the prime ideals is 0. Since every domain is semiprime, so is any ring in the limit closure of any class of domains. The semiprime rings are clearly reflective (factor out the ideal of nilpotent elements), *so it will be convenient to assume that all our rings are semiprime, unless otherwise specified.*

An ideal I of a commutative ring R is called **radical** or **semiprime** if R/I is semiprime. An ideal of a commutative ring is radical if and only if it is an intersection of primes. This is equivalent to saying that if a power of an element lies in the ideal then the element does. If $I \subseteq R$ is any ideal, we denote by $\sqrt{I}$ the set of all elements of R for which some power lies in A. This is the same as the meet of all prime ideals that contain it and is also the least radical ideal containing I.

We say that a category $\mathscr{K}$ of semiprime rings is **domain induced** if $\mathscr{K}$ is the limit closure of a subcategory of domains $\mathcal{A}$ such that every domain (not just those of $\mathcal{A}$) can be embedded into a field in $\mathcal{A}$.

Since every semiprime ring can be embedded into a product of fields and every field can be embedded into a field in $\mathcal{A}$, it follows that $\mathcal{A}$ cogenerates $\mathcal{SPR}$.

2.1.1. Examples. Here are the main examples of the subcategories of domains we will be studying in this paper. In most, although not all, of these cases we will characterize the limit closure of these subcategories.

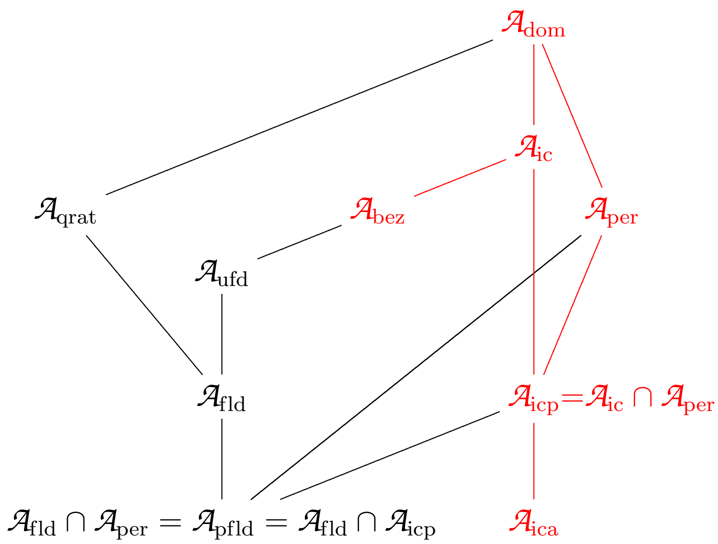
1. $\mathcal{A}_{\mathrm{dom}}$, the category of domains;
2. $\mathcal{A}_{\mathrm{fld}}$, the category of fields;
3. $\mathcal{A}_{\mathrm{pfld}}$, the category of perfect fields;
4. $\mathcal{A}_{\mathrm{ic}}$, the category of domains integrally closed in their field of fractions;
5. $\mathcal{A}_{\mathrm{bez}}$, the category of Bézout domains;
6. $\mathcal{A}_{\mathrm{ica}}$, the category of domains integrally closed in the algebraic closure of their field of fractions;
7. $\mathcal{A}_{\mathrm{icp}}$, the category of domains integrally closed in the perfect closure of their field of fractions;
8. $\mathcal{A}_{\mathrm{per}}$, the category of perfect domains: those that are either of characteristic 0 or are characteristic p and every element has a pth root;

9. $\mathcal{A}_{\mathrm{qrat}}$, the category of domains in which every integer has a quasi-inverse, that is, for each integer d, there is an element d', such that $d^2d' = d$;

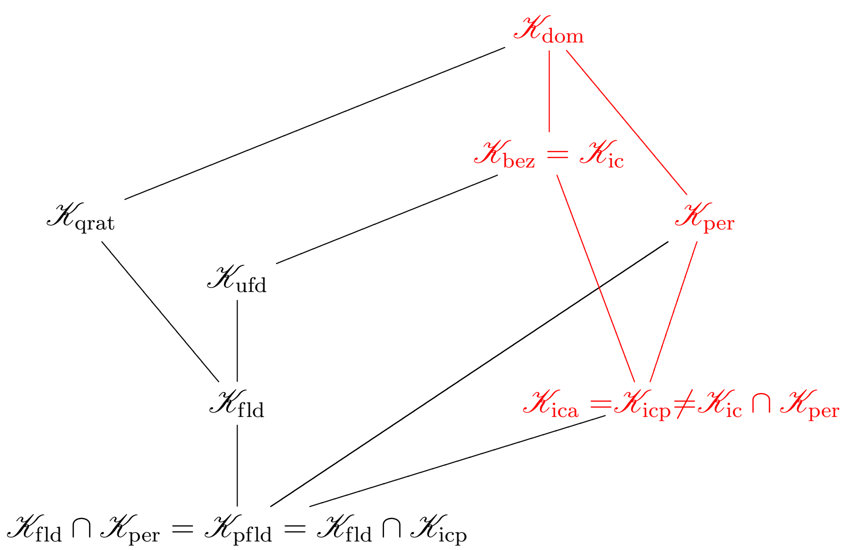
10. $\mathcal{A}_{\mathrm{noe}}$, the category of Noetherian domains;

11. $\mathcal{A}_{\mathrm{ufd}}$, the unique factorization domains.

There are some relations among these subcategories as shown in the following poset of inclusions. The red spine marks the $\mathcal{D}om$-**invariant** categories which will be defined in the sentence preceding Theorem 2.3.1.



When we form the limit completions of these categories, we get a somewhat different diagram (in which $\mathscr{K}_{\mathrm{xx}}$ is the limit completion of $\mathcal{A}_{\mathrm{xx}}$).



2.2. Preliminary results: G and K. We will let $\mathcal{A}$ denote a category of domains that satisfy the conditions in 2.1 and let $\mathscr{K}$ be its limit closure. *If D is a domain, we denote by $\mathbf{Q}(D)$ its field of fractions.*

2.2.1. Proposition. *Suppose $D \subseteq D_1$ and $D \subseteq D_2$ are domains. Then there is a commutative square*

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, hook] & D_1 \arrow[d, hook] \\ D_2 \arrow[r, hook] & F \end{tikzcd}$$

in which $F \in \mathcal{A}$ is a field.

Proof. We can assume without loss of generality that D_1 and D_2 are fields, in which case they both contain copies of $\mathbf{Q}(D)$. Then $D_1 \otimes_{\mathbf{Q}(D)} D_2 \neq 0$ since both factors are non-zero vector spaces over $\mathbf{Q}(D)$. Let M be a maximal ideal of $D_1 \otimes_{\mathbf{Q}(D)} D_2$ and let $F \in \mathcal{A}$ be a field containing $(D_1 \otimes_{\mathbf{Q}(D)} D_2)/M$. ∎

2.2.2. Notation. Let D be a domain and let $D \hookrightarrow F$ be an embedding into a field $F \in \mathcal{A}$. Let $G(D)$ denote the meet of all $\mathscr{K}$-subobjects of F that contain D. For each domain D, we let $\alpha_D : D \hookrightarrow G(D)$ denote the embedding. The operation G is not generally a functor. If it is, we will see later that it is actually the reflector on domains. In general, we have:

2.2.3. Proposition. *The domain $G(D)$ is independent of the choice of F. Furthermore, if $D \subseteq B$ where B is a domain in $\mathscr{K}$, then $G(D)$ is isomorphic to the meet of all $\mathscr{K}$-subobjects of B which contain D.*

Proof. Suppose F_1, F_2 are two fields containing D and belonging to $\mathcal{A}$ and $G_1(D)$ and $G_2(D)$ are the corresponding subobjects of F_1 and F_2 as above. By the previous proposition, up to isomorphism, both F_1 and F_2 are subfields of a field $F \in \mathcal{A}$ that contain D and hence so is $F_1 \cap F_2$. But $G_1(D) \cap G_2(D)$ is a $\mathscr{K}$-subobject of F_1 and of F_2 that contains D and, by minimality, they are equal. Since every $B \in \mathscr{B}$ is contained in some field in $\mathcal{A}$ the second conclusion follows. ∎

2.2.4. Notation. We denote by $\overline{\mathbf{Q}}(D)$ the perfect closure of the field $\mathbf{Q}(D)$ of fractions of D. If D has characteristic 0, this means that $\overline{\mathbf{Q}}(D) = \mathbf{Q}(D)$, while if D has characteristic $p > 0$, it consists of all the elements of the algebraic closure for which a p^eth power lies in $\mathbf{Q}(D)$ for some positive integer e.

2.2.5. Lemma. *The map $\alpha_D : D \longrightarrow G(D)$ is epic in $\mathcal{SPR}$.*

Proof. If not, then there exist maps $g, h : G(D) \longrightarrow F$ where F is a field and $g\alpha_D = h\alpha_D$ but $g \neq h$. We may assume that $F \in \mathcal{A}$, otherwise we can embed F in such a field. It follows that the equalizer of g and h is in $\mathscr{K}$ and is also a proper subring of $G(D)$ which contains D. But this contradicts the definition of $G(D)$. ∎

2.2.6. Proposition. *Suppose $D \subseteq D'$ are domains such that $D' \subseteq G(D)$. Then $G(D') = G(D)$.*

Proof. Embed D' into a field $F \in \mathcal{A}$. By definition $G(D)$ is the meet of all $\mathscr{K}$-subobjects of F that contain D and similarly for D'. Clearly any $\mathscr{K}$-subobject of F that contains D' also contains D, while any $\mathscr{K}$-subobject of F that contains D also contains $G(D)$ and therefore contains D'. ∎

2.2.7. Lemma.

1. *Let F be a field and D a domain. A map $F \longrightarrow D$ is epic in $\mathcal{SPR}$ if and only if $D \subseteq \overline{\mathbf{Q}}(F)$.*

2. *Every perfect field is in $\mathscr{K}$.*

3. *For each domain D, we have $D \subseteq G(D) \subseteq \overline{\mathbf{Q}}(D)$.*

Proof.

1. Since $D \longrightarrow \mathbf{Q}(D)$ is epic, it suffices to show that a map $F \longrightarrow E$ between between fields is epic in $\mathcal{SPR}$ if and only if E is a purely inseparable extension of F. One direction is trivial in $\mathcal{SPR}$. So suppose $F \longrightarrow E$ is epic. Factor the map as $F \longrightarrow F_1 \longrightarrow F_2 \longrightarrow E$ so that F_1 is an extension of F by a transcendence basis for E over F, F_2 is a purely inseparable extension of F_1 and E is a separable extension of F_2. Since epics are left cancellable, we have the $F_2 \longrightarrow E$ is also epic. But F_2 is the equalizer of all the maps of E into its algebraic closure that fix F_2, so that we have $F_2 = E$. Since the map from F_2 to its perfect closure is purely inseparable and epic, we can reduce to the case that F_2 is perfect. If T is a transcendence basis of F_1, the automorphism $\sigma : F_1 \longrightarrow F_1$ defined by $\sigma(t) = t+1$ for each $t \in T$ can have no fixed point outside of F since $\sigma(a) = a$, with $a \notin F$ would give a polynomial relation on T. Thus F is the equalizer of σ and id. For $a \in F_2$, there is some positive integer k such that $a^{p^k} \in F_1$. Since F_2 is perfect, the element $\sigma(a^{p^k})$ has a unique p^kth root in F_2 which we call $\overline{\sigma}(a)$. If we do this for each element of F_2, this results in an endofunction $\overline{\sigma}$ of F_2, which is easily seen to be an automorphism. Thus if $F \neq F_1$, there are non-trivial automorphisms of F_2 over F which contradicts the map's being epic. Thus $F = F_1$ and E is a purely inseparable extension.

2. We know from 2.2.5 that if F is a field, $F \longrightarrow G(F)$ is epic in $\mathcal{SPR}$. But a perfect field has no proper epic extension.

3. This is now immediate from Proposition 2.2.3. ∎

2.2.8. Corollary. *If $C \subseteq D$ is an inclusion of domains that is epic in $\mathcal{SPR}$, then D is contained in the perfect closure of the field of fractions of C.* ∎

2.2.9. Definition. *Given an element d in a domain D, we say that d^ℓ is a* **characteristic power** *of d if $\ell = 1$ or the characteristic p of D is positive and $\ell = p^e$ for some $e > 0$.*

2.2.10. Corollary. *Let D be a domain. If $z \in G(D)$ then there exist $w, v \in D$, with $w \neq 0$ such that $wz^\ell - v = 0$ where z^ℓ is a characteristic power of z.*

Proof. If $z \in G(D)$ then $z \in \overline{\mathbf{Q}}(D)$ and hence some characteristic power, $z^\ell \in \mathbf{Q}(D)$. Thus $z^\ell = v/w$ for some fraction $v/w \in G(D)$, the result follows. ∎

2.2.11. Proposition. *The set of all $G(R/P)$ taken over all the prime ideals $P \subseteq R$ is a solution set for maps $R \longrightarrow A$ where $A \in \mathcal{A}$.*

Proof. Suppose that $f : R \longrightarrow A$ is a homomorphism with $A \in \mathcal{A}$. Let $P = \ker(f)$. Since all objects of $\mathcal{A}$ are domains, P is prime. Clearly f factors through R/P. By Proposition 2.2.1 we have a commutative square

$$\begin{tikzcd} R/P \arrow[r, hook] \arrow[d, hook] & G(R/P) \arrow[d, hook] \\ A \arrow[r, hook] & F \end{tikzcd}$$

with $F \in \mathcal{A}$. But then the pullback $A \times_F G(R/P)$ is a $\mathscr{K}$-subobject of $G(R/P)$ that contains R/P. But then the pullback is $G(R/P)$ which gives the required map. ∎

It is well known that this implies:

2.2.12. Theorem. *The inclusion $\mathscr{K} \hookrightarrow \mathcal{SPR}$ has a left adjoint.* ∎

We will denote the left adjoint by K and the inner adjunction by $\eta : \mathrm{Id} \longrightarrow K$.

2.2.13. Proposition. *A map $g : R \longrightarrow S$ is the reflection of R into $\mathscr{K}$ if and only if $S \in \mathscr{K}$ and g has the unique extension property with respect to every $A \in \mathcal{A}$.*

Proof. The subcategory of objects with respect to which that property holds includes $\mathcal{A}$ by hypothesis and is clearly closed under limits and therefore includes the limit closure of $\mathcal{A}$. ∎

2.2.14. Proposition. *For every semiprime ring R, ηR is an epimorphic embedding in $\mathcal{SPR}$.*

Proof. Suppose we have $R \xrightarrow{\eta R} K(R) \overset{f}{\underset{g}{\rightrightarrows}} S$ with $S \in \mathcal{SPR}$ and $f.\eta R = g.\eta R$. Since S can be embedded in a product of fields in $\mathcal{A}$, we can easily reduce to the case that S is a field in $\mathcal{A}$ and then the uniqueness of the map $K(R) \longrightarrow S$ that extends $f.\eta R = g.\eta R$ implies that $f = g$.

Clearly ηR is an embedding when R is a field. More generally, let $R \subseteq \prod F_i$, a product of fields. From the commutativity of

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d] & \prod F_i \arrow[d, hook] \\ K(R) \arrow[r] & \prod K(F_i) \end{tikzcd}$$

we see that $\eta R : R \hookrightarrow K(R)$. ∎

2.2.15. Proposition. *Suppose $f : R \longrightarrow S$ is an epimorphism in $\mathcal{SPR}$. Then the induced map* $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ *is injective.*

Proof. Suppose that $Q_1, Q_2 \subseteq S$ were distinct prime ideals of S lying above P. From Proposition 2.2.1, we have a commutative diagram

$$\begin{tikzcd} R/P \arrow[r, hook] \arrow[d, hook] & S/Q_1 \arrow[d, hook] \\ S/Q_2 \arrow[r, hook] & F \end{tikzcd}$$

with $F \in \mathcal{A}$. But this gives two distinct maps—since they have different kernels—from S to an object of $\mathcal{A}$ that agree on R, which is not possible. ∎

2.2.16. Proposition. *For any semiprime ring R, the map $\eta R : R \longrightarrow K(R)$ induces a bijection* $\mathrm{Spec}(K(R)) \longrightarrow \mathrm{Spec}(R)$.

Proof. Injectivity follows from the preceding proposition. To see that it is surjective, let $P \in R$ be prime. From the diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dd] & K(R) \arrow[d] \\ & K(R/P) \arrow[d] \\ R/P \arrow[r, hook] & G(R/P) \end{tikzcd}$$

and the fact that $G(R/P)$ is a domain, we see that the kernel of $K(R) \longrightarrow G(R/P)$ is a prime of $K(R)$ lying above P. ∎

Note that this bijection does not, in general, preserve order. The case that it does is special, see Theorem 2.3.1.7.

In the following $P \subseteq R$ is a prime and $P^{@} = \ker(K(R) \longrightarrow K(R/P))$. Since $K(R/P)$ is not always a domain (see Theorem 2.3.1 below), it is not always the case that $P^{@}$ is prime, but we do have:

2.2.17. Proposition. *If $P \subseteq Q$ are primes of R, then $P^{@} \subseteq Q^{@}$.*

Proof. This follows since we can fill in the diagonal map in the diagram

$$\begin{tikzcd} K(R) \arrow[r, two heads] \arrow[dd] & K(R)/P^{@} \arrow[d] \\ & K(R/P) \arrow[d] \\ K(R)/Q^{@} \arrow[r, hook] & K(R/Q) \end{tikzcd}$$

since the top arrow is surjective and the bottom arrow is injective.

2.2.18. Proposition. *Suppose $R \xhookrightarrow{f} S \xrightarrow{g} K(R)$ factors ηR with $S \in \mathscr{K}$. Then the meet of all $\mathscr{K}$-subobjects of S that contain R is isomorphic to $K(R)$.*

Proof. We can suppose without loss of generality that S has no proper $\mathscr{K}$-subobject that contains R. Given $h : R \longrightarrow A$ with $A \in \mathscr{A}$, there is a map $\widehat{h} : K(R) \longrightarrow A$ such that $\widehat{h}gf = h$. Then $\widehat{h}g : S \longrightarrow A$ is a map such that $\widehat{h}gf = h$. If there were more that one map $S \longrightarrow A$ with that property, then their equalizer would be a proper $\mathscr{K}$-subobject of S containing R, a contradiction. Thus S has the universal mapping property that defines $K(R)$. ∎

If $f : R \hookrightarrow S$ is a ring homomorphism, we say that f is an **essential ring homomorphism** if, for every homomorphism $g : S \longrightarrow T$, gf injective implies g is injective. Clearly this is the same as saying that when $I \subseteq S$ is a non-zero ideal, then $I \cap R \neq 0$.

2.2.19. Corollary. *Suppose ηS is essential. Then whenever $R \subseteq S \subseteq K(R)$, we have $K(S) = K(R)$.*

Proof. By adjunction we have a map $K(S) \longrightarrow K(R)$ such that

$$\begin{tikzcd} S \arrow[rr, hook] \arrow[dr, hook] & & K(R) \\ & K(S) \arrow[ur] & \end{tikzcd}$$

commutes. Essentiality implies that $K(S) \longrightarrow K(R)$ is injective. If $K(S) \hookrightarrow K(R)$ were a proper inclusion, this would contradict the proposition since $R \hookrightarrow S \hookrightarrow K(R)$ factors ηR. ∎

2.2.20. Definition. *If R is a ring, we will be considering four topologies on* $\mathrm{Spec}(R)$*. One is the familiar* **Zariski topology** *that has a base consisting of the sets* $Z(r) = \{P \in \mathrm{Spec}(R) \mid r \notin P\}$ *for* $r \in R$*. The second we will call the* **domain topology** *and has as a subbase the sets* $N(r) = \{P \in \mathrm{Spec}(R) \mid r \in P\}$ *for* $r \in R$*. The third, usually called the* **patch topology***, takes all the sets* $Z(r)$ *and* $N(r)$ *as a subbase This topology is known to be compact and Hausdorff (see [Hochster 1969]; see also [Barr et al. (2011), Proof of Theorem 2.4] where Hochster's argument is given in greater detail) and it follows that the other two are compact (but not usually Hausdorff). The fourth topology will be defined in 3.2.1 and lies between the domain and patch topologies and therefore is also compact.*

2.2.21. Lemma. *A subset of* $W \in \mathrm{Spec}(R)$ *is open in the domain topology if and only if it is open in the patch topology and up-closed in the subset ordering (meaning that* $P \in W$ *and* $P \subseteq Q$ *implies* $Q \in W$*).*

Proof. A base for the domain topology is given by the sets of the form $N(r_1, \ldots, r_n) = N(r_1) \cap \ldots \cap N(r_n)$ for $r_1, \ldots, r_n \in R$. Since all of these sets are open in the patch topology and up-closed in the subset ordering, it follows that if W is open in the domain topology, then W, being a union of basic subsets, is patch-open and up-closed in the subset ordering.

Conversely, assume that W is patch-open and up-closed in the subset ordering, but not open in the domain topology. Then there exists $P \in W$ such that there is no basic set with $P \in N(r_1, \ldots, r_n) \subseteq W$. So whenever $r_1, \ldots, r_n \in P$ we have $N(r_1, \ldots, r_n) - W$ is non-empty. It follows that there exists an ultrafilter $\mathbf{u}$ on $\mathrm{Spec}(R)$ such that $N(r_1, \ldots, r_n) - W \in \mathbf{u}$ whenever $r_1, \ldots, r_n \in P$. Let Q be the limit of $\mathbf{u}$ in the patch topology. Then whenever $r \in P$ we see that $N(r) \in \mathbf{u}$ so $r \in Q$ (otherwise $Z(r)$ is a patch-neighbourhood of Q which is not in $\mathbf{u}$). This implies that $P \subseteq Q$, so, by hypothesis, $Q \in W$. This leads to a contradiction because $\mathbf{u}$ converges to Q in the patch topology, while $\mathbf{u}$ does not contain W which is a neighbourhood of Q. ∎

2.2.22. Proposition. *Let Q be a prime ideal of R and U be a subset of* $\mathrm{Spec}(R)$ *that is compact in any topology in which sets $N(r)$ are open for all $r \in R$. If* $\bigcap_{P \in U} P \subseteq Q$*, then there is a* $P \in U$ *with* $P \subseteq Q$*.*

This applies, in particular, to the domain topology and the patch topology.

Proof. If not, let $r_P \in P - Q$ for each $P \in U$. The open sets $N(r_P)$ cover U and hence there is a finite set $P_1, \ldots, P_m$ such that every $P \in U$ contains at least one of $r_{P_1}, \ldots, r_{P_m}$ and then $r = r_{P_1} \cdots r_{P_m} \in \bigcap_{P \in U} P$, while the fact that Q is prime implies that $r \notin Q$. ∎

In the next part of this section, we will need what is described on the first page of [Dobbs (1981)] as a "Folk theorem".

2.2.23. Theorem. *For rings $R \subseteq T$, we have that T is an integral extension of R if and only if whenever* $R \subseteq S_1 \subseteq S_2 \subseteq T$*,*

1. $\mathrm{Spec}(S_2) \longrightarrow \mathrm{Spec}(S_1)$ *is surjective; and*
2. *If* $P \subseteq Q$ *are primes of* S_2 *with* $P \cap S_1 = Q \cap S_1$*, then* $P = Q$*.*

2.2.24. Corollary. *Assume $R \subseteq T$. If $R \subseteq S_1 \subseteq S_2 \subseteq T$ implies* $\mathrm{Spec}(S_2) \longrightarrow \mathrm{Spec}(S_1)$ *is bijective, then T is an integral extension of R.* ∎

2.2.25. Proposition. *Suppose $R \hookrightarrow S$ is epic and integral. Then* $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ *is an order isomorphism.*

Proof. We know it is injective from Proposition 2.2.15, while surjectivity follows from [Zariski & Samuel (1958), Vol. I, Theorem V.3]. The corollary to the same theorem implies that given any primes $P \subseteq Q$ of R and a prime $P^\sharp$ of S lying above P there is at least one prime $Q^\sharp$ of S lying above Q and such that $P^\sharp \subseteq Q^\sharp$. But since $Q^\sharp$ is the only prime lying above Q, we see that the induced map on Specs reflects order while it obviously preserves it. ∎

2.2.26. Notation. We will use the following convention. If a category of domains is denoted $\mathscr{A}_{\mathrm{xx}}$, we will systematically denote its limit closure by $\mathscr{K}_{\mathrm{xx}}$, the reflector by K_{xx} and the construction introduced in 2.2.2 by G_{xx}. For future reference, we also denote by $\mathscr{B}_{\mathrm{xx}}$ the full subcategory consisting of the domains in $\mathscr{K}_{\mathrm{xx}}$.

2.3. When does $G = K$? The main theorem of this section classifies domain induced subcategories that are characterized by $G = K$ on domains. We call them **$\mathscr{D}om$-invariant** for reasons that will become clear from the theorem below.

2.3.1. Theorem. *Let G be as in 2.2.2 and K be the reflector. Then the following are equivalent:*

1. *$G(D) = K(D)$ for all domains D.*
2. *For any domain D and any prime $P \subseteq D$, there is a map $G(D) \longrightarrow G(D/P)$ such that*

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & G(D) \arrow[d] \\ D/P \arrow[r, hook] & G(D/P) \end{tikzcd}$$

   commutes.

3. *The map* $\mathrm{Spec}(G(D)) \longrightarrow \mathrm{Spec}(D)$ *is surjective for all domains D.*
4. *G is a functor on domains in such a way that for $D \longrightarrow D'$*

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & G(D) \arrow[d] \\ D' \arrow[r, hook] & G(D') \end{tikzcd}$$

   commutes.

5. $K(D)$ *is a domain for all domains* D.

6. *For any semiprime ring* R *and any prime* $P \subseteq R$, *the kernel* $P^{@}$ *of* $K(R) \longrightarrow K(R/P)$ *is prime.*

7. *For every semiprime ring* R, *the map* $\operatorname{Spec}(K(R)) \longrightarrow \operatorname{Spec}(R)$ *is an order isomorphism and therefore a homeomorphism in the domain topology.*

8. *For every semiprime ring* R, *the adjunction map* $R \longrightarrow K(R)$ *is essential.*

9. *If* $f : R \longrightarrow S$ *is injective, so is* $K(f) : K(R) \longrightarrow K(S)$.

10. *The canonical map* $K(D) \longrightarrow G(D)$ *is injective for all domains* D.

11. *For all semiprime rings* R *and* S, *whenever* $R \subseteq S \subseteq K(R)$, *then* $K(S) = K(R)$.

12. *For all semiprime rings* R *and* S, *if* $R \subseteq S \subseteq K(R)$, *the inclusion* $R \hookrightarrow S$ *is epic.*

13. *For every semiprime ring* R, *we have that* $K(R)$ *is an integral extension of* R.

14. *For every domain* D, *we have that* $G(D)$ *is an integral extension of* D.

15. $\mathscr{A}_{\text{ica}} \subseteq \mathscr{K}$.

16. $\mathscr{A}_{\text{icp}} \subseteq \mathscr{K}$.

Proof. Here is a diagram of the logical inferences we will prove:

$$
\begin{tikzcd}
 & 6 \arrow[r, Rightarrow] & 7 \arrow[dr, Rightarrow] & & \\
5 \arrow[ur, Rightarrow] & & 9 \arrow[ll, Rightarrow] & 8 \arrow[l, Rightarrow] \arrow[dl, Rightarrow] \arrow[dd, Rightarrow] & \\
 & & 10 \arrow[ull, Rightarrow] & & \\
1 \arrow[uu, Rightarrow] \arrow[r, Rightarrow] & 2 \arrow[d, Rightarrow] & 13 \arrow[d, Rightarrow] & 11 \arrow[l, Rightarrow] \arrow[r, Leftrightarrow] & 12 \\
4 \arrow[u, Rightarrow] & 3 \arrow[l, Rightarrow] & 14 \arrow[l, Rightarrow] \arrow[r, Leftrightarrow] & 15 \arrow[r, Leftrightarrow] & 16
\end{tikzcd}
$$

1 $\Rightarrow$ *2* $\Rightarrow$ *3*: Both are obvious.

3 $\Rightarrow$ *4*: Suppose $D \longrightarrow D'$ is a morphism of domains with kernel P. Let $P^{\sharp}$ be a prime of

$G(D)$ lying above P. We construct a diagram

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[d] & G(D) \arrow[d] \\
D/P \arrow[r, hook] \arrow[d, hook] & G(D)/P^\sharp \arrow[dd, hook] \\
D' \arrow[d, hook] & \\
G(D') \arrow[r, hook] & F
\end{tikzcd}$$

by observing that the domain D/P is included in both domains $G(D)/P^\sharp$ and $G(D')$ and applying Proposition 2.2.1 to find F. We can further suppose that $F \in \mathscr{K}$. Then $G(D') \times_F G(D)$ is a $\mathscr{K}$-subobject of $G(D)$ and, by minimality, the pullback must be $G(D)$, which gives a map $G(D) \longrightarrow G(D')$. The uniqueness follows by another application of minimality and the functoriality is then easy. The required commutation follows from the commutation of the square above together with the fact that $G(D') \longrightarrow F$ is monic.

4 $\Rightarrow$ *1*: If we have a map $D \longrightarrow D'$ with $D' \in \mathcal{A}$, we get $G(D) \longrightarrow G(D') = D'$ such that

$$\begin{tikzcd}
D \arrow[rr] \arrow[dr] & & G(D) \arrow[dl] \\
& D' &
\end{tikzcd}$$

commutes. The uniqueness follows from 2.2.5. This shows that G has the adjunction property with respect to maps to objects in $\mathcal{A}$.

1 $\Leftrightarrow$ *5*: Immediate.

5 $\Rightarrow$ *6*: Since $K(R/P)$ is a domain, $P^@$ is a prime. From the diagram

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d] & K(R) \arrow[d] \\
R/P \arrow[r, hook] & K(R/P)
\end{tikzcd}$$

we readily infer that $P^@ \cap R = P$.

6 $\Rightarrow$ *7*: We know from 2.2.16 that it is a bijection. The direct map preserves inclusion while it follows from 2.2.17 that the inverse map, which takes P to $P^@$ also does.

7 $\Rightarrow$ *8*: If $P \subseteq R$ is prime, let $P^\sharp \subseteq K(R)$ denote the unique prime lying above P. We have to show that if $I \subseteq K(R)$ is an ideal such that $I \cap R = 0$, then $I = 0$. First consider the case that I is a radical ideal of $K(R)$, so that $I = \bigcap P^\sharp$, taken over all the primes $P \subseteq R$ for which $I \subseteq P^\sharp$. Hence $0 = I \cap R = \bigcap P$, again over all primes P such that $I \subseteq P^\sharp$. The set $U = \{P^\sharp \in \mathrm{Spec}(K(R)) \mid I \subseteq P^\sharp\}$ is the meet of all the $N(a)$, for $a \in I$. It is therefore compact in the patch topology and hence in the domain topology. From 7 we conclude that the set $V = \{P \in \mathrm{Spec}(R) \mid I \subseteq P^\sharp\}$ is also compact in the domain topology on $\mathrm{Spec}(R)$. But we have just seen that $\bigcap_{P \in V} P = 0$ and hence is contained in every prime $Q \subseteq R$. But since V is compact it follows from Proposition 2.2.22 that $P \subseteq Q$ for some $P \in V$. But then $I \subseteq P^\sharp \subseteq Q^\sharp$ so that I lies in every prime of $K(R)$ and is then 0.

For a general ideal I, let J be the radical of I. Every element of J has a power that lies in I. Therefore every element of $J \cap R$ has a power that lies in $I \cap R = 0$. Since R is semiprime, it follows that such an element is 0. Thus $J \cap R = 0$ and therefore $J = 0$, whence $I = 0$.

8 $\Rightarrow$ *9*: From the diagram

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d, hook] & K(R) \arrow[d] \\
S \arrow[r, hook] & K(S)
\end{tikzcd}$$

we see that the composite $R \hookrightarrow K(R) \longrightarrow K(S)$ is injective and it follows from essentiality that $K(R) \longrightarrow K(S)$ is.

9 $\Rightarrow$ *5*: A domain D can be embedded $D \hookrightarrow F$ where F is a field in $\mathcal{A}$. If K preserves injectivity, we get $K(D) \hookrightarrow K(F) = F$, whence $K(D)$ is a domain.

8 $\Rightarrow$ *10* $\Rightarrow$ *5*: Both are trivial.

8 $\Rightarrow$ *11*: Since 8 holds for S, corollary 2.2.19 gives the result.

11 $\Rightarrow$ *12*: It suffices to show that it is epic with respect to maps into fields. Suppose $f, g : S \rightrightarrows F$ agree on R with F a field, which can be assumed to lie in $\mathcal{A}$. But since $K(S) = K(R)$, each of the maps f and g extends to $K(R)$. But they agree on R and the uniqueness of the maps from the reflector imply they are equal.

12 $\Rightarrow$ *11*: Suppose $f : S \longrightarrow A$ is given with $A \in \mathcal{A}$. Then $g = f|R$ has an extension to $\widehat{g} : K(R) \longrightarrow A$. Then $\widehat{g}|S$ and f have the same restriction g to R. But if $R \longrightarrow S$ is epic, we must have that $\widehat{g}$ extends f. The uniqueness follows since the equalizer of two extensions of f would be a proper $\mathscr{K}$-subobject of $K(R)$ that contains R, contradicting Proposition 2.2.18.

11 $\Rightarrow$ *13*: Proposition 2.2.16 applied to any intermediate ring, say $R \subseteq S \subseteq K(R) = K(S)$ implies that both maps in $\mathrm{Spec}(K(R)) \longrightarrow \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ are bijections. In particular, for any pair of intermediate rings $R \subseteq S_1 \subseteq S_2 \subseteq K(R)$, we have $\mathrm{Spec}(S_2) \longrightarrow \mathrm{Spec}(S_1)$

is a bijection. Then the corollary to Theorem 2.2.23 implies the result.

13 $\Rightarrow$ *14*: We begin by showing that 13 $\Rightarrow$ 5. We already know that this implies 1 which, together with 13 will obviously imply 14. It follows from 2.2.16 that there is a unique prime $P \subseteq K(D)$ such that $P \cap D = 0$. Since 0 is contained in every prime of D, it follows from 2.2.25 that P is contained in every prime of $K(D)$. But $K(D)$ is semiprime so the intersection of all the primes is 0, whence $K(D)$ is a domain.

14 $\Rightarrow$ *3*: This is a standard property of integral extensions.

14 $\Rightarrow$ *15*: Suppose $D \in \mathcal{A}_{\text{ica}}$. Let F be the algebraic closure of the field of fractions of D. From Lemma 2.2.7.2 we know that $F \in \mathcal{K}$. From Proposition 2.2.3, we know that $G(D) \subseteq F$. But $G(D)$ is an integral extension of D and D is integrally closed in F so we conclude that $G(D) = D$. Since $G(D)$ was constructed to be in $\mathcal{K}$, it follows that $D \in \mathcal{K}$.

15 $\Rightarrow$ *14*: From $\mathcal{A}_{\text{ica}} \subseteq \mathcal{K}$, it obviously follows that $\mathcal{K}_{\text{ica}} \subseteq \mathcal{K}$ and we see that for any domain D, $G(D) \subseteq G_{\text{ica}}(D)$ and since $G_{\text{ica}}(D)$ is integral over D, so is $G(D)$.

15 $\Rightarrow$ *16*: If we show that $\mathcal{A}_{\text{icp}} \subseteq \mathcal{K}_{\text{ica}}$ it obviously follows that $\mathcal{K}_{\text{icp}} \subseteq \mathcal{K}_{\text{ica}}$ while the reverse inclusion is obvious. Suppose that $D \in \mathcal{A}_{\text{icp}}$. Let $F \subseteq \overline{F}$ be the perfect closure and algebraic closure, respectively, of the field of fractions of D. Then $\overline{F}$ is a separable algebraic extension of F so that F is the equalizer of all the F-automorphisms of $\overline{F}$ and hence F lies in $\mathcal{K}_{\text{ica}}$. Let $\overline{D}$ be the integral closure of D in $\overline{F}$. Since being integral is transitive, $\overline{D}$ can have no proper integral extension in $\overline{F}$ and hence is integrally closed in its algebraic closure which implies that $\overline{D} \in \mathcal{A}_{\text{ica}}$. An element of $\overline{D} \cap F$ satisfies an integral equation with coefficients in D and belongs to F, hence is in D. It follows that $D = \overline{D} \cap F \in \mathcal{K}_{\text{ica}}$.

16 $\Rightarrow$ *15*: It follows since $\mathcal{A}_{\text{ica}} \subseteq \mathcal{A}_{\text{icp}}$. ■

2.3.2. Examples. We refer to 2.1.1 for the definitions of the subcategories mentioned here.

1. $\mathcal{K}_{\text{fld}}$ (2.1.1.2) is not $\mathcal{Dom}$-invariant. The limit closure can be shown to be just the von Neumann regular rings. The reflection of $\mathbf{Z}$ is not a domain, but is a subring of $\prod \mathbf{Q}_p$, the product of all prime fields. It is called the Fuchs-Halperin ring [Fuchs & Halperin (1964)].

2. From Theorem 2.3.1.15 we have that $\mathcal{K}_{\text{ica}}$ (2.1.16) is $\mathcal{Dom}$-invariant and, in fact, the smallest $\mathcal{Dom}$-invariant subcategory. The injectivity property for totally integrally closed rings from [Enochs (1968)] implies that a totally integrally closed ring is in $\mathcal{K}_{\text{ica}}$. But the converse is not true: the ring of eventually constant sequences of complex numbers is in $\mathcal{K}_{\text{ica}}$, but is not totally integrally closed. What is clear is that $\mathcal{K}_{\text{ica}}$ is the limit closure of the category of totally integrally closed rings. One sees easily from 9 and 13 that for K_{ica}, or any $\mathcal{Dom}$-invariant category $\mathcal{K}$, that an integrally closed subring of a ring in $\mathcal{K}$ is also in $\mathcal{K}$. This holds for example for the integral closure of any R in $K(R)$.

3. Theorem 2.3.1.16 says $\mathscr{K}_{\text{icp}} = \mathscr{K}_{\text{ica}}$ is also $\mathcal{D}om$-invariant. We note that $\mathscr{K}_{\text{ic}}$ (2.1.1.4) is domain invariant since $\mathcal{A}_{\text{icp}} \subseteq \mathcal{A}_{\text{ic}}$.

   It is evident that no non-perfect field can be in the limit closure $\mathscr{K}_{\text{icp}}$, so $\mathscr{K}_{\text{icp}}$ is strictly smaller than $\mathscr{K}_{\text{ic}}$ which implies that $K_{\text{ic}}(R) \subseteq K_{\text{icp}}(R)$ for any ring R. But for any domain D with F the perfect closure of its field of fractions, the meet of all the $\mathscr{K}_{\text{icp}}$-subobjects of F that contain D is still an integral extension of D and thus in $K_{\text{icp}}(D)$. We do not know what the limit closure is in this case, but one thing such rings satisfy is the essentially equational condition that for every prime number p, there is a pth root operation, see 3.8.1 whose domain is $\{r \mid pr = 0\}$ and whose value is the provably unique pth root of that element. We can say that $\mathscr{K}_{\text{icp}} \subseteq \mathscr{K}_{\text{ic}} \cap \mathscr{K}_{\text{per}}$, but 6.3.1 gives an example of a ring in $\mathscr{K}_{\text{ic}} \cap \mathscr{K}_{\text{per}}$ which is not in $\mathscr{K}_{\text{icp}}$.

4. $\mathscr{K}_{\text{dom}}$ (2.1.1.1) is $\mathcal{D}om$-invariant since clearly $K_{\text{dom}}(D) = D$ for every domain D. In Section 4, we will characterize the rings that lie in $\mathscr{K}_{\text{dom}}$.

5. $\mathscr{K}_{\text{noe}}$ (2.1.1.10) is not $\mathcal{D}om$-invariant. This will be shown in Corollary 2.4.2 below.

6. For the next examples, see Theorem 2.3.3 below. They are suggestions due to David Dobbs. The general reference is the multiplicative ideal theory found in [Gilmer 1992], see especially Theorem 19.8.

   In Section 7, we will be looking at certain rings in $\mathscr{K}_{\text{ufd}}$, although we do not characterize the category. We will see in Corollary 2.4.2 below that $\mathscr{K}_{\text{ufd}}$ is neither $\mathcal{D}om$-invariant nor given as models of a first-order theory.

   There is a related class, that of GCD domains, defined as ones in which every pair of elements has a greatest common divisor. Obviously Bézout domains are also GCD domains and models of either theory are UFDs if Noetherian. But not only if since, for example, a polynomial ring in infinitely many variables over a field is a UFD, but not Noetherian.

   Another related class is that of valuation domains. A domain D with field F of fractions is a valuation domain if for each $x \in F$, at least one of x or $1/x$ lies in D. The relevant facts are

2.3.3. Theorem.

1. *Valuation domains are Bézout (and therefore GCD) domains.*

2. *Every valuation domain, every Bézout domain, and every GCD domain is integrally closed [Gilmer 1992, Corollary 9.8].*

3. *Every integrally closed domain is the meet of all the valuation rings, as well as the meet of all the Bézout domains, as well as the meet of all the GCD domains between it and its field of fractions, [Gilmer 1992, Theorem 19.8].*

4. *It follows that the limit closures of the valuation domains, the GCD domains, and the Bézout domains are the same. See Section 5.*

The corollary to the next proposition shows that neither limit closure $\mathscr{K}_{\mathrm{ufd}}$ nor $\mathscr{K}_{\mathrm{noe}}$ contains $\mathscr{K}_{\mathrm{ica}}$. Then we may invoke Theorem 2.3.1 to show that neither is $\mathcal{Dom}$-invariant.15.

2.4. Special rings. Let us temporarily say that a commutative ring is **special** if it has the property that any element with an nth root for all n also has a quasi-inverse. We make the following claims:

2.4.1. Proposition.

1. *The full subcategory of $\mathcal{CR}$ consisting of special rings is limit closed.*
2. *Every UFD is special.*
3. *Every Noetherian ring is special.*
4. *A non-principal ultrapower of $\mathbf{Z}$ is not special.*
5. *The domain $\mathbf{A}$ of algebraic integers is not special but is in $\mathscr{K}_{\mathrm{ica}}$.*

Proof.

1. We use the familiar fact that quasi-inverses, when they exist, can be chosen uniquely as mutual quasi-inverses. An element of a cartesian product of rings has an nth root (respectively, quasi-inverse) if and only if each coordinate does, so the category of special rings has products. If $R \xrightarrow{f} S \overset{g}{\underset{h}{\rightrightarrows}} T$ is an equalizer in which S and T are special, let $r \in R$ have all nth roots. Then $s = f(r)$ obviously has all nth roots and hence has a unique mutual quasi-inverse s'. Since $g(s) = h(s)$ we easily see that both $g(s')$ and $h(s')$ are mutual quasi-inverses for $g(s)$ and, from uniqueness, we see that $g(s') = h(s')$ and so there is an $r' \in R$ with $f(r') = s'$. But then it follows, since f is an injection, that r' is a quasi-inverse for r.

2. In a UFD no non-zero, non-invertible element (hence no non-quasi-invertible element) can have arbitrary nth roots, so it is clear that UFDs are special.

3. In a Noetherian ring, if a non-zero element x has nth roots for all n, we get an infinite descending divisor chain $\cdots x^{1/2^n} \mid x^{1/2^{n-1}} \mid \cdots \mid x^{1/2} \mid x$ which, in a Noetherian ring, is possible only if for some n, we have $x^{1/2^{n-1}} \mid x^{1/2^n}$. Then $x^{1/2^{n-1}}y = x^{1/2^n}$ for some y. Raising both sides to the 2^n power gives $x^2y^{2^n} = x$ so that y^{2^n} is a quasi-inverse for x.

4. The class of the element $(2, 2^2, 2^{3!}, \ldots, 2^{n!}, \ldots)$ clearly has arbitrary nth roots, but the ring is a domain and the only non-zero elements of a domain that have quasi-inverses are invertible elements while this element is clearly not invertible.

5. Every element of $\mathbf{A}$ has roots of all order. But roots of integers > 1 cannot be invertible. In a domain only invertible elements and 0 have quasi-inverses. ■

2.4.2. COROLLARY. *Neither $\mathscr{K}_{\text{ufd}}$ nor $\mathscr{K}_{\text{noe}}$ is a category of models of a first-order theory and neither category is $\mathcal{D}om$-invariant.*

PROOF. The conclusions are immediate from points 4 and 5 above. ■

3. The sheaf representation in the first-order case.

3.1. FIRST-ORDER CONDITIONS. In this section, we show that if $\mathscr{K}$ is a reflective subcategory of commutative rings, as in the previous section, then, under a reasonable additional assumption (given below), there is, for every semiprime ring R, a topology on $\mathrm{Spec}(R)$ that lies between the domain and the patch topology and a sheaf of rings, given by a local homeomorphism $\pi_R : E_R \longrightarrow \mathrm{Spec}(R)$ whose stalks are domains in $\mathscr{K}$. We will show that $\Gamma(E_R)$, the ring of global sections, is also in $\mathscr{K}$. Under various further conditions, $\Gamma(E_R)$ is the reflection of R into $\mathscr{K}$, see 3.5.3.

We use the concept of first-order conditions, which we briefly (and sketchily) review. First-order conditions for commutative rings are built up from basic conditions of the form $p(x_1, x_2, \ldots, x_n) = 0$ where each x_i is a variable and p is a polynomial with integer coefficients. Further conditions can be obtained by using the connectives *or* and *not*. These are treated classically, so we can make conditions such as *If p then q*, which is equivalent to *Not p or q*. We can also quantify variables (but **not** the constants, such as the integer coefficients of the polynomials).

A ring R satisfies a first-order condition C if and only if whenever each free variable (that is, each unquantified variable) of C is replaced by an element of R, then the statement becomes true.

We say that a full subcategory $\mathscr{B}$ of commutative rings is **first order** if there is a set S (possibly infinite) of first-order conditions such that $R \in \mathscr{B}$ if and only if R satisfies each condition in S. It is well known that if $\mathscr{B}$ is first order, then $\mathscr{B}$ is closed under the formation of ultraproducts.

The converse is also true when $\mathscr{B}$ is the class of domains in $\mathscr{K}$ where the conditions of 2.1 are satisfied, see Theorem 3.8.25. We thank Michael Makkai for his suggestions and help.

As an example, a ring is semiprime if and only if it satisfies the first-order condition that $x^2 = 0$ implies $x = 0$ (more precisely, the condition *If* $x^2 = 0$ *then* $x = 0$, but we often use *p implies q* to mean *If p then q*). A ring has characteristic 0 if it satisfies each of the infinitely many conditions that $nx = 0$ implies $x = 0$ for $n = 2, 3, 4, \ldots$. As another example, we mention that being a domain is first order. But the condition of having finite characteristic is not first order. Note that the infinite disjunction $2 = 0$ or $3 = 0$ or $5 = 0$ or $\ldots$ is not a first-order condition. For example a non-principal ultraproduct of fields of finite characteristic will have characteristic 0 provided no one characteristic is present infinitely often.

3.1.1. Notation and Blanket Assumptions. Throughout this section we consider subcategories $\mathcal{A}$, meeting the conditions in 2.1 plus the further condition that $\mathscr{B}$ is first order, where $\mathscr{B}$ is the category of domains in $\mathscr{K}$. (As before, $\mathscr{K}$ denotes the limit closure of $\mathcal{A}$.) So: *In this section, we always suppose that $\mathscr{B}$ is first order.*

We will, of course, use notation and results from the previous section. The following result gives us one criterion that $\mathscr{B}$ be first order.

3.1.2. Proposition. *If $\mathscr{K}$ is $\mathfrak{Dom}$-invariant (see Theorem 2.3.1) and if $\mathcal{A}$ is first order, then $\mathscr{B}$ is also first order.*

We postpone this proof until 3.8 at the end of this section.

3.2. Heuristics for constructing the sheaf. Our goal (which is realized in all cases we know of) is to construct, for each semiprime ring R, a sheaf $E_R = E$ whose stalks are integral domains in $\mathscr{B}$ such that the ring of global sections, $\Gamma(E)$, is canonically isomorphic to the reflection of R into $\mathscr{K}$.

We will start by considering a very rough approximation to the sheaf. As shown in Proposition 2.2.11, the set $\{G(R/P) \mid P \in \mathrm{Spec}(R)\}$ forms a solution set for maps from R to objects in $\mathscr{K}$. So a crude version of the sheaf would be to give $\mathrm{Spec}(R)$ the discrete topology and erect a stalk $G(R/P)$ at each $P \in \mathrm{Spec}(R)$. The ring of global sections for this sheaf is clearly the product $\prod\{G(R/P)\}$. For each prime ideal P, we have a map $R \longrightarrow R/P \longrightarrow G(R/P)$ and so there is an obvious injection $R \rightarrowtail \prod\{G(R/P)\}$.

We claim that any map $f : R \longrightarrow B$, with $B \in \mathscr{B}$ factors through this injection. If $P = \ker(f)$ then f factors through $R \longrightarrow R/P \longrightarrow G(R/P) \longrightarrow B$ and therefore through $R \rightarrowtail \prod\{G(R/P)\} \longrightarrow G(R/P)$. However, this factorization is generally not unique. For example, suppose that $Q \subseteq P$ are prime ideals of R and that there is a homomorphism $G(R/Q) \longrightarrow G(R/P)$ which makes the obvious diagram commute (as in the Definition below). Then there are two maps from the product to $G(R/P)$; one is the projection onto $G(R/P)$, the other map is the projection onto $G(R/Q)$ followed by $G(R/Q) \longrightarrow G(R/P)$.

Thus we need to tighten up the topology on $\mathrm{Spec}(R)$. As we will see, correcting for this possibility involves requiring that every open subset of $\mathrm{Spec}(R)$ be up-closed in the following order on $\mathrm{Spec}(R)$:

3.2.1. Definition. *We define a partial order relation $\sqsubseteq$ on* $\mathrm{Spec}(R)$*, called the* $\mathcal{A}$-**ordering** *on* $\mathrm{Spec}(R)$*, by saying that $Q \sqsubseteq P$ in this ordering if $Q \subseteq P$ and there is a map $G(R/Q) \longrightarrow G(R/P)$ such that*

$$\begin{tikzcd}
R/Q \arrow[r, hook] \arrow[d] & G(R/Q) \arrow[d] \\
R/P \arrow[r, hook] & G(R/P)
\end{tikzcd}$$

commutes.

But there is another potential difficulty. Suppose that $\mathbf{u}$ is an ultrafilter on $\mathrm{Spec}(R)$. Since the set $\{G(R/P)\}$ is indexed by $P \in \mathrm{Spec}(R)$ there is an ultraproduct, which we will temporarily denote by $G_{\mathbf{u}}$, which is obtained as a quotient $q_{\mathbf{u}} : \prod_{P\in\mathrm{Spec}(R)} G(R/P) \longrightarrow G_{\mathbf{u}}$. Now we have one map $R \rightarrowtail \prod G(R/P) \longrightarrow G_{\mathbf{u}}$ which uses $q_{\mathbf{u}}$:

$$R \rightarrowtail \prod_{P\in\mathrm{Spec}(R)} G(R/P) \longrightarrow G_{\mathbf{u}}$$

There is another map (where $P_{\mathbf{u}}$ is the kernel of the above map and $\prod G(R/P) \longrightarrow G(R/P_{\mathbf{u}})$ is the projection associated with the product):

$$R \rightarrowtail \prod_{P\in\mathrm{Spec}(R)} G(R/P) \longrightarrow G(R/P_{\mathbf{u}}) \longrightarrow G_{\mathbf{u}}$$

As we will see, correcting for this possibility involves requiring that every open subset of $\mathrm{Spec}(R)$ be open in the patch topology. It can be shown that $r \in P_{\mathbf{u}}$ if and only if $\{P \mid r \in P\} \in \mathbf{u}$ and $P_{\mathbf{u}}$ is the limit (in the patch topology) of the ultrafilter $\mathbf{u}$.

To correct for both of the problems mentioned above we need the following topology on $\mathrm{Spec}(R)$.

3.2.2. Definition. *The* $\mathcal{A}$-**topology** *on* $\mathrm{Spec}(R)$ *is defined so that a set is* $\mathcal{A}$*-open if it is patch-open and up-closed in the* $\mathcal{A}$*-ordering.*

We note that the $\mathcal{A}$-topology, which lies between the domain and the patch topologies is compact, but unlikely to be Hausdorff. Note that in the $\mathcal{D}om$-invariant case $\sqsubseteq$ is the same as $\subseteq$, and the $\mathcal{A}$-topology is the domain topology by Lemma 2.2.21. The converse is also true, see Example 3.7.1.

As before we let $E_R = E$ be the space over $\mathrm{Spec}(R)$ with stalk $G(R/P)$ over $P \in \mathrm{Spec}(R)$. We give $\mathrm{Spec}(R)$ the above topology. It remains to define a sheaf topology on E. In effect, this means saying when $z \in G(R/P)$ is "close to" $z' \in G(R/P')$. Conceptually, there are two basic ways of being close:

(1) We say that z and z' are close if there exist $r, s \in R$ such that $z = r/s$ (in $G(R/P)$) and $z' = r/s$ (in $G(R/P')$)

(2) We also say that $z \in G(R/P)$ is close to $z' \in G(R/P')$ if $P \sqsubseteq P'$ and the associated map $G(R/P) \longrightarrow G(R/P')$ takes z to z'.

Conceptually, we could use these notions to define a Section $\sigma : U \longrightarrow \pi^{-1}(U)$ to be continuous if :

(1) Whenever $\sigma(P) = r/s$ (in $G(R/P)$) then there is a patch-open neighbourhood of P such that if P' is in the neighbourhood, then $\sigma(P') = r/s$ (in $G(R/P')$).

(2) Whenever $P \sqsubseteq P'$, then $\sigma(P')$ has to be the image of $\sigma(P)$ under the canonical map $G(R/P) \longrightarrow G(R/P')$.

But a definition along these lines would be awkward to work with. (Among other things, we would also have to deal with the case when $\sigma(P)$ is not of the form r/s but is a characteristic root of such a fraction.) We will instead use a less conceptual definition that is technically convenient and prove that the continuous sections are characterized by properties similar to the ones given above. See Proposition 3.4.5.

3.3. Technical results needed to construct the sheaf. We assume that the semiprime ring R is given. Since we will be talking about maps from R to another semiprime ring, it is convenient to use the language of an R-algebra, as established by:

3.3.1. Notation.

1. An R-algebra consists of a ring S, together with a **structure map** $R \longrightarrow S$.

2. In this paper, the term "R-algebra" always refers to a semiprime ring over R.

3. If S and T are R-algebras, then a map $S \longrightarrow T$ is an R-algebra homomorphism if and only if the following triangle commutes (where the maps from R are assumed to be the structure maps):

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & S \arrow[dl] \\ & T & \end{tikzcd}$$

4. If we are given a map $R \longrightarrow S$ and we subsequently refer to the R-algebra structure on S, then, unless the contrary is explicitly stated, we assume the given map $R \longrightarrow S$. is the structure map.

5. We will use Isbell's term **dominion** as interpreted in the category $\mathcal{SPR}$. This means that $s \in S$ is in the dominion of $e : R \longrightarrow S$ if whenever $g, h : S \longrightarrow T$ satisfy $ge = he$, then $g(s) = h(s)$. Note that if $S' \subseteq S$ is an R-subalgebra containing s, it is possible that s be in the dominion of $R \longrightarrow S$ and *not* of $R \longrightarrow S'$.

6. If S is an R-algebra and there is no danger of confusion, we will say that an element of $s \in S$ is in the **dominion of** R if it is in the dominion of the structure map $R \longrightarrow S$.

7. The R-algebra S is a **finitely generated R-algebra** if it is generated as an R-algebra by a finite number of elements. This is equivalent to S being isomorphic as an R-algebra to $R[x_1, \ldots, x_k]/J$ for some ideal $J \subseteq R[x_1, \ldots, x_k]$. The structure map $R \longrightarrow S$ in that case will be an injection if and only if $R \cap J = 0$. The R-algebra S will be semiprime if and only if J is a radical ideal (that is, an intersection of prime ideals).

8. If R is a ring and $A \subseteq R$ we denote by (A) the ideal generated by A and by $\langle A \rangle$ the radical $\sqrt{(A)}$, the least radical ideal containing A. Note that a ring homomorphism vanishes on A if and only it vanishes on (A) and, if the codomain is semiprime, this is also if and only if it vanishes on $\langle A \rangle$.

9. The R-algebra S is **finitely presented as a semiprime ring** (or just "finitely presented") if it is isomorphic to $R[x_1, \ldots, x_k]/\langle J \rangle$ where J is finite.

3.3.2. Definition. *Let $R \in \mathcal{SPR}$ be given. Then (H, h) is an R-**dominator** if H is a finitely presented R-algebra and $h \in H$ is in the dominion of R.*

3.3.3. Lemma. *If S and T are R-algebras and if $f : S \longrightarrow T$ is an R-homomorphism and if $s \in S$ is in the dominion of R, then $t = f(s)$ is also in the dominion of R.*

Proof. Obvious. ∎

3.3.4. Definition. *Let $R \longrightarrow S$ be a map in semiprime rings. The **cokernel pair** of a map $R \longrightarrow S$ (in semiprime rings) is a semiprime ring C together with maps $c_1, c_2 : S \longrightarrow C$ such that the following is a pushout diagram (in semiprime rings):*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d, "c_1"] \\ S \arrow[r, "c_2"'] & C \end{tikzcd}$$

If we regard $R \longrightarrow S$ as a structure map, then the cokernel pair is the coproduct of S with itself in the category of (semiprime) R-algebras.

3.3.5. Lemma. *Let $R \longrightarrow S$ be given and let C together with $c_1, c_2 : S \longrightarrow C$ denote the cokernel pair. Then $s \in S$ is in the dominion of R if and only if $c_1(s) = c_2(s)$.*

Proof. This was shown by Isbell and follows from the definition of a pushout. ∎

3.3.6. Notation. Suppose S is an R-algebra. We can write $S = R[X]/I$, where X is a set of indeterminates and I is a radical ideal. The R-structure map is the obvious composite $R \longrightarrow R[X] \longrightarrow R[X]/I = S$. For any $s \in S$, we can choose a set X and an element $x \in X$ so that this composite takes x to s.

Let X' be a disjoint copy of X so that each element $x \in X$ corresponds to an element $x' \in X'$. By extension, each polynomial $f \in R[X]$ corresponds to a polynomial $f' \in R[X']$. (Note that if $r \in R$ then $r' = r$.) A subset $J \subseteq R[X]$ then corresponds to a subset $J' \subseteq R[X']$.

We regard $R[X]$ and $R[X']$ as subrings of $R[X \cup X']$. Given $J \subseteq R[X]$ we let $\overline{J}$ denote the radical ideal $\langle J \cup J' \rangle \subseteq R[X \cup X']$.

3.3.7. Lemma. *Suppose $S = R[X]/I$ as above. Let $q : R[X] \longrightarrow S$ be the quotient map. Then:*

1. *The cokernel pair for $R \longrightarrow S$ corresponds to $R[X \cup X']/\overline{I}$ (where, as indicated above, $\overline{I} = \langle I \cup I' \rangle$).*
2. *Suppose that $q(x) = s$ for some $x \in X$. Then s is in the dominion of R if and only if $x \equiv x'$ modulo $\overline{I}$.*
3. *Suppose that $q(x) = s$ for some $x \in X$. Then s is in the dominion of R if and only if there exists a finite subset $J \subseteq I$ such that $x \equiv x'$ modulo $\overline{J}$.*

Proof.

1. is obvious.

2. This follows from the above lemma.

3. This follows as $\overline{I}$ is clearly the filtered union of $\{\overline{J}\}$ as J varies over the set of all finite subsets of I. ∎

3.3.8. Proposition. *Let $R \longrightarrow S$ and $s \in S$ be given. Then s is in the dominion of R if and only if there exists an R-dominator (H, h) and an R-homomorphism $H \longrightarrow S$ which sends h to s.*

Proof. Suppose there is an R-dominator (H, h) together with an R-homomorphism $H \longrightarrow S$ which maps h to s. Then s is in the dominion of R by Lemma 3.3.3.

Conversely, suppose $s \in S$ is in the dominion of R. Write $S = R[X]/I$ where the set X of indeterminates is in bijective correspondence with the elements of S. Let $z \in X$ correspond to s. By the above lemma, we have that $z - z' \in \overline{I}$ and, moreover, there exists a finite set $J \subseteq I$ such that $z - z' \in \overline{J}$. Let X_0 be the subset of all members of X which includes z and all indeterminates, which appear in the elements of $J \subseteq R[X]$. Then we can choose $H = R[X_0]/\langle J \rangle$ and let h be the image of z. ∎

3.3.9. Proposition. *Let $R \in \mathcal{SPR}$ be given and let $f(x_1, \ldots, x_k) \in R[x_1, \ldots, x_k]$ be a polynomial in k indeterminates with coefficients in R. Then the set S of all prime ideals $P \in$ Spec(R) such that $f = 0$ has a solution in $G(R/P)$ is open in the $\mathcal{A}$-topology on* Spec(R).

Proof. Since S is obviously up-closed in the $\mathcal{A}$-order, it suffices to show that S is open in the patch topology. We will actually show that $T = \text{Spec}(R) - S$ is patch-closed. If T is not closed, then there exists an ultrafilter $\mathbf{u}$ on T which converges, in the patch topology, to $P \in S$. For each $Q \in T$, we see that $G(R/Q) \in \mathscr{B}$ and we let $B_{\mathbf{u}}$ be the corresponding ultraproduct and let $q_{\mathbf{u}} : \prod\{G(R/Q) | Q \in T\} \longrightarrow B_{\mathbf{u}}$ be the corresponding quotient map. Since $\mathscr{B}$ is the category of models of a first-order theory, it is closed under ultraproducts so that $B_{\mathbf{u}} \in \mathscr{B}$. For each $Q \in T$, there is a map $h_Q : R \longrightarrow R/Q \longrightarrow G(R/Q)$ hence a map $h : R \longrightarrow \prod G(R/Q)$. Let $h_{\mathbf{u}} : R \longrightarrow B_{\mathbf{u}}$ be defined by $h_{\mathbf{u}} = q_{\mathbf{u}} h$.

We claim that $\ker(h_{\mathbf{u}}) = P$. If $a \in P$, then $N(a) = \{Q \mid a \in Q\}$ is a neighbourhood of P in the patch topology and hence belongs to $\mathbf{u}$. Since $h_Q(a) = 0$ for all $Q \in N(a) \in \mathbf{u}$, it follows that $h_{\mathbf{u}}(a) = 0$. Similarly, if $a \notin P$, then $Z(a)$ is a neighbourhood of P in the patch topology and a similar argument shows that $h_{\mathbf{u}}(a) \neq 0$.

Thus we can regard $R/P \subseteq B_{\mathbf{u}}$. Since $B_{\mathbf{u}}$ is a subdomain of some field in $\mathcal{A}$ that contains R/P, we see that $G(R/P) \subseteq B_{\mathbf{u}}$. Since $P \in S$, the equation $f = 0$ has a solution in $G(R/P)$ which is a solution of f in $B_{\mathbf{u}}$. Thus we have $t_1, \ldots, t_k \in B_{\mathbf{u}}$ such that $f(t_1, \ldots, t_k) = 0$. For each prime Q and $i = 1, \ldots, k$, let $t_{iQ} \in G(R/Q)$ be elements such that $(t_{iQ}) \in \prod G(R/Q)$ lies above t_i. Then $f(t_1, \ldots, t_k) = 0$ implies that $U = \{Q \mid f(t_{1Q}, \ldots, t_{kQ}) = 0\} \in \mathbf{u}$. But then $U \neq \emptyset$ and $Q \in T \in U$ contradicts the definition of T. ∎

By making an obvious modification of this argument, we can show:

3.3.10. Proposition. *Suppose $f_1, \ldots, f_n$ is a finite set of polynomials in $R[x_1, \ldots, x_k]$. Then the set of all primes $Q \subseteq R$ for which $f_1 = \cdots = f_n = 0$ has a simultaneous solution in $G(R/Q)$ is open in the $\mathcal{A}$-topology on* $\operatorname{Spec}(R)$. ■

3.3.11. Lemma. *Let S be a finitely presented semiprime R-algebra. The set of all primes $P \subseteq R$ for which there is an R-algebra homomorphism $S \longrightarrow G(R/P)$ is open in the $\mathcal{A}$-topology.*

Proof. Let $S = R[x_1, \ldots, x_k]/J$. Let $J_0 \subseteq J$ be a finite set such that $J = \langle J_0 \rangle$. The existence of an R-algebra homomorphism $S \longrightarrow G(R/P)$ is equivalent to the existence of simultaneous solutions in $G(R/P)$ to the equations $f = 0$ for every $f \in J_0$. By the above proposition, the set of primes for which this happens is open in the $\mathcal{A}$-topology. ■

3.3.12. Corollary. *Let R be a semiprime ring, $P \subseteq R$ be a prime ideal, and $z \in G(R/P)$. Then there exists an R-dominator (H, h), and an R-algebra homomorphism $H \longrightarrow G(R/P)$ for which the image of h is z.*

Proof. $R \longrightarrow R/P \longrightarrow G(R/P)$ is epic, so every $z \in G(R/P)$ is in the dominion of R. ■

3.4. Construction of the Canonical $\mathcal{A}$-Sheaf. Let $\mathcal{A}$ be as above and let $R \in \mathcal{SPR}$. In this subsection, we will construct the **canonical $\mathcal{A}$-sheaf for** R as a local homeomorphism $\pi : E_R \longrightarrow \operatorname{Spec}(R)$ whose stalks are domains in $\mathcal{B}$.

3.4.1. Notation. Let $\mathcal{A}$ satisfy the blanket assumptions of 3.1.1 and let $R \in \mathcal{SPR}$ be given. Then:

1. We let $E_R = E$ be the disjoint union $\bigcup \{G(R/P) \mid P \in \operatorname{Spec}(R)\}$.
2. $\pi_R = \pi : E \longrightarrow \operatorname{Spec}(R)$ denotes the map for which $\pi^{-1}(P) = G(R/P)$ for all $P \in \operatorname{Spec}(R)$.
3. For each R-dominator (H, h) we let $W(H) \subseteq \operatorname{Spec}(R)$ denote the set of all $P \in \operatorname{Spec}(R)$ for which there exists an R-homomorphism $h : H \longrightarrow G(R/P)$.
4. Given an R-dominator (H, h), we let $\zeta_{(H,h)} : W(H) \longrightarrow E$ denote the function for which $\zeta_{(H,h)}(P) = f(h)$ where $f : H \longrightarrow G(R/P)$ is an R-homomorphism. Note that f need not be uniquely determined but $f(h)$ is uniquely determined as h is in the dominion of R.

3.4.2. Definition. *Let $\mathcal{A}$, R and $E = E_R$ and $\pi = \pi_R$, etc. be as above. We give* $\operatorname{Spec}(R)$ *the $\mathcal{A}$-topology and for each R-dominator (H, h) give $W(H) \subseteq$* $\operatorname{Spec}(R)$ *the relative topology. We then give E the largest topology for which $\zeta_{(H,h)}$ is continuous for every R-dominator (H, h).*

We will call the maps $\zeta_{(H,h)}$ the **canonical local sections**.

We will prove that $\pi : E \longrightarrow \operatorname{Spec}(R)$ is a local homeomorphism and therefore defines a sheaf which we will call the **canonical $\mathcal{A}$-sheaf for** R.

The tensor products used in the following argument should be understood as taking place in the category $\mathcal{SPR}$. This means take the ordinary tensor product and factor out by the ideal of nilpotents. This tensor product is then a coproduct in the category of semiprime R-algebras.

3.4.3. THEOREM. *Let $R \in \mathcal{SPR}$ be given and let $E = E_R$ and $\pi = \pi_R : E \longrightarrow \mathrm{Spec}(R)$ be as defined in 3.4.1 and 3.4.2. Then π is a local homeomorphism that defines a sheaf of R-algebras all of whose stalks are in $\mathcal{B}$.*

The theorem readily follows from the steps below.

Step 1: *For each $z \in E$, there is an R-dominator (H, h) such that $\zeta_{(H,h)}(\pi(z)) = z$.*

PROOF. Given $z \in E$, let $P = \pi(z)$. Then $z \in G(R/P)$. By Corollary 3.3.12, there is an R-dominator (H, h) and a map $H \longrightarrow G(R/P)$ for which the image of h is z. It is readily shown that (H, h) has the required property. ■

Step 2: *The canonical local sections, $\{\zeta_{(H,h)}\}$ defined in Definition 3.4.2 agree on open sets.*

PROOF. Let (H_1, h_1) and (H_2, h_2) be R-dominators. Let $\zeta_1 = \zeta_{(H_1,h_1)}$ and $\zeta_2 = \zeta_{(H_2,h_2)}$. Let $R[x] \longrightarrow H_i$ be the R-homomorphism which sends x to h_i (for $i = 1, 2$). Let $H = H_1 \otimes_{R[x]} H_2$ and let $h = h_1 \otimes 1 = 1 \otimes h_2 \in H$. Clearly H is finitely presented. An R-algebra map $g : H \longrightarrow T$ is a pair of R-algebra maps (g_1, g_2) of maps $g_1 : H_1 \longrightarrow T$ and $g_2 : H_2 \longrightarrow T$ for which $g_1(h_1) = g_2(h_2)$. From this it readily follows that (H, h) is an R-dominator and that there is an R-algebra map $f : H \longrightarrow G(R/P)$ whose components are $f_1 : H_1 \longrightarrow G(R/P)$ and $f_2 : H_2 \longrightarrow G(R/P)$. Let $\zeta = \zeta(H, h)$. Clearly, $\zeta = \zeta_1$ and $\zeta = \zeta_2$ on the intersection of their domains. Moreover, it follows that whenever $\zeta_1(Q) = \zeta_2(Q)$ then Q is in the domain of ζ. So ζ is a canonical local section on which ζ_1 and ζ_2 agree. ■

Note: The above two steps show that $\pi : E \longrightarrow \mathrm{Spec}(R)$ is a sheaf (that is, a local homeomorphism). The next two steps will show that π is a sheaf of R-algebras.

Step 3: *The sum and product of any two sections are continuous.*

PROOF. Let (H_1, h_1) and (H_2, h_2) be R-dominators and let $\zeta_1 = \zeta_{(H_1,h_1)}$, and $\zeta_2 = \zeta_{(H_2,h_2)}$. The intersection of the two domains is open and we will define a section representing the sum on that intersection. Let $H = (H_1 \otimes_R H_2)$. By Let $h = (h_1 \otimes 1) + (1 \otimes h_2)$. By Lemma 3.3.3, we see that $h_1 \otimes 1$ and $1 \otimes h_2$ are in the dominion of R and it follows that h is also in the dominion of R. Moreover, H is clearly finitely presented as an R-algebra. Let $\zeta = \zeta_{(H,h)}$. It is readily shown that ζ is well defined and the domain of ζ includes the intersection of the domains of ζ_1 and ζ_2. Furthermore, on that intersection, ζ is the sum of ζ_1 and ζ_2. A similar proof works for the product. ■

Step 4: *The constant sections are continuous.*

PROOF. We have to show that for each $r \in R$ the section $\overline{r}$ for which $\overline{r}(P)$ is the image of r in $G(P/R)$ is continuous. Let $H = R$ and $h = r$. Then $\overline{r} = \zeta_{(H,h)}$. ■

3.4.4. Definition. *We let $\nu : R \longrightarrow \Gamma(E)$ denote the canonical injection such that $\nu(r)$ is, for each $r \in R$, the* **constant section** *$\overline{r}$ defined at the end of the above proof. If there is no danger of confusion, we will sometimes write r for $\nu(r)$ or $\overline{r}$.*

3.4.5. Proposition. *Let R be a semiprime ring and let $\pi : E \longrightarrow \mathrm{Spec}(R)$ be the canonical $\mathcal{A}$-sheaf. Let $U \subseteq \mathrm{Spec}(R)$ be open in the $\mathcal{A}$-topology. Then a function $f : U \longrightarrow E$, with $f(P) \in \pi^{-1}(P) = G(R/P)$ for all $P \in U$ is a continuous section if and only if the following two conditions are met:*

1. *Whenever there exist $r, s \in R$ with $sf(P)^\ell = r$ where ℓ is a characteristic power in the domain $G(R/P)$, then there is a patch-open neighbourhood W of P such that for all $P' \in W$, we have $sf^\ell = r$ (in $G(R/P')$).*
2. *Whenever $Q \sqsubseteq P$ for $Q, P \in U$ then $f(P) = q(f(Q))$ where $q : G(R/Q) \longrightarrow G(R/P)$ is the unique R-homomorphism.*

Proof. Assume that f is a continuous section on U. Suppose $sf(P)^\ell = r$. Then the local sections sf^ℓ and r agree at P so must agree on an $\mathcal{A}$-open set, and therefore on a patch-open set. Similarly, given $Q \sqsubseteq P$, we have that f must agree with a local section σ (as defined in the above proof) on a neighbourhood (in the $\mathcal{A}$-topology) of Q. But every such neighbourhood is up-closed in the $\sqsubseteq$-order and therefore must contain P. A straightforward verification shows that every local section σ satisfies $\sigma(P) = q(\sigma(Q))$ so the same must be true of f.

Conversely, assume that $f : U \longrightarrow E$ is such that $f(P) \in \pi^{-1}(P)$ for all $P \in U$ and that the above two conditions are satisfied. We will show that f is continuous at Q for an arbitrary $Q \in U$. Let σ be a local section, defined on an $\mathcal{A}$-open neighbourhood W of Q with $f(Q) = \sigma(Q)$. It suffices to show that f and σ agree on an $\mathcal{A}$-open set. Let $V = \{P' \in U \cap W \mid f(P') = \sigma(P')\}$. Since both f and σ satisfy condition (2), it is clear that V is up-closed in the $\sqsubseteq$-order. It remains to show that V is open in the patch topology. Let $P' \in V$ be given.

We first consider the case when $\mathrm{char}(R/P) = p > 0$ (so $p \in P'$). Then there exists ℓ, a power of p, and $r, s \in R$ such that $f(P')^\ell = r/s$ (in $G(R/P')$) and $s \notin P'$. Clearly, we have $sf(P')^\ell = r$. Since $P' \in V$ we see that $\sigma(P') = f(P')$ so $s\sigma(P')^\ell = r$. By hypothesis, we see that $sf(P')^\ell = r$ for all P' in a patch-neighbourhood of P and, by the above argument, $s\sigma(P')^\ell = r$ for all P' in a patch-neighbourhood of P. So there is a patch-open neighbourhood W_0 with $P \in W_0$ and on which both f and σ are defined and such that for all $P' \in W_0$ we have both $sf(P')^\ell = r$ and $s\sigma(P')^\ell = r$. Since $p \in P$, we have $N(p)$ is a patch-neighbourhood of P. Since s is non-zero mod P, we have that $Z(s)$ is also a patch-neighbourhood of P. It follows that $sf^\ell = r = s\sigma^\ell$ on the patch-open set $W_1 = W_0 \cap N(p) \cap Z(s)$. Since $s \neq 0$ in $G(R/P'')$ for all $P'' \in Z(s)$, we see that $f^\ell = \sigma^\ell$ on W_0. Since ℓ is a power of p, and since $\mathrm{char}(R/P'') = p$ for all $P'' \in N(p)$, we see that pth roots are unique for all $P'' \in W_1$. It follows that $f(P'') = \sigma(P'')$ for all $P'' \in W_1$. So W_1 is the required neighbourhood where $f = \sigma$, Finally we consider the case where $\mathrm{char}(R/P') = 0$. A simpler version of the above argument now applies, as we must have $\ell = 1$. In both cases, f and σ agree on an $\mathcal{A}$-open subset of U so f is continuous. ∎

3.4.6. Remark. As shown in the above proof, we may assume that all prime ideals P'' in the patch-open neighbourhood mentioned in condition 1 satisfy $s \neq 0$ modulo P'' , and further assume that either $\ell = 1$ or that ℓ is a power of a prime number p and the patch-open set in condition 1 contains only prime ideals for which $\mathrm{char}(R/P') = p$.

3.5. The canonical sheaf representation property: when is $\Gamma(E_R) = K(R)$? Let $R \in \mathcal{SPR}$ be given. We want to know when the canonical map $\nu : R \longrightarrow \Gamma(E_R)$ is the adjunction map of the reflection of R into $\mathscr{K}$. We will show there are several cases in which $\Gamma(E_R)$ is the reflection, but the general question of whether this is always true remains open.

3.5.1. Definition. *We say that $R \in \mathcal{SPR}$ has the* **canonical sheaf representation property** *with respect to $\mathcal{A}$, if $\nu R : R \longrightarrow \Gamma(E_R)$ is the reflection of R into $\mathscr{K}$.*

We say that $\mathcal{A}$ has the **canonical sheaf representation property** *if every $R \in \mathcal{SPR}$ has this property with respect to $\mathcal{A}$.*

3.5.2. Notation.

1. In what follows, we assume that $R \in \mathcal{SPR}$ is given and use E for E_R and π for π_R.

2. We identify $r \in R$ with the corresponding constant section in $\Gamma(E)$, so ν denotes the canonical embedding $R \hookrightarrow \Gamma(E)$.

3. Let $P \in \mathrm{Spec}(R)$ be given. Define $P^* \in \mathrm{Spec}(\Gamma(E))$ by $P^* = \{\psi \in \Gamma(E) \mid \psi(P) = 0\}$. Clearly P^* is a prime ideal of $\Gamma(E)$ and lies over P.

Our objective is to find and prove several conditions which are equivalent to the canonical sheaf representation property. We first show that $\Gamma(E)$ lies in the limit closure $\mathscr{K}$. We actually prove the following more general result:

3.5.3. Theorem. *Let $\mathscr{R}$ be any class of rings that is closed under ultraproducts. Let the local homeomorphism $\pi : F \longrightarrow X$ be a sheaf of rings such that, for each $x \in X$, the stalk $F_x = \pi^{-1}(x)$ is in $\mathscr{R}$. Then the ring of global sections of the sheaf is in the limit closure of $\mathscr{R}$.*

Proof. If $\mathbf{u}$ is an ultrafilter on X, we denote by $F_{\mathbf{u}}$ the corresponding ultraproduct of the $\{F_x\}$. Thus there is a quotient map $q_{\mathbf{u}} : \prod F_x \longrightarrow F_{\mathbf{u}}$. If $\mathbf{u}$ is an ultrafilter, let cnv ($\mathbf{u}$) denote the set of limits of $\mathbf{u}$.

We find it convenient to use the definition of an ultraproduct as a colimit of products taken over members of the ultrafilter: $F_{\mathbf{u}} = \mathrm{colim}_{U \in \mathbf{u}} \left(\prod_{x \in U} F_x\right)$, taking the colimit over inclusions. We use this to define a map $m_{\mathbf{u},y} : F_y \longrightarrow F_{\mathbf{u}}$ whenever $\mathbf{u}$ converges to y. For $z \in F_y$, choose a local section $\psi : U \longrightarrow F$ defined on a neighbourhood U of y such that $\psi(y) = z$. Then we let $m_{\mathbf{u},y}(z) = (\psi(x))_{x \in U}$. This is an element of $\prod_{x \in U} F_x$ which is one of the components of $F_{\mathbf{u}}$ since U is a neighbourhood of x. If ψ' is another local section defined on a neighbourhood U' of y such that $\psi'(y) = z$, then $\psi = \psi'$ on a neighbourhood of y which is also in $\mathbf{u}$ and so gives the same element of the ultrafilter.

Now let $H = \prod_{x \in X} F_x$ and $K = \prod_{\mathbf{u}} (F_{\mathbf{u}})^{\mathrm{cnv}(\mathbf{u})}$. For $x \in X$, let $p_x : H \longrightarrow F_x$ be the product projection. For an ultrafilter $\mathbf{u}$ and $y \in \mathrm{cnv}(\mathbf{u})$, we let $p_{\mathbf{u},y} : K \longrightarrow F_{\mathbf{u}}$ be that product projection. We define two maps $\beta, \gamma : H \longrightarrow K$ whose equalizer is, we claim, the set of global sections. Define β so that $p_{\mathbf{u},y}\beta = m_{\mathbf{u},y} p_y$ and define γ so that $p_{\mathbf{u},y}\gamma = q_{\mathbf{u}}$. Clearly the equalizer L of β and γ is in the limit closure of $\mathscr{R}$. We now show that L is isomorphic to the ring of global sections of the sheaf. Note that each element of H can be thought of as a map $\psi : X \longrightarrow F$ for which $\pi\psi = 1_X$. Moreover $\psi \in L$ if and only if $q_{\mathbf{u}}(\psi) = m_{\mathbf{u},y}(\psi(y))$ whenever $\mathbf{u}$ converges to y. But this is equivalent to saying that ψ preserves the convergence of the ultrafilter $\mathbf{u}$ and ψ preserves the convergence of all ultrafilters if and only if ψ is continuous, or is a global section. ∎

3.5.4. Remark. The above theorem can clearly be generalized further, for example to algebras other than rings. See [Kennison 1976, Lemma 2.5].

3.5.5. Definition. *We say that R has the* **unique prime-lifting property** *if for every $P \in \mathrm{Spec}(R)$, the ideal P^* is the unique prime ideal of $\Gamma(E)$ that lies over P.*

3.5.6. Definition. *R has the* **fractional root property** *if for every global section $\phi \in \Gamma(E_R)$, every $P \in \mathrm{Spec}(R)$, and every $P_1 \in \mathrm{Spec}(\Gamma(E_R))$ lying over P, there exist $w, v \in R$ and a characteristic power $\ell \in \mathbf{N}$ such that:*

1. *$w \notin P$.*
2. *$w\phi^\ell - v \in P_1$*
3. *If $\phi \notin P^*$, then $v \notin P$.*

Note that these conditions say that, modulo P_1, the section ϕ is the ℓ-th root of the fraction v/w with $v \notin P$ unless $\phi \in P^*$.

3.5.7. Theorem. *Let $R \in \mathcal{SPR}$ be given. Then the following conditions are equivalent:*

1. *R has the canonical sheaf representation property (that is, the ring of global sections of the canonical sheaf is $K(R)$, the reflection of R into $\mathscr{K}$).*
2. *The natural map $\nu : R \longrightarrow \Gamma(E)$ is epic in $\mathcal{SPR}$.*
3. *The map $\nu : R \longrightarrow \Gamma(E)$ has the unique prime lifting property.*
4. *R has the fractional root property.*

Proof.

1 $\Rightarrow$ *2*: Trivial since $R \longrightarrow K(R)$ is epic.

2 $\Rightarrow$ *1*: As noted earlier, every map $f : R \longrightarrow A$ with $A \in \mathcal{A}$ factors through ν (as $\Gamma(E) \subseteq \prod\{G(R/P\}$, see the discussion in 3.2). By hypothesis, ν is epic, so ν has the unique extension property with respect to every object in $\mathcal{A}$. Since $\Gamma(E) \in \mathscr{K}$, by 3.5.3 and 2.2.13, we conclude that ν is the reflection map.

2 $\Rightarrow$ *3*: Assume $\nu : R \longrightarrow \Gamma(E)$ is epic in $\mathcal{SPR}$. Let $P_1, P_2 \in \mathrm{Spec}(\Gamma(E))$ be two ideals lying over $P \in \mathrm{Spec}(R)$. Then, by Proposition 2.2.1, there is a field F and a commutative diagram

$$\begin{tikzcd}
R/P \arrow[r, hook] \arrow[d, hook] & \Gamma(E)/P_1 \arrow[d, hook] \\
\Gamma(E)/P_2 \arrow[r, hook] & F
\end{tikzcd}$$

Since ν is epic, we have that

$$\Gamma(E) \longrightarrow \Gamma(E)/P_1 \hookrightarrow F = \Gamma(E) \longrightarrow \Gamma(E)/P_2 \hookrightarrow F$$

and this implies that $P_1 = P_2$.

3$\Rightarrow$*4*: Suppose $\phi \in \Gamma(E)$, $P \subseteq R$ is prime and P^* is the unique prime ideal of $\Gamma(E)$ lying above P. Then $\phi(P) \in G(R/P)$ which, by 2.2.10, means that there is a characteristic power $\phi(P)^\ell$ such that $w\phi(P)^\ell - v = 0$ in $G(R/P)$, which implies that $w\phi(P)^\ell - v \in P^*$. The remaining details are straightforward.

4 $\Rightarrow$*2*: We first prove that 4$\Rightarrow$ 3 and then show that 4 + 3 $\Rightarrow$2. Assume the fractional root property and let P_1 be a prime ideal of $\Gamma(E)$ lying over $P \in \mathrm{Spec}(R)$. We claim that $P_1 \subseteq P^*$. If not, there exists $\phi \in P_1 - P^*$. By 4, there exist $w, v \in R$ and a characteristic power ℓ such that $w \notin P$ and $w\phi^\ell - v \in P_1$ and, since $\phi \notin P^*$, we also have $v \notin P$. Then, since $\phi \in P_1$ and $w\phi^\ell - v \in P_1$, we see that $v \in P_1$ so $v \in R \cap P_1 = P$, a contradiction.

We next claim that $P^* \subseteq P_1$. Assume $\phi \in P^*$ is given. By 4, there exist $w, v \in R$ and a characteristic power ℓ such that $w \notin P$ and $w\phi^\ell - v \in P_1$. But, as shown above, $P_1 \subseteq P^*$ so $w\phi^\ell - v \in P^*$. Since $\phi \in P^*$ and $w\phi^\ell - v \in P^*$, we see that $v \in P^*$ so $v \in R \cap P^* = P$. From $w\phi^\ell - v \in P_1$ and $v \in P \subseteq P_1$, we get $w\phi^\ell \in P_1$ and, since $w \notin P_1$ (as this would imply $w \in P$) we get $\phi \in P_1$.

Therefore $\nu : R \longrightarrow \Gamma(E)$ has the unique prime lifting property and it remains to show that ν is epic. Let $g_1, g_2 : \Gamma(E) \longrightarrow F$ be maps in $\mathcal{SPR}$ for which $g_1\nu = g_2\nu$. Since every semiprime ring is a subring of a product of fields, we may assume that F is a field. Let $P_i = \ker(g_i)$ for $i = 1, 2$. Let $P = \ker(g_1\nu) = \ker(g_2\nu)$. Then, by the unique prime lifting, we see that $P_1 = P_2 = P^*$.

Let $\phi \in \Gamma(E)$ be given. By the fractional root property, we see that there are $w, v \in R$, with $w \notin P$ and a characteristic power $\ell \in \mathbf{N}$ such that $w\phi^\ell - v \in P^*$. But this clearly implies that any R-homomorphism $R \longrightarrow F$ takes ϕ to the unique root of $wx^\ell - v$ as $w \notin \ker(g_i)$. ∎

3.6. When is $Q \sqsubseteq P$? Assume we are given $\mathcal{A}$, a class of domains satisfying the assumptions of 3.1.1. Let R be a semiprime ring and let Q, P be prime ideals of R with $Q \subseteq P$. In this subsection, we will find conditions under which $Q \sqsubseteq P$, then use these and related conditions to show that certain classes of reflections have the fractional root property.

3.6.1. Lemma. *Let $Q, P \subseteq R$ be prime ideals of the semiprime ring R with $Q \subseteq P$. Then $Q \sqsubseteq P$ if and only if $G(R/Q)$ has a prime ideal P' which lies over P.*

Proof. If $Q \sqsubseteq P$ then we can let P' be the kernel of $G(R/Q) \longrightarrow G(R/P)$. Conversely, suppose that P' is a prime ideal of $G(R/Q)$ lying over P. Let $C = G(R/Q)/P'$. Then the map $R/Q \hookrightarrow G(R/Q) \longrightarrow C$ clearly factors through an injection $R/P \rightarrowtail C$. By Proposition 2.2.1, there exists a field $F \in \mathcal{A}$ together with maps $C \rightarrowtail F$ and $G(R/P) \rightarrowtail F$ such that the diagram formed by these maps and the maps from R/P to C and to $G(R/P)$ commutes. Define B so that the following diagram is a pullback:

$$\begin{tikzcd}
B \arrow[r, tail] \arrow[dd] & G(R/Q) \arrow[d] \\
 & C \arrow[d, tail] \\
G(R/P) \arrow[r, tail] & F
\end{tikzcd}$$

We see that $B \in \mathscr{K}$, as $\mathscr{K}$ is limit closed. By the pullback property, there exists a map $R/Q \rightarrowtail B$ making everything commute. By definition of $G(R/Q)$, and the fact that $B \in \mathscr{K}$, we see that the map $B \rightarrowtail G(R/Q)$ is invertible and we may assume that $B = G(R/Q)$. The map $B \longrightarrow G(R/P)$ is then the required R-homomorphism which shows that $Q \sqsubseteq P$. ∎

3.6.2. Lemma. *Let $Q, P \subseteq R$ be prime ideals of the semiprime ring R with $Q \subseteq P$. Then there exists at most one prime ideal P' of $G(R/Q)$ satisfying the above conditions.*

Proof. Suppose that P' and P'' are two such prime ideals. Let $C' = G(R/Q)/P'$ and $C'' = G(R/Q)/C''$. The map $R \longrightarrow R/Q \longrightarrow G(R/Q) \longrightarrow C'$ is clearly epic (as $R/Q \longrightarrow G(R/Q)$ is epic, etc.) and has kernel P so it factors as $R \longrightarrow R/P \longrightarrow C'$ where $R/P \longrightarrow C'$ is epic. Similarly, there is an epic map $R/P \longrightarrow C''$. By the corollary to Lemma 2.2.7, this implies that the inclusion $R/P \hookrightarrow \overline{\mathbf{Q}}(R/P)$ factors through $R/P \longrightarrow C'$ by a map $C' \longrightarrow \overline{\mathbf{Q}}(R/P)$. Similarly, there is a map $C'' \longrightarrow \overline{\mathbf{Q}}(R/P)$. Since $R \longrightarrow G(R/Q)$ is epic and since $R \longrightarrow G(R/Q) \longrightarrow C' \longrightarrow \overline{\mathbf{Q}}(R/P)$ is the canonical embedding as is $R \longrightarrow G(R/Q) \longrightarrow C'' \longrightarrow \overline{\mathbf{Q}}(R/P)$ it follows that the map from $G(R/Q) \longrightarrow C' \longrightarrow \overline{\mathbf{Q}}(R/P)$ is the same as $G(R/Q) \longrightarrow C'' \longrightarrow \overline{\mathbf{Q}}(R/P)$, which obviously implies that $P' = P''$. ∎

3.6.3. Definition. *Let $Q, P \subseteq R$ be prime ideals of the semiprime ring R with $Q \subseteq P$. Then $(p_1, \ldots, p_n; z_1, \ldots, z_n)$ is a (Q, P)-**obstacle** if each $p_i \in P$ and each $z_i \in G(R/Q)$ and $p_1z_1 + \cdots + p_nz_n$ is in the image of $R - P$ under the map $R \longrightarrow R/Q \longrightarrow G(R/Q)$.*

3.6.4. Proposition. *Let $Q, P \subseteq R$ be prime ideals of the semiprime ring R. Then $Q \sqsubseteq P$ if and only if $Q \subseteq P$ and no (Q, P)-obstacle exists. Moreover, $y \in G(R/Q)$ is in the kernel of the unique R-homomorphism $G(R/Q) \longrightarrow G(R/P)$ if and only if there exists $r \in R - P$ and a positive integer m and $(p_1, \ldots, p_n; z_1, \ldots, z_n)$ with each $p_i \in P$ and each $z_i \in G(R/Q)$ such that $ry^m = p_1z_1 + \cdots + p_nz_n$.*

Proof. Suppose $(p_1, \ldots, p_n; z_1, \ldots, z_n)$ is a (Q, P)-obstacle. Then we claim that $Q \not\sqsubseteq P$. Assume $h : G(R/Q) \longrightarrow G(R/P)$ is an R-homomorphism. Let $s = p_1z_1 + \cdots + p_nz_n$ be in the image of $R - P$. Since $s \in R - P$, we see that $h(s) \neq 0$. But, clearly, $h(p_i) = 0$ for all i so $h(s) = 0$ which is a contradiction.

Conversely, assume that no (Q, P)-obstacle exists. Then regard P as an ideal of R/Q, which is clearly possible as $Q \subseteq P$. Let $\widehat{P}$ be the ideal of $G(R/Q)$ generated by $P \subseteq R/Q \subseteq G(R/Q)$. (Note that $\widehat{P}$ need *not* be a prime ideal of $G(R/Q)$, just an ideal.) Then:

$$\widehat{P} = \{p_1z_1 + \cdots + p_nz_n \mid p_i \in P \text{ and } z_i \in G(R/Q) \text{ for all } i\}$$

Let M be the image of $R-P$ in $G(R/Q)$. Clearly, M is a multiplicative subset of $G(R/Q)$. Since no (Q, P)-obstacle exists, we see that the ideal $\widehat{P}$ is disjoint from M. Let P' be an ideal which is maximal among the ideals of $G(R/Q)$ that contain $\widehat{P}$ and are disjoint from the multiplicative set M. It is well known (and readily proven) that P' is a prime ideal of $G(R/Q)$ and it obviously satisfies the conditions of Lemma 3.6.1.

If there are elements $r, m, p_1, \ldots, p_n, z_1, \ldots z_n \in R$ with $p_i \in P$, $r \notin P$ and $ry^m = p_1z_1 + \cdots + p_nz_n$, then clearly $y \in P'$. But if there exists no such expression, then the multiplicative system M_y generated by $R - P$ and y, which consists of all elements of the form ry^m, does not meet $\widehat{P}$, so we can choose P' disjoint from M_y and so $y \notin P'$. Since P' is unique by 3.6.2, it clearly follows that P' consists precisely of those elements y satisfying $ry^m = p_1z_1 + \cdots + p_nz_n$ as required. ∎

3.6.5. Definition. *Let R be semiprime ring and assume $R \subseteq S \subseteq \Gamma(E)$. For each prime ideal P of R let $P^\# = P^* \cap S$. Note that for every prime ideal Q of R we can regard $R/Q \subseteq S/Q^\# \subseteq G(R/Q)$. We then say that $R \subseteq S \subseteq \Gamma(E)$ is* **admissible** *if:*

1. *For all Q, the map $R/Q \longrightarrow S/Q^\#$ is epic.*
2. *Whenever $Q \subseteq P$ are prime ideals of R then $Q \sqsubseteq P$ if there exists a prime ideal P' of $S/Q^\#$ such that $P' \cap R/Q$ is P (where P is regarded as a prime ideal of R/Q).*

3.6.6. Definition. *If $R \subseteq S \subseteq \Gamma(E)$ is admissible, we will say that $(p_1, \ldots, p_n; z_1, \ldots, z_n)$ is an S-**obstacle from** Q **to** P if each $p_i \in P$ and each $z_i \in S/Q^\#$ and $p_1z_1 + \cdots + p_nz_n$ is in the image of $R - P$ under the map $R \longrightarrow R/Q \longrightarrow S/Q^\#$.*

3.6.7. Lemma. *Let $R \subseteq S \subseteq \Gamma(E)$ be admissible. Then the following three conditions are equivalent:*

1. *$Q \sqsubseteq P$.*

2. *There is no S-obstacle from Q to P.*

3. *There is an R-homomorphism $S/Q^{\#} \longrightarrow S/P^{\#}$.*

Moreover, if the above equivalent conditions hold, then P', the kernel of the R-homomorphism $S/Q^{\#} \longrightarrow S/P^{\#}$ is the set of $y \in S/Q^{\#}$ such that $ry^m = p_1 z_1 + \cdots + p_n z_n$ for some $z_i \in S/Q^{\#}$ and $p_i \in P$.

Proof. Assume $Q \sqsubseteq P$ then clearly $Q^* \subseteq P^*$ and $Q^{\#} \subseteq P^{\#}$ so 3 follows. The arguments given in the proofs of Lemma 3.6.2 and Proposition 3.6.4 can now be modified slightly to complete the proof of the equivalence of 1,2 and 3.

The proof about the characterization of the kernel of $S/Q^{\#} \longrightarrow S/P^{\#}$ also follows by adapting the arguments in 3.6.2 and 3.6.4. ∎

3.6.8. Definition. *Let R be a semiprime ring and let $\Gamma(E)$ be the ring of global sections of its canonical sheaf. We say that R satisfies the* **fractional root property at** $\phi \in \Gamma(E)$*, if whenever P_1 is a prime ideal of $\Gamma(E)$ lying over some prime P of R there exist $w, v \in R$ and a characteristic power ℓ such that $w \notin P$, $w\phi^\ell - v \in P_1$ and if $\phi \notin P^*$ then $v \notin P$.*

3.6.9. Proposition. *If $R \subseteq S \subseteq \Gamma(E)$ is admissible, then R has the fractional root property with respect to ϕ whenever $\phi \in S$.*

Proof. Let $\phi \in S$ and let $P_1 \in \mathrm{Spec}(\Gamma(E))$ be a prime lying over $P \in \mathrm{Spec}(R)$. We must find $w, v \in R$ and a characteristic power ϕ^ℓ such that the conditions in Definition 3.5.6 are satisfied. Our approach will be to find, for each $Q \in \mathrm{Spec}(R)$, an element $\zeta_Q \in \Gamma(E)$ such that $\zeta_Q(Q) = 0$ and, if $\zeta_Q \in P_1$, then the desired elements w, v can readily be found. So let $Q \in \mathrm{Spec}(R)$ be given. We consider the following cases:

Case 1: $Q \not\subseteq P$. In this case, we can choose $r \in Q - P$. We can choose w, v, ℓ so that $\phi(Q)$ is determined by the equation $\phi(Q)^\ell = v/w$ with $w \notin Q$. We can further assume that $w \notin P$ and $v \notin P$. For if $w \in P$, we can replace w by $w + r$ and similarly v can be replaced by $v + r$, if necessary. Let $\zeta_Q = w\phi^\ell - v$.

Case 2: $Q \subseteq P$. Let $e_Q : S \longrightarrow G(R/Q)$ be defined by evaluation at Q, so that $e_Q(\phi) = \phi(Q)$. Since $Q^{\#} = \ker(e_Q)$, we can regard it as a map $S \longrightarrow S/Q^{\#} \hookrightarrow G(R/Q)$ where $S/Q^{\#}$ is identified with a subring of $G(R/Q)$.

Subcase 2a: $Q \sqsubseteq P$. Note that $\phi(Q) \in S/Q^{\#}$. Let α be the image of $\phi(Q)$ in $S/P^{\#}$. Then α is determined by a condition of the form $w\alpha^\ell - v = 0$. It follows that for some $r \in R - P$ we can write $r(w\phi(Q)^\ell - v)^m = p_1 d_1 + \cdots + p_n d_n$. For each i, let $\delta_i \in S$ be such that $\delta_i(Q) = d_i$. Let $\zeta_Q = r(w\phi^\ell - v)^m - (p_1\delta_1 + \cdots + p_n\delta_n)$. Note that if $v \in P$ then $\alpha = 0$ and since $Q \sqsubseteq P$, we see that $\phi(P) = 0$ so $\phi \in P^*$

Subcase 2b $Q \not\sqsubseteq P$: Then there must exist an obstacle given by $r = p_1 d_1 + \cdots + p_n d_n$ with $r \in R - P$ and each $p_i \in P$. Again, we let $\delta_i \in S$ be such that $\delta_i(Q) = d_i$. Let $\zeta_Q = r - (p_1\delta_1 + \cdots + p_n\delta_n)$. Then $\zeta_Q(Q) = 0$ and we cannot possibly have $\zeta_Q \in P_1$. So it follows trivially that if $\zeta_Q \in P_1$, then R has the fractional root property at ϕ. We call this the "impossible case".

For each $Q \in \mathrm{Spec}(R)$, let W_Q be the open set (in the $\mathcal{A}$-topology on $\mathrm{Spec}(R)$) where $\zeta_Q = 0$. This is clearly an open neighbourhood of Q. So the neighbourhoods $\{W_Q\}$ cover $\mathrm{Spec}(R)$. Compactness implies that there is a finite subcover $W_{Q_1}, \ldots, W_{Q_k}$. It follows that the finite product $\zeta_{Q_1} \cdots \zeta_{Q_k} = 0$. Since P_1 is prime, there exists i with $\zeta_{Q_i} \in P_1$ (so Q_i cannot be in the impossible case) and this readily implies that we have the required global section of the form $w\phi^\ell - v \in P_1$. ∎

3.6.10. Corollary. *If $\Gamma(E) \longrightarrow G(R/Q)$ is always surjective then R has the fractional root property.*

Proof. Clearly, $S = \Gamma(E)$ is admissible, so the result follows from the above proposition. ∎

3.6.11. Corollary. *In the $\mathcal{D}om$-invariant case, every R has the fractional root property.*

Proof. Let $\phi \in \Gamma(E)$ and let $S = R[\phi] \subseteq \Gamma(E)$. Then $R \subseteq S \subseteq \Gamma(E)$ is admissible because $R/Q \longrightarrow S^{\#}$ is always epic by Theorem 2.3.1.12 and, because $Q \sqsubseteq P$ whenever $Q \subseteq P$. The proposition now implies that R has the fractional root property with respect to ϕ and the result follows as ϕ is arbitrary. ∎

Putting this together with Theorem 3.5.7 we conclude:

3.6.12. Theorem. *In the $\mathcal{D}om$-invariant case, every ring R has the canonical sheaf representation property, that is $\nu : R \longrightarrow \Gamma(E)$ is the reflection of R into $\mathscr{K}$.* ∎

3.7. Examples. Our examples for this section illustrate several ways of proving the canonical sheaf representation property. We do not have any example where it fails and the question of whether it always holds remains open as far as we know.

Recall the convention of 2.2.26 that if $\mathcal{A}_{\mathrm{xx}}$ denotes a category of domains, then $\mathscr{K}_{\mathrm{xx}}$ denotes its limit closure and $\mathscr{B}_{\mathrm{xx}}$ denotes the domains in $\mathscr{K}_{\mathrm{xx}}$.

3.7.1. Example. Suppose that for every $R \in \mathcal{SPR}$, the $\mathcal{A}$-topology on $\mathrm{Spec}(R)$ coincides with the domain topology. We will then prove that we are in the $\mathcal{D}om$-invariant case, so, as noted above, the canonical sheaf representation property holds. We first claim that whenever Q, P are prime ideals of R with $Q \subseteq P$, there is an R-homomorphism $h : G(R/Q) \longrightarrow G(R/P)$. To prove this claim, let $z \in G(R/Q)$ be given. Let $E \longrightarrow \mathrm{Spec}(R)$ be the canonical sheaf. Then there is a local section $\zeta : U \longrightarrow E$ with $\zeta(Q) = z$, where U is a neighbourhood of Q. But in the domain topology, every neighbourhood of Q is up-closed, so $P \in U$. Define $h(z) = \zeta(P)$. Since local sections must agree on open sets, we see that $h(z)$ is uniquely defined. Since sums and products of local sections are sections, we see that h preserves addition and multiplication. Similarly, h preserves the elements of R as for each $r \in R$, there is a "constant" section $\nu(r)$. Applying this map h to the case where $R = D$, a domain and Q is the zero ideal, we get a proof of Theorem 2.3.1.2, so we are in the $\mathcal{Dom}$-invariant case.

3.7.2. Example. Let $R \in \mathcal{SPR}$ be given and suppose the $\mathcal{A}$-topology on $\mathrm{Spec}(R)$ coincides with the patch topology. We claim that in this case $\Gamma(E) \longrightarrow G(R/P)$ is surjective for all prime ideals P, which, by Corollary 3.6.10, proves the canonical representation property. To prove the claim, let $z \in G(R/P)$ be given. There is a neighbourhood U of z and a local section $\zeta_0 \in \Gamma(U)$ with $\zeta_0(P) = z$. Since the patch topology has a base of clopen sets, we can choose U to be clopen. We can then define $\zeta \in \Gamma(E)$ so that $\zeta(Q) = \zeta_0(Q)$ whenever $Q \in U$ and define $\zeta(Q) = 0$ whenever $Q \notin U$.

For example, this happens for all R if $\mathcal{A}$ is the category of fields or the category of perfect fields.

3.7.3. Example. If $R \in \mathcal{SPR}$, then there is clearly a unique morphism $\mathbf{Z} \longrightarrow R$ where $\mathbf{Z}$ is the ring of integers. For each $n \in \mathbf{Z}$, we let n_R denote the image of n in R. If there is no danger on confusion, we simply write n for n_R. We call the elements of the form n_R the **integers** of R.

We say that $R \in \mathcal{SPR}$ is **quasi-rational** if every integer of R has a quasi-inverse. We let $\mathcal{A}_{\mathrm{qrat}}$ be the class of quasi-rational domains. It is clear that the class of all quasi-rational rings is closed under limits, hence $\mathcal{K}_{\mathrm{qrat}}$ is contained in the class of quasi-rational rings. But there are quasi-rational rings which are not in $\mathcal{K}_{\mathrm{qrat}}$, see 4.4.5 for a quasi-rational ring that is not in the limit closure of any class of domains. A description of $\mathcal{K}_{\mathrm{qrat}}$ is given in 4.4.3.

If D is a domain, then $G(D)$ is obtained by adjoining inverses for each non-zero integer of D. If $E = E_R$ for some $R \in \mathcal{SPR}$ then $\Gamma(E) \in \mathcal{K}_{\mathrm{qrat}}$ and hence is rational and the evaluation map $\Gamma(E) \longrightarrow G(R/P)$ sends integers to integers and therefore sends the quasi-inverses of integers in $\Gamma(E)$ to the quasi-inverses of integers in $G(R/P)$. Since $G(R/P)$ is generated by these quasi-inverses, it follows that $\Gamma(E) \longrightarrow G(R/P)$ is surjective for all P and so $\mathcal{A}_{\mathrm{qrat}}$ has the canonical sheaf representation property by Corollary 3.6.10.

3.7.4. Example. Recall that $\mathcal{A}_{\mathrm{per}}$ denotes the full subcategory of all perfect domains. A description of $\mathcal{K}_{\mathrm{per}}$ is given in section 6 and it is clearly first order. Since $\mathcal{A}_{\mathrm{icp}} \subseteq \mathcal{A}_{\mathrm{per}}$, it follows from Theorem 2.3.1.15 that the limit closure of $\mathcal{A}_{\mathrm{per}}$ is $\mathcal{Dom}$-invariant and by Corollary 3.6.11 it has the canonical sheaf representation property.

3.8. Polynomial operations. As in Section 2 we assume that $\mathcal{K}$ is the limit closure of a full subcategory, $\mathcal{A}$, of domains such that the conditions in 2.1 are satisfied.

Many of our examples involve essentially algebraic operations (see 3.8.1) such as the polynomial operations, see 3.8.4. In this subsection we will show that such operations often arise, particularly in the $\mathcal{Dom}$-invariant case and when $\mathcal{A}$ is first order.

3.8.1. Remark. An **essentially algebraic theory** consists of an equational theory in the usual sense, with operations and equations, which may be infinitary, augmented by partial operations whose domains are given by equational conditions in the operations, both in the total operations and in the other partial operations as well as equations that may involve both the operations and partial operations. If all operations and partial operations depend on only finitely many variables, then the essentially algebraic theory is called **finitary**. But even a finitary theory might have infinitely many operations and partial operations.

Morphisms of the algebras for such a theory are required to preserve all operations and partial operations. However, *in all the examples in this article, the values of the partial operations are uniquely determined by the equations they satisfy.* Thus the algebras for the theory form a full subcategory of the algebras for the subtheory consisting of the total operations and all the equations they satisfy.

As an example, we look at the theory whose operations and equations are those of commutative rings. Add a partial operation ω whose domain is $\{x \mid x^2 = 0\}$ that satisfies the equations $\omega(x) = x$ and $\omega(x) = 0$. The algebras for this theory is the category of semiprime rings. The value of this operation is uniquely determined by being 0.

For an example of a non-finitary theory, consider the following infinitary partial operation ω. The domain is described as $\{x, y_2, y_3, \ldots, y_n, \ldots \mid y_2^2 = x, y_3^3 = x, \ldots, y_n^n = x, \ldots\}$ and subject to the equations that for $z = \omega(x, y_2, \ldots)$, then $x^2 z = x$ and $xz^2 = z$. It is well known that x and z uniquely determine each other. A UFD satisfies this, since an element with all nth roots is either 0 or invertible. The limit closure of UFDs is not closed under ultraproducts (see 2.4), which illustrates the results of the following proposition, whose proof we leave to the reader. (We use, and prove, a more detailed version of this proposition.)

3.8.2. PROPOSITION. *The algebras for an essentially algebraic theory are limit closed in the category of algebras for the total operations and equations. If the theory is finitary, it is also closed under filtered colimits and, in particular, ultraproducts.* ∎

3.8.3. NOTATION.

1. For any ring R, we let $|R|$ denote its underlying set. So a **function** h from R to S, which is not necessarily a homomorphism, will usually be denoted by $h : |R| \longrightarrow |S|$.

2. In this subsection we assume that X is a set, possibly infinite, whose elements will be called **independent variables**.

3. We assume that y denotes any entity which is not in X and which will be called the **dependent variable**.

4. As usual, $\mathbf{Z}[X]$ is the ring of polynomials in the variables X with integer coefficients, and $\mathbf{Z}[X \cup \{y\}]$ is the ring of polynomials in $X \cup \{y\}$.

5. If R is a ring and $i : X \longrightarrow |R|$ is a function, and if $f \in \mathbf{Z}[X]$, then $f(i) \in R$ is obtained by replacing each variable $x \in X$ with $i(x) \in R$ and evaluating in the usual way.

6. Similarly if i is as above and if $g \in \mathbf{Z}[X \cup \{y\}]$ and $r \in R$, then $g(i,r) \in R$ is obtained by replacing each variable $x \in X$ with $i(x)$ and replacing y with r and evaluating.

   We then say that (i,r) is an **extension** of i.

7. If $F \subseteq \mathbf{Z}[X]$, then an **interpretation of** (X,F) **in** R is a function $i : X \longrightarrow |R|$ such that for $f(i) = 0$ for all $f \in F$.

8. Similarly, if $G \subseteq \mathbf{Z}[X \cup \{y\}]$, then an **interpretation of** $(X \cup \{y\}, G)$ **in** R is a function $i : X \longrightarrow |R|$ together with an element $r \in R$ such that for $g(i,r) = 0$ for all $g \in G$.

3.8.4. Definition. *Let X and $X \cup \{y\}$ be as in the above notation. A* **polynomial operation** *is determined by sets $F \subseteq \mathbf{Z}[X]$ and $G \subseteq \mathbf{Z}[X \cup \{y\}]$ such that for every semiprime ring R and every interpretation $i : X \longrightarrow |R|$ there exists at most one $r \in R$ such that (i,r) is an interpretation of $(X \cup \{y\}, G)$ in R.*

We say that R is a model of (F,G) if for every interpretation i of (X,F) in R there exists $r \in R$ (necessarily unique) such that (i,r) is an interpretation of $(X \cup \{y\}, G)$ in R. A polynomial theory is given by a possibly infinite set Ω of polynomial operations. The semiprime ring R is a model of the theory Ω if and only if R is a model of each operation $\omega \in \Omega$.

3.8.5. Definition. *The polynomial operation (F,G) will be called* **finitistic** *if F and G are finite.*

We can always assume that X is the set of indeterminates that actually appear in F and G, so we may assume that X is finite in the finitistic case.

3.8.6. Proposition. *Let $\mathcal{A}$ be a first-order class of domains such that $\mathscr{K}$, the limit closure of $\mathcal{A}$, is in the $\mathfrak{Dom}$-invariant case. Then there is a finitistic polynomial theory Ω such that $\mathscr{K}$ is the class of all models of Ω.*

Before proving this Proposition, we introduce some definitions and lemmas.

3.8.7. Definition. *Let $\mathcal{A}$, $\mathscr{K}$, and the reflector K be as in section 2. We say that the polynomial operation (F,G) is a* **polynomial operation on $\mathcal{A}$** *if every $A \in \mathcal{A}$ is a model of (F,G).*

3.8.8. Lemma. *If (F,G) is a polynomial operation on $\mathcal{A}$, then every ring in $\mathscr{K}$ is a model of (F,G).*

Proof. It is easily shown that the class of all rings which are models of (F,G) is closed under products and equalizers. ∎

3.8.9. Lemma. *If (F,G) is a polynomial operation on $\mathcal{A}$, then, using the above notation, the map $k : \mathbf{Z}[X]/\langle F\rangle \longrightarrow \mathbf{Z}[X \cup \{y\}]/\langle F \cup G\rangle$ is epic in $\mathcal{SPR}$.*

Proof. Suppose that $m, n : \mathbf{Z}[X \cup \{y\}]/\langle F \cup G\rangle \longrightarrow R$ are given with $mk = nk$. Since we can embed R into $K(R)$ it suffices to prove this for $R \in \mathscr{K}$. But, in view of the above lemma, we must have $m(y) = n(y)$ but this clearly implies that $m = n$. ∎

3.8.10. LEMMA. *There exists a finite set $H_1 \subseteq F \cup G$ such that the homomorphism $\mathbf{Z}[X]/\langle F\rangle \longrightarrow \mathbf{Z}[X \cup \{y\}]/\langle H_1\rangle$ is epic.*

PROOF. The map $k : \mathbf{Z}[X]/\langle F\rangle \longrightarrow \mathbf{Z}[X \cup \{y\}]/\langle F \cup G\rangle$ was shown to be epic in the above lemma. Clearly, for any $H \subseteq F \cup G$, the map $k : \mathbf{Z}[X]/\langle F\rangle \longrightarrow \mathbf{Z}[X \cup \{y\}]/\langle H\rangle$, will be epic if and only if y is in the dominion of $\mathbf{Z}[X]/\langle F\rangle$. Now, by Lemma 3.3.7 there is a finite set $H_1 \subseteq F \cup G$ for which $y \in \mathbf{Z}[X \cup \{y\}]/\langle H_1\rangle$ is in the dominion of $\mathbf{Z}[X]/\langle F\rangle$. ∎

3.8.11. LEMMA. *Let $\mathscr{U}$ be a collection of subsets of a set S. Recall that $\mathscr{U}$ has the bf finite intersection property (f.i.p.) if every finite set $\{U_1, U-_2, \ldots, U_n\} \subseteq \mathscr{U}$ has non-empty intersection. Then there exists an ultrafilter $\mathbf{u}$ on S with $\mathscr{U} \subseteq \mathbf{u}$ if and only if $\mathscr{U}$ has f.i.p.*

PROOF. If $\mathscr{U}$ is contained in an ultrafilter it is immediate that $\mathscr{U}$ has f.i.p. Conversely, suppose that $\mathscr{U}$ has f.i.p. Then let $\mathscr{F}$ be the family of all subsets of S that contain a finite intersection of sets from $\mathscr{U}$. Clearly, $\mathscr{F}$ is a filter of subsets and therefore it can be extended to an ultrafilter. ∎

3.8.12. LEMMA. *Assume that $\mathcal{A}$ is closed under ultraproducts and let (F, G) be a polynomial operation on $\mathcal{A}$. Then there are finite subsets $F_0 \subseteq F$ and $G_0 \subseteq G$ such that (F_0, G_0) is a polynomial operation on $\mathcal{A}$.*

Moreover, whenever a semiprime ring R is a model of (F_0, G_0) then it is also a model of (F, G).

PROOF. Let $H_1 \subseteq F \cup G$ be as in the above lemma. Let $F_1 = H_1 \cap F$ and $G_0 = H_1 \cap G$. Now say that a subset $S \subseteq F$ is **adequate** if $F_1 \subseteq S$ and every map $\mathbf{Z}[X]/\langle S\rangle \longrightarrow A$ with $A \in \mathcal{A}$ has an extension (necessarily unique) to a map $\mathbf{Z}[X \cup \{y\}]/\langle S \cup G_0\rangle \longrightarrow A$. It follows that S, with $F_1 \subseteq S \subseteq F$ is adequate if and only if for every map $\mathbf{Z}[X]/\langle S\rangle \longrightarrow A$ with $A \in \mathcal{A}$, there exists a unique $a \in A$ such that $g(a) = 0$ for all $g \in G_0$, as we can then map $\mathbf{Z}[X \cup \{y\}] \longrightarrow A$ by sending y to a. (Our notation suppresses the variables from X that might appear in g as these variables have already been assigned to values in A by the given map $\mathbf{Z}[X]/\langle S\rangle \longrightarrow A$.)

For example, $S = F$ is clearly adequate as every map $\mathbf{Z}[X]/\langle F\rangle \longrightarrow A$ has an extension to a map $\mathbf{Z}[X \cup \{y\}]/\langle F \cup G\rangle \longrightarrow A$ because A is a model of the polynomial operation (F, G). We claim that F has a finite adequate subset. Assume the claim is false. Let $\mathscr{E}$ be the set of all finite subsets E with $F_1 \subseteq E \subseteq F$. Since the claim is false, it follows that for every $E \in \mathscr{E}$ there is a map $m_E : Z[X]/\langle E\rangle \longrightarrow A_E$ with $A_E \in \mathcal{A}$ such that there is no element $a \in A_E$ which satisfies $g(a) = 0$ for every $g \in G_0$.

For each $f \in F$ let $U_f = \{E \in \mathscr{E} \mid f \in E\}$. It is clear that this family of subsets of $\mathscr{E}$ has the finite intersection property because if $C \subseteq F$ is a finite subset, then $F_1 \cup C \in \bigcap\{U_f \mid f \in C\}$. Therefore, by Lemma 3.8.11, there exists an ultrafilter $\mathbf{u}$ on $\mathscr{E}$ such that $U_f \in \mathbf{u}$ for all $f \in F$. Let $A_{\mathbf{u}}$ be the ultraproduct of the family $\{A_E \mid E \in \mathscr{E}\}$. We can regard each map m_E as a function from $\mathbf{Z}[X]/\langle F_1\rangle \longrightarrow A_E$ hence there is an obvious induced map $m : \mathbf{Z}[X]/\langle F\rangle \longrightarrow \prod\{A_E \mid E \in \mathscr{E}\}$. Let $m_{\mathbf{u}}$ be $q_{\mathbf{u}}m$ where $q_u : \prod\{A_E\} \longrightarrow A_{\mathbf{u}}$ is the quotient map associated with the ultraproduct. But then $m_{\mathbf{u}}(f) = 0$ for all $f \in F$

because this condition is true on $U_f \in \mathbf{u}$. So we can regard $m_u : \mathbf{Z}[X]/\langle F\rangle \longrightarrow A_{\mathbf{u}}$. But $A_{\mathbf{u}} \in \mathcal{A}$ and, since (F, G) is a polynomial operation on $\mathcal{A}$, it follows that there exists $a \in A_{\mathbf{u}}$ such that $g(a) = 0$ for all $g \in G_0$ (in fact, for all $g \in G$).

Since $a \in A_{\mathbf{u}}$ we can write $a = q_{\mathbf{u}}(a')$ where $a' \in \prod\{A_E\}$. For each $E \in \mathscr{F}$, let a_E be the projection of a' onto A_E. Since $g(m_{\mathbf{u}}, a) = 0$, there must exist $V_g \in \mathbf{u}$ such that $g(m_E, a_E) = 0$ for all $E \in V_g$. Let $V = \bigcap\{V_g \mid g \in G_0\}$. Since $V \in \mathbf{u}$, we see that V is non-empty. If we choose $E \in V$, then a_E satisfies $g(a_E) = 0$ for all $g \in G_0$, which contradicts the assumption that no such element of A_E exists. By the claim we just proved, there is a finite adequate subset $E \in \mathscr{E}$. Letting $F_0 = E$, it easily follows that (F_0, G_0) is a polynomial operation on $\mathcal{A}$

Finally, suppose R is a model of (F_0, G_0). Let i be an interpretation in R of (X, F). Then i is an interpretation of (X, F_0), so there is a unique $r_0 \in R$ such that (i, r_0) is an interpretation of $(X \cup \{y\}, G_0)$. There is also a unique $r_1 \in K(R)$ such that (i, r_1) is an interpretation of $(X \cup \{y\}, G)$. Therefore both r_0 and r_1 are interpretations of $(X \cup \{y\}, G_0)$ in $K(R)$ and, by the choice of G_0 there is only one such interpretation in $K(R)$, so $r_1 = r_0$. But this shows that $r_1 \in R$ is such that (i, r_1) is an interpretation of $(X \cup \{y\}, G)$. ■

The following proposition is clearly equivalent to Proposition 3.8.6.

3.8.13. Proposition. *Assume that $\mathcal{A}$ is closed under ultraproducts and that $\mathscr{K}$ is $\mathfrak{Dom}$-invariant. Then a semiprime ring is in $\mathscr{K}$ if and only if it is a model of every finitistic polynomial operation on $\mathcal{A}$.*

Proof. By Lemma 3.8.8 every ring in $\mathscr{K}$ is a model for every finitistic polynomial operation on $\mathcal{A}$. Conversely, assume that R is a model of every such polynomial operation. We claim that $R \in \mathscr{K}$. Assume the contrary. Then there exists $\zeta \in K(R) - R$. Let $R[\zeta]$ be the subring of $K(R)$ generated by R and ζ. By Theorem 2.3.1.12, we see that $R \longrightarrow R[\zeta]$ is epic. Suppose $A \in \mathcal{A}$. By adjointness, every homomorphism $h : R \longrightarrow A$ has a unique extension to $K(R) \longrightarrow A$. By restricting that extension to $R[\zeta]$, we see that every homomorphism $h : R \longrightarrow A$ extends uniquely to a homomorphism $\overline{h} : R[\zeta] \longrightarrow A$.

We proceed to restate this as a polynomial operation on $\mathcal{A}$. For each $r \in R$, we introduce a variable x_r and let $X = \{x_r \mid r \in R\}$. We next define a set $F \subseteq \mathbf{Z}[X]$ such that a function $h : |R| \longrightarrow |A|$ interprets (X, F) if and only if h is a homomorphism. Define F to consist of $x_1 - 1$, and, for every pair of elements $r, s \in R$, the polynomials $x_{r-s} - x_r + x_s$ and $x_{rs} - x_r x_s$. Clearly h interprets F if and only if h is a homomorphism.

Next given a homomorphism $h : R \longrightarrow A$, we consider what it means to have an extension of h to $\overline{h} : R[\zeta] \longrightarrow A$. Since ζ generates $R[\zeta]$ as an R-algebra, we can write $R[\zeta] = R[y]/I$ where y is an indeterminate and I is an ideal. For each polynomial $g(y) \in I$, we let $g'(y) \in \mathbf{Z}[Y]$ be the polynomial obtained from g by replacing each coefficient r of g by the corresponding variable x_r. Let $G = \{g' \mid g \in I\}$. It is readily shown that if $h : R \longrightarrow A$ is a homomorphism, then h extends to $\overline{h} : R[\zeta] \longrightarrow A$ for which $\overline{h}(\zeta) = r_0$ if and only if (h, r_0) is an interpretation of $(\overline{X}, G)$. It follows that (F, G) is a polynomial operation on $\mathcal{A}$. By Lemma 3.8.12, there exists a finitistic polynomial operation (F_0, G_0)

on $\mathcal{A}$ such that any semiprime ring which is a model of (F_0, G_0) is a model of (F, G). But this implies that R is a model of (F, G). Consider the interpretation $i : X \longrightarrow R$ for which $i(x_r) = r$. Clearly this is an interpretation of (X, F) so there exists a unique $r_0 \in R$ such that (i, r_0) is an interpretation of $(X \cup \{y\}, G)$. But (i, ζ) is the unique such extension of i in the ring $K(R)$, so $r_0 = \zeta$. This implies $\zeta \in R$, which contradicts the choice of ζ. ∎

3.8.14. Corollary. *If we are in the* $\mathfrak{Dom}$*-invariant case and if* $\mathcal{A}$ *is closed under ultraproducts, then* $\mathscr{K}$ *(and therefore* $\mathscr{B}$*) are both first order.*

Proof. Given a finitistic polynomial operation (F, G), there clearly exists a first-order condition which R satisfies if and only if R is a model of (F, G). ∎

3.8.15. $\mathscr{B}$ is first order if it is ultraproduct closed.

In what follows, we assume that $\mathscr{B}$ is closed under ultraproducts and we assume that D is a domain which is not in $\mathscr{B}$, but which satisfies every first-order condition that is satisfied by every $B \in \mathscr{B}$. We will show that these assumptions lead to a contradiction.

3.8.16. Lemma. *Let* $\mathscr{B}$ *and* $D \notin \mathscr{B}$ *be as above. Then there exist* $z \in G(D) - D$ *and* $c, d \in D$*, with* $d \neq 0$ *and a characteristic power* ℓ*, (a power of* $\mathrm{char}(D)$*) such that* $dz^\ell = c$*.*

Proof. We may as well assume that $D \subseteq G(D) \subseteq \overline{Q}(D)$ where $\overline{Q}(D)$ is the perfect closure of the field of fractions of D. Since $D \notin \mathscr{B}$, there exists $z \in G(D) - D$. By definition of $\overline{Q}(D)$ there exists ℓ, a power of $\mathrm{char}(D)$ together with $c, d \in D$, with $d \neq 0$ such that $z^\ell = c/d$. ∎

3.8.17. Lemma. *If* $h : D \longrightarrow B$ *is an injection and* $B \in \mathscr{B}$ *then there exists* $b \in B$ *such that* $h(d)b^\ell = h(c)$*.*

Proof. We can define $G(D)$ by first embedding D in any field in $\mathcal{A}$ then taking the meet of all domains in $\mathscr{B}$ that contain D. It follows that we could also start by embedding D in any $B \in \mathscr{B}$ as any such B can be embedded in a field in $\mathcal{A}$. Since $h : D \longrightarrow B$ is such an embedding, it follows that we can assume that (up to isomorphism) $G(D) \subseteq B$ and the result follows. ∎

3.8.18. Notation. Let X be any set of indeterminates that is in bijective correspondence with $|D|$. To be specific, let us say that for every $r \in D$ we have $x_r \in X$. There is an isomorphism $i : \mathbf{Z}[X]/I \longrightarrow D$ which takes $x_r \in X$ to $r \in D$ for every $r \in D$, and where I is a semiprime ideal. In what follows, we will identify D with $\mathbf{Z}[X]/I$.

3.8.19. Definition. *Given the above notation, we say that a* **function** $h : X \longrightarrow B$*, with* $B \in \mathscr{B}$*, is* **admissible** *if the extension of* h *to a homomorphism* $\overline{h} : \mathbf{Z}[X] \longrightarrow B$ *is such that* $\ker(\overline{h}) = I$*.*

3.8.20. Notation. Let $h : X \longrightarrow B$ with $B \in \mathscr{B}$ be a function. For every polynomial $f \in I$, we let $f(h) \in B$ be the element defined by replacing each indeterminate $x \in X$ with $h(x) \in B$. Equivalently, $f(h) = \overline{h}(f)$ where $\overline{h}$ is the extension of h to a homomorphism from $\mathbf{Z}[X] \longrightarrow B$.

3.8.21. Lemma. *The function $h : X \longrightarrow B$ with $B \in \mathscr{B}$ is admissible if and only if for every $f \in I$ we have $f(h) = 0$ and for every $x_s \in X$, with s a* **non-zero** *member of D, we have $h(x_s) \neq 0$.*

Proof. The first condition implies that the kernel of $\overline{h}$ contains all of I, where I is the above radical ideal for which $D = \mathbf{Z}[X]/I$. We claim that the second condition implies that $\overline{h}$ contains nothing which is not in I. For if $w \notin I$ then, w is equivalent, modulo I, to a non-zero element $s \in D$ which in turn, is equivalent to x_s. ∎

3.8.22. Corollary. *If $h : X \longrightarrow B$ with $B \in \mathscr{B}$ satisfies the above conditions for being admissible, then there exists $b \in B$ with $h(x_d)b^\ell = h(x_c)$.*

Proof. By the above results, if h is admissible, then $\overline{h} : D \longrightarrow B$ is an injection and the conclusion follows from Lemma 3.8.17. ∎

3.8.23. Notation. Let X and I, with $D = \mathbf{Z}[X]/I$ be as above. Given $I_0 \subseteq D$ and $S_0 \subseteq D$, we say that the function $h : D \longrightarrow B$, with $B \in \mathscr{B}$, is an (I_0, S_0)-function if;

1. $f(h) = 0$ for all $f \in I_0$
2. $h(x_s) \neq 0$ for all non-zero $s \in S_0$.

3.8.24. Proposition. *With the above notation, there is a finite set $I_0 \subseteq I$ and a finite set $S_0 \subseteq D$ such that whenever $h : X \longrightarrow B$, with $B \in \mathscr{B}$, is an (I_0, S_0)-function, there exists $b \in B$ with $h(x_d)b^\ell = h(x_c)$.*

Proof. Regard I and D as disjoint sets. We say that $W \subseteq I \cup D$ is **adequate** if, letting $I_0 = W \cap I$ and $S_0 = W \cap D$, it is the case that whenever $h : X \longrightarrow B$, with $B \in \mathscr{B}$, is an (I_0, S_0) function, there exists $y \in B$ with $h(x_d)y^\ell = h(x_c)$. We observe that $W = I \cup D$ is adequate.In this case, $W \cap I = I$ and $W \cap D = D$ and we can apply 3.8.22.

We claim that there exists a finite adequate set W_0. Assume the contrary that no finite adequate set exists. Let $\mathscr{F}$ be the family of all finite subsets of $I \cup D$. Then, for each $F \in \mathscr{F}$, we let $I_F = F \cap I$ and $S_F = F \cap D$. By assumption there must be an (I_F, S_F)-function $h_F : X \longrightarrow B_F$ with $B \in \mathscr{B}$ such that there is no $b \in B$ for which $h_F(d)b^\ell = h_F(c)$. For each $w \in I \cup D$, let $U_w = \{F \in \mathscr{F} \mid w \in F\}$. Then the family $\{U_w \mid w \in I \cup D\}$ has the finite intersection property because, if $C \subseteq I \cup D$ is finite, then $\bigcap\{U_w \mid w \in C\}$ is non-empty, as it contains C. By Lemma 3.8.11 there is an ultrafilter $\mathbf{u}$ on $\mathscr{F}$ such that $U_w \in \mathbf{u}$ for all $w \in I \cup D$. Let $B_\mathbf{u}$ be the associated ultraproduct of $\{B_F \mid F \in \mathscr{F}\}$. Let $q_\mathbf{u} : \prod\{B_\beta \mid \beta < \alpha\}$ be the canonical quotient map. Let $h : X \longrightarrow \prod\{B_\beta\}$ be the function whose projections are the above functions, $h_F : X \longrightarrow B_F$. Also let $h_\mathbf{u} : X \longrightarrow B_\mathbf{u}$ be given by $h_\mathbf{u} = hq_\mathbf{u}$. Since the condition associated with each $w \in I \cup D$ is true on U_w, we see that $h_\mathbf{u}$ satisfies all the conditions in 3.8.22. Thus there exists $b \in \prod B_F$ such that $q_\mathbf{u}(b)$ satisfies $h(d)q_\mathbf{u}(b)^\ell = h(c)$. But, letting b_F be the projection of b onto B_F, we have that b_F satisfies $h_F(d)b^\ell = h_F(c)$ for all F in some $V \in \mathbf{u}$. But, choosing $F \in V$ we find that the element b_F contradicts the assumption that there is no $b \in B_F$ with $h_F(d)b^\ell = h_F(c)$.

Therefore we have proven the claim that a finite adequate subset W_0 exists and if we let $I_0 = W_0 \cap I$ and $S_0 = W \cap D$ the result follows. ∎

3.8.25. Theorem. *$\mathscr{B}$ is closed under ultraproducts if and only if it is first order.*

Proof. If $\mathscr{B}$ is first order, it is well known that it is closed under ultraproducts. Conversely, assume that $\mathscr{B}$ is closed under ultraproducts. Let D be a domain which satisfies all first-order conditions that are satisfied by every $B \in \mathscr{B}$. It suffices to prove that $D \in \mathscr{B}$. If not, then, by the above discussion, there exists $c, d \in D$, with $d \neq 0$ such that whenever $B \in \mathscr{B}$ and $h : X \longrightarrow B$ is an (I_0, S_0)-function, then there exists $b \in B$ with $h(x_d)b^\ell = h(x_c)$. But by the above proposition, we can assume that I_0 and S_0 are finite sets. If we let X_0 be the set of indeterminate of X involved in the polynomials $g \in I_0$ and the non-zero elements of D in S_0, then this is a first-order condition involving the elements of X_0 as variables, which is satisfied by every $B \in \mathscr{B}$. It follows that it is satisfied by D. But consider the function $h : X \longrightarrow D$ for which $h(x_t) = t$ for all $t \in D$. It easily follows that h satisfies all the conditions in 3.8.22 so h certainly is an (I_0, S_0)-function. This implies that there exists $e \in D$ with $de^\ell = c$. But, regarding $D \subseteq G(D)$, the element $z \in G(D)$ is determined by the fact that $dz^\ell = c$. Thus $z = e \in D$ contradicting the assumption that $z \in G(D) - D$. ■

3.9. Local representations of sections. Until the end of this section, we let E denote the sheaf on $\mathrm{Spec}(R)$, when the latter is equipped with the domain topology.

Assumptions and Notation. Throughout this subsection, we assume that:

1. *We are in the $\mathfrak{Dom}$-invariant case* which implies that the $\mathscr{A}$-topology is the same as the domain topology (see 3.2.1).

2. We further assume that $\mathscr{B}$ is first order, so, for every semiprime ring R, we can assume that its reflection, $K(R)$ is $\Gamma(E)$, see Theorem 3.6.12.

3. We regard $R \subseteq \Gamma(E)$ by identifying R with its image, under $\nu : R \longrightarrow \Gamma(E)$.

4. Unless otherwise specified, a reference to the topology on $\mathrm{Spec}(R)$ refers to the domain topology.

3.9.1. Definition. *Let ζ in $\Gamma(E)$ and $r \in R$ and $P \in \mathrm{Spec}(R)$ be given. We will abuse notation by saying that $\zeta(P) = r$ when $\zeta(P) = r + P$ (the image of r in the stalk R/P).*

Let $U \subseteq \mathrm{Spec}(R)$ be closed in the patch topology. We say that $\zeta \in \Gamma(E)$ is **grounded on** *U if for every $P \in U$, there exists $r \in R$ such that $\zeta(P) = r$.*

We further say that ζ is **uniformly grounded on** *U if there exists $r \in R$ such that $\zeta(P) = r$ for all $P \in U$. Note that ζ is uniformly grounded on $\mathrm{Spec}(R)$ if and only if $\zeta \in R$. In view of our identification of R with a subring of $\Gamma(E)$, this means that $\zeta = \nu(r)$ for some $r \in R$.*

3.9.2. Definition. *Let $\zeta \in \Gamma(E)$ be given. We say that $\mathscr{U} = \{U_1, \ldots, U_n\}$ is a* **grounded representation of** *ζ* **on** *$U = U_1 \cup \cdots \cup U_n$ if each U_i is a patch-closed subset on which ζ is uniformly grounded.*

3.9.3. Proposition. *$\zeta \in \Gamma(E)$ has a grounded representation on the patch-closed set $U \subseteq \operatorname{Spec}(R)$ if and only if ζ is grounded on U.*

Proof. Assume the ζ is grounded on U. Then, for each $P \in U$, there exists $r_P \in R$ such that ζ and the constant section $\nu(r_P)$ agree at P. Since global sections agree on open sets, there is a neighbourhood U_P of P on which ζ and $\nu(r_P)$ agree. Moreover, since the domain topology has a base of patch-clopen subsets, we can assume that each U_P is patch-clopen. Letting $\{U_1, \ldots, U_n\}$ be a finite subcover of $\{U_P \cap U \mid P \in U\}$ gives us the desired grounded representation.

The converse implication is immediate. ∎

3.9.4. Proposition. *Suppose $\zeta, \tau \in \Gamma(E)$. If $P \subseteq Q$ in $\operatorname{Spec}(R)$ and $\zeta(P) = \tau(P)$, then $\zeta(Q) = \tau(Q)$.*

Proof. Two sections agree on an open set, Theorem 3.4.3, Step 2. But every open set in the domain topology is up-closed. ∎

3.9.5. Theorem. *Let $\zeta \in \Gamma(E)$ be given and let $\mathscr{U} = \{U_1, U_2, \ldots, U_n\}$, with $n \geq 2$, be a grounded representation of ζ on $U = U_1 \cup U_2 \cup \cdots \cup U_n$. Then there exists $r \in R$ such that for all sufficiently large w, $\mathscr{U}' = \{U_1 \cup U_2, U_3, \ldots, U_n\}$ is a grounded representation of $(\zeta - r)^w$. Note that $\mathscr{U}'$ has cardinality one less than $\mathscr{U}$.*

Proof. Choose $r_i \in R$ such that $\zeta(P) = r_i$ for all $P \in U_i$. We may assume that $r_1 = 0$; otherwise we can replace ζ by $\zeta - r_1$. We will then prove that for all sufficiently large w there is an $a_w \in R$ such that $\zeta^w(P) = a_w$ for all $P \in U_1 \cup U_2$ from which it will easily follow that $\mathscr{U}'$ is a grounded representation of ζ^w on U.

For $i = 1, 2$, let $J_i = \bigcap\{P \mid P \in U_i\}$. We claim that $r_2^w \in J_1 + J_2$ for all sufficiently large w. This is equivalent to showing that the image of r_2 belongs to every prime of $R/(J_1 + J_2)$ or equivalently, that r_2 belongs to every prime of R that contains both J_1 and J_2.

So suppose that Q is such a prime. By Proposition 2.2.22, there exist $P_1 \in U_1$ and $P_2 \in U_2$ with $P_1 \subseteq Q$ and $P_2 \subseteq Q$. But $\zeta(P_1) = r_1 = 0$ which implies that $\zeta(Q) = 0$ by the preceding proposition. Similarly $\zeta(P_2) = r_2$ which implies that $\zeta(Q) = r_2 = 0$ and thus $r_2 \in Q$, as claimed. For sufficiently large w, we can write $r_2^w = a_w + b_w$ with $a_w \in J_1$ and $b_w \in J_2$.

If $P \in U_1$, we have that $0 = \zeta^w(P) = a_w$ since $a_w \in J_1 \subseteq P$. If $P \in U_2$, then $\zeta^w(P) = a_w + b_w = a_w$ since $b_w \in J_2 \subseteq P$ and so we see that $\zeta^w = a_w$ on all of $U_1 \cup U_2$ as required. ∎

3.9.6. Corollary. *Suppose that $\zeta \in \Gamma(E)$ is grounded on all of $\operatorname{Spec}(R)$. Suppose also that R has the property that $\zeta^k, \zeta^{k+1} \in R$ implies $\zeta \in R$. Then $\zeta \in R$.*

Proof. By Proposition 3.9.3, there exists $\mathscr{U} = \{U_1, U_2, \ldots, U_n\}$ which is a grounded representation of ζ on $\operatorname{Spec}(R)$. We proceed by induction on n, the cardinality of $\mathscr{U}$. If $n = 1$ then it is immediate that $\zeta \in R$. Assume that the result holds whenever ζ has a grounded representation on $\operatorname{Spec}(R)$ of cardinality n. Let $\mathscr{U}$ be a grounded representation of ζ on $\mathrm{Spec}(R)$ of cardinality $n+1$. Then by the above theorem, we see that there exists $r \in R$ such that $(\zeta - r)^w$, for sufficiently large w, has a grounded representation of cardinality n. By our induction hypothesis, $(\zeta - r)^w \in R$ for large enough w, which implies, for large k, that $(\zeta - r)^k, (\zeta - r)^{k+1} \in R$ which implies $\zeta - r \in R$, which implies $\zeta \in R$. ■

3.9.7. PROPOSITION. *Let $n > 1$ and $k > 1$ be relatively prime integers. Suppose that ζ is a global section such that $\zeta(P) \in R$ for all primes P. Assume that $\zeta^n, \zeta^k \in R$ implies $\zeta \in R$. Then $\zeta^w \in R$ for all sufficiently large w also implies $\zeta \in R$.*

PROOF. Let w_0 be the least integer such that $\zeta^w \in R$ for all $w > w_0$, while $\zeta^{w_0} \notin R$ and so $w_0 > 0$. But then $\zeta^{nw_0}, \zeta^{kw_0} \in R$ and it follows from our hypothesis that $\zeta^{w_0} \in R$, which contradicts the choice of w_0. ■

3.9.8. COROLLARY. *Let $n > 1$ and $k > 1$ be relatively prime integers. Suppose that ζ is a global section such that $\zeta(P) \in R$ for all primes P. Assume that $\zeta^n, \zeta^k \in R$ implies $\zeta \in R$. Then $\zeta \in R$.* ■

4. Limit closure of domains.

4.1. THE BACKGROUND. In this section, we will characterize the limit closure of integral domains in the category of commutative rings. An early version was done in [Kennison 1976] but the treatment in that paper suffers from being somewhat opaque and having made a needless detour into ordered rings. Sifting out the proof for the case of integral domains is difficult as some of it is in a general section and the rest in a special section on domains. Here is a more direct approach, which expands on and corrects the brief treatment of this material at the end of [Kennison & Ledbetter 1979].

In this section, we denote the limit closure of domains by $\mathscr{K}_{\mathrm{dom}}$ and denote the reflector by $K_{\mathrm{dom}} : \mathcal{SPR} \longrightarrow \mathscr{K}_{\mathrm{dom}}$. We briefly discussed this example in 2.3.2.4, in which we showed that $\mathscr{K}_{\mathrm{dom}}$ was $\mathcal{Dom}$-invariant. Thus all the consequences of Theorems 2.3.1 and 3.6.12 are available. In particular, when $R \subseteq S \subseteq K(R)$ then $R \longrightarrow S$ is epic and essential, and $K_{\mathrm{dom}}(S) = K_{\mathrm{dom}}(R)$.

The purpose of this section is to characterize the category $\mathscr{K}_{\mathrm{dom}}$. We begin with some things that will be needed in this section and the next.

4.1.1. DEFINITION.

1. *If $R \in \mathcal{SPR}$, we will say that R is* **(2,3)-closed** *if whenever $a^3 = b^2 \in R$ there is a $c \in R$ with $c^2 = a$ and $c^3 = b$. The uniqueness of c follows from Proposition 4.1.6.*

2. *If $R \in \mathcal{SPR}$, we will say that R is* **DL-closed** *if whenever $a^3 = b^2$ and if, moreover, a is square mod every prime $P \subseteq R$, then there is a (unique) $c \in R$ with $c^2 = a$ and $c^3 = b$.*

After reading an earlier draft of this article David Dobbs pointed out that the notion of being (2,3)-closed is known in the literature as **seminormality**.

4.1.2. Notation. As in the previous section, we use E_R (or just E if R is understood) to denote the canonical sheaf. Note that the stalk of E_R lying over $P \in \operatorname{Spec}(R)$ is R/P as $G(R/P) = R/P$. *This implies that the results of 3.9 are valid for all sections ζ on a* $\operatorname{Spec}(R)$ *or on any subset.*

Here is the main result of this section:

4.1.3. Theorem. *Let R be a commutative semiprime ring. Then the following are equivalent:*

DL-1. *R is DL-closed.*

DL-2. *R is isomorphic, under the canonical map, to the ring of global sections of the sheaf E_R.*

DL-3. *R is isomorphic to a ring of global sections of sheaf whose stalks are domains.*

DL-4. *R is in the limit closure of the domains.*

Proof.

DL-1 $\Rightarrow$ DL-2: It is easily proven, when R is DL-closed, that if $\zeta \in K(R)$ then $\zeta^2, \zeta^3 \in R$ implies $\zeta \in R$. It is immediate, using Corollary 3.9.8, that $R = K_{\mathrm{dom}}(R)$ in light of the fact that for every prime P and every global section ζ, we have that $\zeta(P) \in R$.

DL-2 $\Rightarrow$ DL-1: Assume that $a, b \in R$ are such that a has a square root mod every prime ideal and that $a^3 = b^2$. Then mod every prime ideal P, there exists a unique c_P such that mod P, we have $c_P^2 = a$ and $c_P^3 = b$. Since $N(c_P^2 - a) \cap N(c_P^3 - b)$ is open in the domain topology, these equations hold in a neighbourhood of P and the elements c must agree on overlaps by uniqueness. So they determine a section ζ. But by DL-2, $\zeta \in R$ and so R satisfies DL-1.

DL-2 $\Rightarrow$ DL-3: Obvious.

DL-3 $\Rightarrow$ DL-4: This follows from 3.5.3.

DL-4 $\Rightarrow$ DL-2: This is 3.6.12. ∎

4.1.4. Remark. For further conditions equivalent to being DL-closed, and involving kth powers instead of squares, see Definition 6.1.9 and Lemma 6.1.10.

4.1.5. Proposition. *An element a in a semiprime ring R is a square mod every prime if and only if there are elements $r_1, \dots, r_n \in R$ such that $(a - r_1^2) \cdots (a - r_n^2) = 0$.*

Proof. One direction is obvious. To go the other way, suppose that a is a square mod every prime. For each prime P, let $r_P \in R$ be such that $a - r_P^2 \in P$. The sets $N(a - r_P^2)$ are open and cover $\operatorname{Spec}(R)$ so that $\operatorname{Spec}(R)$ is covered by finitely many of them, say $N(a - r_{P_1}^2), \dots, N(a - r_{P_n}^2)$ whence $(a - r_{P_1}^2) \cdots (a - r_{P_n}^2)$ lies in every prime. In a semiprime ring, the only element that lies in every prime is 0. ∎

4.1.6. Proposition. *If $s^2 = t^2$ and $s^3 = t^3$ in a semiprime ring R, then $s = t$.*

Proof. This is immediate by cubing $s - t$. Alternatively, Proposition 5.1.2 in the next section says that in a semiprime ring R if $s^n = t^n$ and $s^k = t^k$ for relatively prime k, n then $s = t$. ∎

4.1.7. Proposition. *The limit closure of domains is an essentially algebraic category.*

Proof. We adjoin to the equational theory of commutative rings a countable family of partial operations whose domain is given equationally. We begin with a unary partial operation ω whose domain is the set of all a such that $a^2 = 0$, subject to the equations $\omega(a) = a$ and $\omega(a) = 0$. The commutative ring models of this partial operation are just the semiprime rings. For each $n > 0$, define an $(n+2)$-ary operation ω_n whose domain consists of all $a, b, r_1, \ldots, r_n$ such that $(a - r_1^2)(a - r_2^2) \cdots (a - r_n^2) = a^3 - b^2 = 0$. The operation ω_n must satisfy the equations

$$\omega_n(a, b, r_1, \ldots, r_n)^2 = a \qquad \text{and} \qquad \omega_n(a, b, r_1, \ldots, r_n)^3 = b$$

It is clear from the preceding development that the models of this theory are the rings in the limit closure of domains. ∎

4.2. DL-extensions and a step-by-step construction of K_{dom}.

4.2.1. Definition.

1. *If $R \subseteq S$ in $\mathcal{SPR}$, we will say that S* **is a simple (2,3)-extension of** *R if there is an $s \in S$ such that $S = R[s]$ and $s^2, s^3 \in R$. We say that S is a* **(2,3)-extension** *of R if for some ordinal α there is an ordinal-indexed sequence $S_0 \subseteq S_1 \subseteq S_2 \cdots \subseteq \cdots S_\omega \subseteq \cdots \subseteq S_\alpha$ such that:*

   (a) $R = S_0$;

   (b) $S_{\beta+1}$ is a simple (2,3)-extension of S_β for each $\beta < \alpha$;

   (c) if $\beta \leq \alpha$ is limit ordinal, then $S_\beta = \bigcup_{\gamma<\beta} S_\gamma$; and

   (d) $S = S_\alpha$.

   A field is (2,3)-closed—just let $c = b/a$ except when $a = b = 0$—and then a product of fields is also (2,3)-closed. Since a semiprime ring is embedded in a product of fields, it is easy to see that R is (2,3)-closed if and only if it has no proper (2,3)-extension.

2. *If $R \subseteq S$ in $\mathcal{SPR}$, we will say that S* **is a simple DL-extension of** *R if it is a simple (2,3)-extension by the element $s \in S$ and if, moreover, s is a square mod P for every prime ideal $P \subseteq R$. We say that S is a* **DL-extension** *of R if there is an ordinal-indexed sequence of simple DL-extensions from R to S, as above. Similarly to the (2,3) case, an SPR is DL-closed if it has no proper DL-extension.*

4.2.2. Proposition. *If S is a (2,3)-extension of R, then the first four of the statements below hold and if it is a DL-extension, then all five do.*

1. *The inclusion $R \hookrightarrow S$ is integral.*

2. *The inclusion $R \hookrightarrow S$ is an epimorphism in $\mathcal{SPR}$.*

3. *The induced map $\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$ is an order isomorphism.*

4. *The induced map $\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$ is a homeomorphism in the patch topology and the domain topology.*

5. *If $P \subseteq R$ is prime and $P^\sharp \subseteq S$ is the unique prime lying above P, then the canonical map $R/P \longrightarrow S/P^\sharp$ is an isomorphism.*

Proof. Since a DL-extension is also a (2,3)-extension we may prove the first four for (2,3)-extensions.

1. A simple (2,3)-extension is obviously integral and the end of an ordinal sequence of integral extensions is integral.

2. For a simple (2,3)-extension, it follows from the uniqueness of s that the inclusion is an epimorphism and the general case follows by transfinite induction.

3. Since a (2,3)-extension is integral Proposition 2.2.25 applies here.

4. For the patch topology, which is compact and Hausdorff, this is clear. For the domain topology it follows from 2.2.21.

5. If $S = R[s]$ is a simple DL-extension, then $S/P^\sharp$ is generated by R/P and the image of s. But since s has a square root in R/P, that image is already there. The case of a general DL-extension follows by transfinite induction. ∎

4.2.3. Theorem. *Let R be a commutative semiprime ring and $R \hookrightarrow \Gamma(E_R)$ be the adjunction homomorphism.*

1. *The rings R and $\Gamma(E_R)$ have the same Krull dimension.*

2. *The inclusion $R \subseteq \Gamma(E_R)$ induces an isomorphism on the set of idempotents.*

3. *The inclusion $R \subseteq \Gamma(E_R)$ is the inclusion into a ring of quotients in the sense of [Lambek (1986)] and is therefore an essential extension of rings.*

4. *Every artinian ring is DL-closed.*

PROOF.

1. This is immediate since, by Proposition 4.2.2.3, they have isomorphic prime ideal lattices.

2. This will follow if we show that there is a 1-1 correspondence between idempotents in R and clopens in the domain topology on $\mathrm{Spec}(R)$. In fact, if $e \in R$ is idempotent, then $\mathrm{Spec}(R) = N(e) \cup N(1-e)$ is a disjoint union of open sets. Conversely, if $\mathrm{Spec}(R) = Y \cup Z$ is a disjoint union of open sets, let $I = \bigcap_{P \in Y} P$ and $J = \bigcap_{Q \in Z} Q$. Then $I \cap J = 0$ since it is the intersection of all the primes of the semiprime ring R. If M were a prime ideal of R that contains $I + J$, then it would follow from 2.2.22 that some prime $P \in Y$ is contained in M and some prime $Q \in Z$. Since open sets in the domain topology are up-closed, this implies that $M \in Y \cap Z$ which contradicts the disjointness of Y and Z. Thus $R = I + J$. Write $1 = e + d$ with $e \in I$ and $d \in J$. Then $e = e^2 + ed = e^2$ since $ed \in I \cap J$ and similarly $d^2 = d$.

3. A little background is needed here. See [Lambek (1986), Chapter 2] for details. An ideal $D \subseteq R$ is called **dense** if its annihilator, denoted $\mathrm{ann}(D) = \mathrm{ann}_R(D)$, is 0. Clearly, any ideal containing a dense ideal is dense. If $R \subseteq S$ and $s \in S$, we denote by $s^{-1}R = \{r \in R \mid sr \in R\}$. Proposition 2.6 of [Lambek (1986)] shows that the embedding $R \hookrightarrow S$ is an embedding into a complete ring of quotients if and only if for all $0 \neq s \in S$, the ideal $s^{-1}R$ is a dense ideal of R and $s(s^{-1}R) \neq 0$. This condition clearly implies that S is an essential ring extension of R.

   In section [Lambek (1986), Proposition 1 of Section 2.4], it is shown that if R is semiprime, the complete ring of quotients $Q(R)$ is regular. By 2.3.1.8, $K_{\mathrm{dom}}(R)$ is a subring of $Q(R)$, which means that $\Gamma(E_R)$ is a ring of quotients of R and hence an essential ring extension.

   The same considerations show that $K_{\mathrm{dom}}(R)$ is a subring of the epimorphic hull of R, the (regular) intersection of all the regular subrings of $Q(R)$ that contain R (see [Storrer 1968]).

4. By Wedderburn's structure theorem on artinian rings, a commutative semiprime artinian ring is a product of fields. ∎

4.3. PERMANENCE PROPERTIES OF DL-CLOSED RINGS. The class of rings which are DL-closed is closed under products and equalizers because they are a reflective subcategory of $\mathscr{C}$. They are evidently not closed under homomorphic images or even semiprime homomorphic images since free commutative rings are, by the following theorem, DL-closed. DL-closed rings do have some other permanence properties however.

4.3.1. THEOREM. *The ring of polynomials in any set of variables over a DL-closed ring is again DL-closed.*

PROOF. Let $E \longrightarrow \mathrm{Spec}(R)$ be the canonical sheaf for R associated with $\mathcal{A}_{\mathrm{dom}}$, so that the stalk over each prime ideal P of R is R/P. We will construct a sheaf $E[x] \longrightarrow \mathrm{Spec}(R)$ whose stalk over P is $R/P[x]$. Given a polynomial $a(x) \in R[x]$ and $U \subseteq \mathrm{Spec}(R)$, which is open in the domain topology, we define a basic open subset $(a(x), U)$ of $E[x]$ to consist of all polynomials $a_P(x) \in R/P[x]$ where $P \in U$ and $a_P(x)$ is the image of $a(x)$ under the obvious map $R[x] \longrightarrow (R/P)[x]$.

We claim that this defines a base for a topology on $E[x]$ and, moreover, that the resulting map $E[x] \longrightarrow \mathrm{Spec}(R)$ is a sheaf map (that is, a local homeomorphism). The main step in proving this claim is to show that given $a(x), b(x) \in R[x]$ and open subsets $U, V \subseteq \mathrm{Spec}(R)$, there exist $c(x) \in R[x]$ and an open set $W \subseteq \mathrm{Spec}(R)$ such that $(a(x), U) \cap (b(x), V) = (c(x), W)$. Choose n so that $a(x), b(x)$ both have degrees no bigger than n (for this purpose, we may suppose that the degree of the zero polynomial is -1). Write $a(x) = \sum_{0 \le i \le n} \alpha_i x^i$ and $b(x) = \sum_{0 \le i \le n} \beta_i x^i$. Then, for $0 \le i \le n$, let $W_i = N(\alpha_i - \beta_i)$ be the open subset of $\mathrm{Spec}(R)$ consisting of all primes P for which $\alpha_i = \beta_i$ modulo P. Let $W = U \cap V \cap \bigcap_{0 \le i \le n} W_i$. Letting $c(x)$ be either $a(x)$ or $b(x)$, then we have $(a(x), U) \cap (b(x), V) = (c(x), W)$. The remaining steps in showing that $E[x] \longrightarrow \mathrm{Spec}(R)$ is a local homeomorphism are now straightforward.

We next claim that $R[x]$ is isomorphic to $\Gamma(E[x])$. Let $\gamma \in \Gamma(E[x])$ be given. At each prime ideal P of R we can find a polynomial $a_P(x) \in (R/P)[x]$ such that $\gamma(P) = a_P(x)$ in a neighbourhood of P. By compactness, there is a finite cover of $\mathrm{Spec}(R)$ by open sets U_j such that for each j there exists $a_j(x) \in R[x]$ such that $a_j(x)$ and γ have the same restriction to U_j. Now choose n such that every $a_j(x)$ has degree no bigger than n. Write $a_j(x) = \sum_{0 \le i \le n} r_{i,j} x^i$. Note that $E \longrightarrow \mathrm{Spec}(R)$, the canonical sheaf for R, is easily shown to be an open subsheaf of $E[x]$. It follows that, for a fixed i with $0 \le i \le n$, the coefficients $r_{i,j}$ form a global section with values in $E \longrightarrow \mathrm{Spec}(R)$, hence, as R is DL-closed, there exists $r_i \in R$ such that for $P \in U_j$ we have $r_i = r_{i,j}$ modulo P. It follows that $\gamma = \sum_{0 \le i \le n} r_i x^i$ which is in $R[x]$. Therefore, by Theorems 3.5.3 and 4.1.3, we have that $R[x]$ is DL-closed.

It immediately follows that the polynomial ring in any finite set of indeterminates is DL-closed. For the general case, one easily verifies that the category DL-closed rings is closed under filtered colimits. That fact is generally true for models of finitary essentially algebraic theories. ∎

4.3.2. THEOREM. *Let R be DL-closed and suppose that $S \subseteq R$ is a multiplicatively closed subset that contains no zero divisors. Then $S^{-1}R$ is also DL-closed.*

PROOF. Suppose that $a/s, b/t, r_1/u_1, \ldots, r_n/u_n$ are elements of $S^{-1}R$ such that $(a/s - (r_1/u_1)^2) \cdots (a/s - (r_n/u_n)^2) = 0$ and $(a/s)^3 = (b/t)^2$. Let $v = stu_1 \cdots u_n$, $a' = (v^2/s)a$, $b' = (v^3/t)b$, and, for $i = 1, \ldots, n$, $r'_i = (v/u_i)r_i$. Note that these new elements are all elements of R since the denominator is one factor of the numerator. Then from $a' - r_i'^2 = v^2 a/s - v^2 r_i^2/u_i^2 = v^2(a/s - r_i^2/u_i^2)$, we see that $(a' - r_1'^2) \ldots (a' - r_n'^2) = 0$ and $a'^3 = v^6 a^3/s^3 = v^6 b^2/t^2 = (v^3 b/t)^2 = b'^2$ so that there is a $c \in R$ such that $a' = c^2$ and $b' = c^3$. Thus $v^2 a/s = c^2$ or $a/s = (c/v)^2$, while $v^3 b/t = c^3$ or $b/t = (c/v)^3$, as required. ∎

4.4. Examples.

4.4.1. Proposition. *Every von Neumann regular ring is DL-closed.*

Proof. It is well known that every von Neumann regular ring is in the limit closure of fields, hence in the limit closure of domains. ∎

4.4.2. Example. It was proved in [Niefield & Rosenthal 1987] that a commutative ring R is a Pierce sheaf of integral domains if and only if the annihilator of each element is generated by a set of idempotents. It follows from Theorem 4.1.3 that these rings are DL-closed. It also follows that commutative semiprime Baer rings are DL-closed, which means that if we adjoin to a commutative semiprime ring R the idempotents of its complete ring of quotients, [Lambek (1986)], we get a DL-closed ring. Thus although the reflection of R cannot adjoin idempotents it is contained in a subring of the complete ring of quotients of R obtained by adjoining idemptents.

4.4.3. Example. The notion of a quasi-rational ring is introduced in Example 3.7.3 where it is shown that $\mathscr{K}_{\mathrm{qrat}}$, the limit closure of $\mathcal{A}_{\mathrm{qrat}}$, the class of quasi-rational domains, has the canonical sheaf representation property. Here we claim that the semiprime ring R is in $\mathscr{K}_{\mathrm{qrat}}$ if and only if R is quasi-rational and DL-closed.

Proof. The necessity that R be quasi-rational is shown in 3.7.3 and the necessity that R be DL-closed is obvious. Conversely, suppose that R is quasi-rational and DL-closed. Then R is isomorphic to the ring of global sections of its canonical $\mathcal{A}_{\mathrm{dom}}$ sheaf. But since R is quasi-rational, it is obvious that every quotient ring R/P, for P a prime ideal of R, is quasi-rational, and therefore the canonical $\mathcal{A}_{\mathrm{dom}}$ sheaf coincides with the canonical $\mathcal{A}_{\mathrm{qrat}}$ sheaf. So R is isomorphic to the ring of global sections of its canonical $\mathcal{A}_{\mathrm{qrat}}$ sheaf, which implies that $R \in \mathscr{K}_{\mathrm{qrat}}$. Example 4.4.5 shows that a quasi-rational ring need not be DL-closed. ∎

The following is an immediate consequence of Example 5.3.9.

4.4.4. Example. For any topological space X, the ring $C(X)$, respectively $C(X, \mathbf{C})$, of continuous real-valued, respectively complex-valued, functions on X is DL-closed.

On the other hand, we have:

4.4.5. Example. The ring $C^1(\mathbf{R})$ of continuously differentiable functions in $C(\mathbf{R})$ is not DL-closed.

For let f be the function that is x^2 for $x > 0$ and 0 otherwise. Then clearly $f \in C^1(\mathbf{R})$. We have that $(f - 0^2)(f - x^2) = 0$ and f^3 has a square root, namely $xf \in C^1(\mathbf{R})$. But no square root of f can have a derivative at 0.

Therefore the following is slightly surprising.

4.4.6. Example. The ring $C^\infty(\mathbf{R})$ of infinitely differentiable functions $\mathbf{R} \longrightarrow \mathbf{R}$ is DL-closed.

We let $A = C^\infty(\mathbf{R})$ denote the ring of all C^∞ functions from $\mathbf{R}$ to $\mathbf{R}$. To show that A is in the limit closure of all domains it suffices, by Theorem 3.5.3, to show that A is isomorphic to the ring of global sections of a sheaf of domains. We first establish some notation.

4.4.7. Notation. In the list below, we assume $f \in A$.

1. For $f \in A$ and $n \in \mathbf{N}$, we let $f^{(n)}$ denote the nth derivative of f (with $f^{(0)} = f$).
2. We say that f is **flat at** $r \in R$ if f and all its derivatives vanish at r.
3. For each $r \in \mathbf{R}$, we denote by $P_r \subseteq A$ the set of all functions that are flat at r.
4. We denote by $f^\flat$ the set of $r \in \mathbf{R}$ at which f is flat.
5. If $\mathbf{u}$ is an ultrafilter on R, we let $P_\mathbf{u}$ denote the set of all f such that $f^\flat \in \mathbf{u}$. We note that if $\mathbf{u}$ is the principal ultrafilter at $r \in \mathbf{R}$ then $P_\mathbf{u} = P_r$. We will show below that $P_\mathbf{u}$ (and hence P_r) is a prime ideal of A.
6. We let X denote the set of all pairs $(r, \mathbf{u})$ where $r \in \mathbf{R}$ and $\mathbf{u}$ is an ultrafilter on $\mathbf{R}$ which converges to r.
7. For each $r \in \mathbf{R}$, we let $\widehat{r} \in X$ denote the pair $(r, (r))$ where (r) is the principal ultrafilter generated by r.
8. The elements of the form $\widehat{r}$ will be called the **principal elements** of X.
9. For all $U \subseteq \mathbf{R}$ we let $\widehat{U} \subseteq X$ denote the set $\{\widehat{r} \mid r \in U\}$.

4.4.8. Proposition. *For any ultrafilter $\mathbf{u}$, the set $P_\mathbf{u}$ is a prime ideal of A.*

Proof. We begin by showing that P_r is prime for $r \in \mathbf{R}$. It is clearly an ideal. So let $f, g \in A$ such that $f \notin P_r$ and $g \notin P_r$. Suppose $m, n \in \mathbf{N}$ are least indices such that $f^{(m)}(r) \neq 0$ and $g^{(n)}(r) \neq 0$. Then our supposition shows that in the computation of

$$(fg)^{(m+n)}(r) = \sum_{i=0}^{m+n} \binom{m+n}{i} f^{(i)}(r) g^{(m+n-i)}(r)$$

every term is 0 except for $\binom{m+n}{m} f^{(m)}(r) g^{(n)}(r)$ and that one is non-zero.

Now suppose that $\mathbf{u}$ is an ultrafilter. One easily sees that $(f+g)^\flat \supseteq f^\flat \cap g^\flat$ which belongs to $\mathbf{u}$ assuming both sets do. The preceding paragraph shows that $(fg)^\flat = f^\flat \cup g^\flat$ and if it belongs to $\mathbf{u}$, then either $f^\flat \in \mathbf{u}$ or $g^\flat \in \mathbf{u}$. ∎

4.4.9. Lemma. *Suppose the ultrafilter* $\mathbf{u}$ *converges to* $r \in \mathbf{R}$*. Then* $P_{\mathbf{u}} \subseteq P_r$*.*

Proof. If $f \in P_{\mathbf{u}}$, then $f^\flat \in \mathbf{u}$. If $\mathbf{n}(r)$ is the neighbourhood filter of r, then $\mathbf{n}(r) \subseteq \mathbf{u}$. Thus if $V \in \mathbf{n}(r)$, then $V \cap f^\flat \in \mathbf{u}$. That means that every neighbourhood of r contains a point at which f is flat. But all the derivatives of a C^∞ function are continuous and thus they all vanish at r. ∎

4.4.10. Definition.

1. *We topologize* X *so that* $W \subseteq X$ *is open if whenever* $(r, \mathbf{u}) \in W$*, then there is a* $U \in \mathbf{u}$ *such that* $M_{(r,\mathbf{u})}(U) = \{(r, \mathbf{u})\} \cup \{\widehat{r}\} \cup \widehat{U} \subseteq W$*. Note that each* $\widehat{r} \in X$ *is an isolated point, that is the singleton set* $\{\widehat{r}\}$ *is open. But* $\{\widehat{r}\}$ *is not closed because its closure is the set of all points of the form* $(r, \mathbf{u})$*.*

2. *We define a set* F *and a map* $\pi : F \longrightarrow X$ *such that for all* $(r, \mathbf{u}) \in X$ *we have* $\pi^{-1}((r, \mathbf{u})) = A/P_{\mathbf{u}}$*.*

3. *For each* $f \in A$ *we define a global section* $\sigma(f) : X \longrightarrow F$ *so that* $\sigma(f)((r, \mathbf{u}))$ *is the image of* f *in* $A/P_{\mathbf{u}}$*.*

4. *We give* F *the largest topology for which* $\sigma(f)$ *is continuous for each* $f \in A$*.*

We aim to prove that A is isomorphic (in the obvious way) to the ring of global sections of F. We first need some lemmas:

4.4.11. Lemma. *For each* $(r, \mathbf{u}) \in X$*, sets of the form* $\{M_{(r,\mathbf{u})}(U) \mid U \in \mathbf{u}\}$ *form a neighbourhood base at the point* $(r, \mathbf{u})$*. It readily follows that* F *is a sheaf over* X*.*

Proof. We must first show that $M_{(r,\mathbf{u})}(U)$ is open in the topology on X. But this readily follows from the observation that all elements of $M_{(r,\mathbf{u})}(U)$ except $(r, \mathbf{u})$ are principal elements which are isolated.

The main step in showing that F is a sheaf over X is to observe that the when two global sections $\sigma(f), \sigma(g)$ agree at a point $(r, \mathbf{u})$, then there exists $U \in \mathbf{u}$ such that they agree on $M_{(r,\mathbf{u})}(U)$. The remaining details are straightforward. ∎

4.4.12. Lemma. *For each* $r \in \mathbf{R}$*, the ring* A/P_r *is isomorphic to the ring* $\mathbf{R}[\![x]\!]$ *of formal power series in* x *with real coefficients.*

Proof. Let $t : A \longrightarrow \mathbf{R}[\![x]\!]$ be given by Taylor's formula $t(f) = \sum (f^{(n)}(r)/n!)x^n$. It is clear that the kernel of t is P_r. It is surjective by [Borel (1895)]. ∎

4.4.13. NOTATION. If $\zeta : X \longrightarrow E$ is a global section of E, then for $r \in \mathbf{R}$, we can, by Lemma 4.4.12, consider $\zeta(P_r)$ to be a power series in $\mathbf{R}[\![x]\!]$. Then write $\zeta(P_r) = \sum_{n\geq 0} a_n(r)x^n$. Let $\zeta_n(r) = n!a_n(r)$. This assigns to each global section ζ a sequence $\zeta_0, \zeta_1, \ldots$ of functions $\mathbf{R} \longrightarrow \mathbf{R}$.

4.4.14. LEMMA. *Let $\zeta \in \Gamma(F)$ be given. Let $(\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots)$ be the corresponding sequence of functions. Then for each $n \geq 0$ the function ζ_{n+1} is the derivative of ζ_n.*

PROOF. It suffices to show that, for each $r \in \mathbf{R}$, the differential quotient $(\zeta_n(r + \Delta r) - \zeta_n(r))/\Delta r$ must approach $\zeta_{n+1}(r)$ as Δr approaches zero (without actually being zero). More precisely, along every non-principal ultrafilter $\mathbf{u}$ which converges to r the differential quotient of ζ_n must converge to $\zeta_{n+1}(r)$. But we claim this follows as there must be an $f \in A$ such that ζ agrees with $\sigma(f)$ on a neighbourhood of $(r, \mathbf{u})$. This means that there exists $U \in \mathbf{u}$ such that $f^{(n)}(s) = \zeta_n(s)$ for all $s \in U$ and $f^{(n+1)}(r) = \zeta_{n+1}(r)$. Since the differential quotient of $f^{(n)}$ approaches $f^{(n+1)}(r)$, the same is true of ζ_n on U and $\zeta_{n+1}(r)$. ∎

4.4.15. THEOREM. *The ring A is DL-closed.*

PROOF. As noted before, it suffices to show that the canonical embedding $A \longrightarrow \Gamma(F)$, which maps $f \in A$ to $\sigma(f) \in \Gamma(F)$ is an isomorphism. Let $\zeta \in \Gamma(F)$ be given. Then in view of the above lemma, ζ_0 is in $C^\infty(\mathbf{R})$ and it is easily shown that $\zeta = \sigma(\zeta_0)$. It is also obvious that the map $A \longrightarrow \Gamma(F)$ is an injection and an isomorphism. ∎

5. Limit closure of integrally closed domains.

We remind the reader that in standard usage the phrase "integrally closed domain" does not describe a domain that is integrally closed in every containing domain, but rather a domain that is integrally closed in its field of fractions. We use the standard meaning here.

In this section we will denote by $\mathcal{A}_{\mathrm{ic}}$ the category of integrally closed domains, by $\mathcal{K}_{\mathrm{ic}}$ the limit closure of $\mathcal{A}_{\mathrm{ic}}$, and by $K_{\mathrm{ic}} : \mathcal{SPR} \longrightarrow \mathcal{K}_{\mathrm{ic}}$ the reflector.

We will characterize the rings in $\mathcal{K}_{\mathrm{ic}}$ but first we need some preliminary results.

5.1. INTEGRALLY CLOSED DOMAINS AND (n, k)-CLOSURES.

The following result is well known, but we include a proof.

5.1.1. PROPOSITION. *Recall that $\mathbf{N}$, set of non-negative integers is a monoid under addition. A submonoid of $\mathbf{N}$ that contains a relatively prime pair of integers contains all sufficiently large integers.*

PROOF. Denote the submonoid by S and suppose $n, k \in S$ are relatively prime. Any integer m (possibly negative) has a unique representation $m = xn+yk$ with $0 \leq x < k$. In fact, starting with any representation $m = un+vk$, let $x = u-\lfloor u/k \rfloor k$ and $y = v+\lfloor u/k \rfloor n$ (where $\lfloor u/k \rfloor$ is the greatest integer that is less than or equal to u/k). Now suppose m_1, m_2 are two integers such that $m_1+m_2 = nk-n-k$ and write them in the above form $m_1 = x_1n+y_1k$ and $m_2 = x_2n+y_2k$. Adding them, we get $(x_1+x_2)n+(y_1+y_2)k = nk-n-k$

or $(x_1 + x_2 + 1)n + (y_1 + y_2 + 1)k = kn$. Now $0 < x_1 + x_2 + 1 < 2k$ and must be divisible by k since k and n are relatively prime. This implies that $x_1 + x_2 + 1 = k$ and hence that $y_1 + y_2 + 1 = 0$ which implies that one of them is negative and the other one isn't. It follows that one of $m_1, m_2 \in S$ (and it is easily seen that the other one is not). In particular, $m_2 < 0$, we cannot have $m_2 \in S$ and so every $m_1 > nk - k - n$ belongs to S. ■

The following proposition will be helpful.

5.1.2. Proposition. *Suppose r, s are elements of a (semiprime) ring S and k, n are a relatively prime pair of positive integers such that $r^k = s^k$ and $r^n = s^n$. Then $r = s$.*

Proof. It readily follows from the preceding that $r^\ell = s^\ell$ for all sufficiently large integers. Let m be the least integer for which it fails. In the expansion of $(r - s)^{2m+1}$ each term $r^i s^j$ must have either $i > m$ or $j > m$. In the first case, $r^i s^j = s^{i+j} = r^{i+j}$ and in the second, $r^i s^j = r^{i+j}$. The result is that $(r - s)^{2m+1}$ coincides with the binomial expansion of $(r - r)^{2m+1}$ so $(r - s)^{2m+1} = 0$. But in a semiprime ring, this implies that $r = s$. ■

5.1.3. Definition. *If $n > 1$ and $k > 1$ are a pair of relatively prime integers we say that a commutative ring R is* (n, k)**-closed** *if, whenever $r, s \in R$ satisfy $r^n = s^k$, then there is a $t \in R$ such that $t^k = r$ and $t^n = s$. The uniqueness of t follows from the preceding proposition.*

If $R \subseteq S$, we say that R **is** (n, k)**-closed in** *S if whenever $s \in S$ is such that $s^n \in R$ and $s^k \in R$, then also $s \in R$.*

We will say that R **is absolutely** (n, k)**-closed** *if it is (n, k)-closed in every containing semiprime ring.*

5.1.4. Proposition. *If n and k are relatively prime integers, a ring R is (n, k)-closed if and only if it is absolutely (n, k)-closed.*

Proof. Suppose $R \subseteq S$ and R is (n, k)-closed. Suppose $s \in S$ is such that $s^n, s^k \in R$. Then $(s^k)^n = (s^n)^k$ so there is a $t \in R$ with $t^n = s^n$ and $t^k = s^k$. Since S is semiprime, it follows that $t = s \in R$. For the converse, simply embed R into a product of fields, which certainly is (n, k)-closed. ■

5.1.5. Lemma. *Let R be a semiprime ring. Then the following conditions are equivalent:*

1. *R is (2,3)-closed.*
2. *R is (n, k)-closed for some relatively prime pair $n > 1$, $k > 1$.*
3. *Whenever $R \hookrightarrow S$ embeds R into a semiprime ring, then $s^\ell \in R$ for all sufficiently large ℓ implies $s \in R$.*
4. *R is (n, k)-closed whenever $n > 1$ and $k > 1$ and $\gcd(n, k) = 1$.*

Proof. (1) ⇒ (2) and (4) ⇒ (1) are immediate.
(2) ⇒ (3): Assume that $s^\ell \in R$ for all sufficiently large ℓ and suppose that m is the least integer such that $s^m \notin R$. Then $a = s^{mk}$ and $b = s^{mn}$ are both in R and $a^n = b^k$. By (n,k)-closure, there is a $c \in R$ such that $c^k = a = s^{mk}$ and $c^n = b = s^{mn}$. If we write $t = s^m$, then $c^k = t^k$ and $c^n = t^n$, whence by Proposition 5.1.2, we conclude that $c = t$, which contradicts the hypothesis that $t = s^m \notin R$. ∎

(3) ⇒ (4): If $R \hookrightarrow S$ is an embedding into a semiprime ring and $s \in S$ is such that $s^k \in R$ and $s^n \in R$, then Proposition 5.1.1 implies that $s^\ell \in R$ for all sufficiently large ℓ and hence from (3) that $s \in R$. ∎

5.1.6. Lemma. *Every integrally closed domain is (2,3)-closed.*

Proof. Let D be integrally closed with field of fractions F. Let $r, s \in D$ such that $r^3 = s^2$. We can leave aside the trivial case that $r = 0$. Otherwise, $\alpha = s/r \in F$ satisfies $\alpha^2 = r$, $\alpha^3 = s$ and is a root of $x^2 - r$ and therefore lies in D. ∎

5.1.7. Example. The converse of the preceding is false. Let $\mathbf{Z}[i][x]$ denote the ring of polynomials over the Gaussian integers. It is a UFD by standard theorems. Let $R \subseteq \mathbf{Z}[i][x]$ be the subring consisting of the polynomials whose constant term is real. Since $i = ix/x$ is integral over R, R is not integrally closed in its field of fractions. But in the following diagram,

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \mathbf{Z}[i][x] \arrow[d] \\ \mathbf{Z} \arrow[r] & \mathbf{Z}[i] \end{tikzcd}$$

in which the vertical maps are evaluation at $x = 0$, R is a pullback of three UFDs and so is (2,3)-closed.

Another example is the domain constructed in 6.3.1.

5.1.8. Proposition. *R is absolutely (2,3)-closed if and only if it is (2,3)-closed in $K_{\mathrm{ic}}(R)$.*

Proof. Suppose $R \hookrightarrow S$ is given and $s \in S$ is such that $s^2 = a \in R$ and $s^3 = b \in R$. Let $R[c] = R[x]/\langle x^2 - a, x^3 - b\rangle$ with c the image of x. Since $c^2 = a = s^2$ and $c^3 = b = s^3$, we see that the inclusion of R into S can be extended to $R[c]$ by the map that sends $c \mapsto s$. Moreover, any map $R \longrightarrow D$ with D an integrally closed domain can be extended uniquely to $R[c]$ by the previous proposition. So $R \longrightarrow R[c]$ has the unique extension property with respect to all $A \in \mathscr{A}_{\mathrm{ic}}$. But this implies that $R \longrightarrow R[c]$ has the unique extension property with respect to all $T \in \mathscr{K}_{\mathrm{ic}}$ (as the class of all objects with respect to which $R \longrightarrow R[c]$ has the unique extension property is limit-closed). It follows that $K_{\mathrm{ic}}(R[c])$ coincides with $K_{\mathrm{ic}}(R)$ and we can regard $R \subseteq R[c] \subseteq K(R)$. But if R is (2,3)-closed in $K_{\mathrm{ic}}(R)$, it follows that $c \in R$. ∎

5.2. The main theorem. The main theorem of this section states that a semiprime ring is in $\mathscr{K}_{\rm ic}$ if and only if it is (2,3)-closed.

Before stating and proving this theorem, we need some definitions and lemmas.

5.2.1. Notation. As usual, we identify R/P with its image in $K_{\rm ic}(R/P)$. If $\zeta \in K_{\rm ic}(R)$ and if P is a prime ideal of R, we let ζ_P denote the image of ζ in $K_{\rm ic}(R/P)$. We know from Theorem 2.3.1.15, together with the obvious fact that $\mathcal{A}_{\rm ica} \subseteq \mathcal{A}_{\rm ic}$ that $K_{\rm ic}$ is $\mathcal{Dom}$-invariant and hence by Theorem 3.6.12 that $K_{\rm ic}(R)$ is isomorphic to $\Gamma(E)$ where $E = E_R$. So each $\zeta \in K_{\rm ic}(R)$ is associated with a global section, also denoted by ζ in $\Gamma(E)$. In this case, ζ_P coincides with $\zeta(P)$, the value of the global section ζ at the prime ideal P.

5.2.2. Definition. *Recall from 3.9.1 that ζ is grounded on $U \subseteq$ Spec if $\zeta_P \in R/P$ for all $P \in U$. If ζ is grounded over all of* Spec(R) *we will simply say it is* **grounded.** *If ζ is not grounded (on U) we will say that it is* **ungrounded (on U)**.

5.2.3. Lemma. *Let R be a semiprime ring that is (2,3)-closed. If $\zeta \in K_{\rm ic}(R)$ is such that $\zeta^2 \in R$ and $\zeta^3 \in R$, it follows that $\zeta \in R$.*

Proof. This immediately follows from the fact that R is (2,3)-closed in $K_{\rm ic}(R)$. ■

5.2.4. Corollary. *Let R be a semiprime ring which is (2,3)-closed and $\zeta \in K_{\rm ic}(R)$ be given. It follows that:*

1. *if $\zeta^w \in R$ for all sufficiently large w, then $\zeta \in R$;*
2. *if ζ is grounded, then $\zeta \in R$.*

Proof. These results follow from Lemma 5.1.5, Proposition 3.9.7, and Corollary 3.9.8. ■

Outline of the proof of the Main Theorem. We recall that $\mathcal{A}_{\rm ic}$ denotes the class of integrally closed domains, and $\mathscr{K}_{\rm ic}$ denotes its limit closure. It follows from Theorem 2.3.1.15, and the obvious fact that $\mathcal{A}_{\rm ica} \subseteq \mathcal{A}_{\rm ic} \subseteq \mathscr{K}_{\rm ic}$ that we are in the $\mathcal{Dom}$-invariant case so that all the equivalent properties in Theorem 2.3.1 are available to us. We will start the proof by showing that $\mathcal{A}_{\rm ic}$ is first order, so, by Proposition 3.1.2, we can construct the associated canonical sheaf. By 3.6.12, the reflection of a ring into $\mathscr{K}_{\rm ic}$ is isomorphic to $\Gamma(E)$, the ring of global sections of the canonical sheaf.

As usual, we regard $R \subseteq \Gamma(E)$ by identifying $r \in R$ with the "constant" global section $\nu(r) \in \Gamma(E)$. The main part of the proof will be showing that if R is a (2,3)-closed semiprime ring, then every $\zeta \in \Gamma(E)$ is in R.

If $\zeta \in \Gamma(E)$ is grounded, then, as shown in the corollary above, $\zeta \in R$. What if ζ is ungrounded at some prime ideals? We will use an inductive technique starting with the largest prime ideals then working our way down to the smaller primes. In fact, if P is a maximal ideal, then ζ is automatically grounded at P, because we can always find $r, s \in R$ such that $\zeta(P) = r/s$ with $s \notin P$. But when P is maximal, $s \notin P$ implies that s has an inverse, mod P, so that $r/s = rs^{-1} \pmod{P}$).

If P is not quite maximal, but big enough so that if $\zeta(P) = r/s$ then for any ideal Q with $P \subseteq Q$ and $s \in Q$, we have already proven that ζ is grounded at Q. Then, using

Corollary 3.9.8 and Lemma 5.2.11 below, we will prove that we can reduce the problem of whether $\zeta \in R$ to the equivalent problem of whether $(\zeta - c)^m$ and $(\zeta - c)^{m+1}$ are both in R (for $c \in R$). We can solve this problem as these powers of $\zeta - c$ will be grounded at such ideals P. More precisely, we will use induction on the "depth" of the ideal P to complete the proof. The "depth" is essentially a measure of the size of P with the maximal ideals being of maximal depth. The induction works by considering, in effect, an ideal P of greatest depth for which ζ is not grounded at P.

The exact notion of depth is related to the concept of a fractional cover—see Definition 5.2.9). The idea is that for a given prime ideal P, we can write $\zeta(P) = r/s \pmod{P})$ for $r, s \in R$. Then there is a neighbourhood U of P such that for all $P' \in U$, we have $\zeta(P') = r/s \pmod{P'}$. We cover $\mathrm{Spec}(R)$ by finitely many such U. For technical reasons, to make our arguments work, we need to make the cover "gridded" (see Definition 5.2.14).

5.2.5. Proposition. *The class $\mathcal{A}_{\mathrm{ic}}$ of all integrally closed domains is first order.*

Proof. The domain D is integrally closed if and only if whenever c/d is a fraction, with $c, d \in D$ and $d \neq 0$, such that c/d is a root of a monic polynomial $p(x) \in D[x]$, then $c/d \in D$ (meaning that there exists $e \in D$ with $de = c$). If n is the degree of $p(x)$ then this condition is first order; we will denote by C_n the sentence:

$$(\forall a_{n-1}, a_{n-2}, \dots, a_0, c, d).(c^n + c^{n-1}da_{n-1} + \cdots + a_1cd^{n-1} + a_0d^n) = 0 \Rightarrow \exists e.(de = c)$$

It follows that D is integrally closed if and only if D satisfies C_n for each $n > 1$. ■

5.2.6. Remark. By the above proposition, and 3.1.2, the canonical sheaf associated with the limit closure of $\mathcal{A}_{\mathrm{ic}}$ exists for every semiprime ring. In what follows, we let R be a given semiprime ring and we let $E = E_R$ be the associated canonical sheaf. As usual, we regard $R \subseteq \Gamma(E)$, by identifying each $r \in R$ with the global section $\nu(r) \in \Gamma(E)$. By 3.6.12, we can identify $K_{\mathrm{ic}}(R)$ with $\Gamma(E)$.

5.2.7. Notation. For $\zeta \in \Gamma(E)$ and $r, s \in R$, we write $\zeta(P) = r/s$ for P a prime ideal of R if $s \notin P$ and $\zeta(P) = q_P(r)/q_P(s)$ where $q_P : R \longrightarrow R/P$ is the quotient map.

5.2.8. Definition. *Let R be a semiprime ring and let $\zeta \in \Gamma(E)$ be given.*

If $r, s \in R$ we say that (r, s) **fractionally represents ζ at the prime ideal** *$P \in \mathrm{Spec}(R)$ if $\zeta(P) = r/s$.*

We say that $U \subseteq \mathrm{Spec}(R)$ is **fractional for** *ζ if there exist $r, s \in R$ such that (r, s) fractionally represents ζ at P for every $P \in U$.*

5.2.9. Definition. *Let $\zeta \in \Gamma(E)$ be given where R is a semiprime ring. Then a* **fractional cover for** *ζ is a finite cover of $\mathrm{Spec}(R)$ by subsets which are clopen in the patch topology and fractional for ζ.*

5.2.10. Proposition. *Every $\zeta \in \Gamma(E)$ has a fractional cover.*

Proof. For every $P \in \mathrm{Spec}(R)$, there is some $r, s \in R$ such that $\zeta(P) = r/s$ and $s \notin P$. Then (r, s) represents ζ on $N(s\zeta - r) \cap Z(s)$. Clearly we can cover $\mathrm{Spec}(R)$ by a collection of such sets. Since the patch topology is compact, this cover has a finite subcover. ■

5.2.11. Lemma. *Suppose $\zeta \in \Gamma(E)$ and $r, s \in R$ are such that $s \neq 0$ and $s\zeta = r$ in $\Gamma(E)$. Assume that, for all primes $P \in R$, $s \in P$ implies $\zeta \in P^*$. Then there exist $t \in R$ and a positive integer m such that $\zeta^m = st$ and $\zeta^{m+1} = rt$.*

Proof. We start by claiming that $\zeta^n \in sR[\zeta]$ for some positive integer n. This is equivalent to saying that ζ is in every prime ideal of $R[\zeta]$ that contains s. By Theorem 2.3.1.11 and Proposition 2.2.16 the prime ideals of $R[\zeta]$ are all of the form $P^* \cap R[\zeta]$ for a prime $P \subseteq R$. By hypothesis, it easily follows that $s \in P^* \cap R[\zeta]$ implies $\zeta \in P^* \cap R[\zeta]$ which proves the claim. Since $\zeta^n \in sR[\zeta]$, there is a polynomial $g(x) \in R[x]$ such that $\zeta^n = sg(\zeta)$. Let $f(x) \in R[x]$ be a monic polynomial satisfied by ζ (Theorem 2.3.1.13) and suppose that f has degree w. Then $\zeta^{nw} = s^w g(\zeta)^w$. Let $h(x)$ be the remainder when $g(x)^w$ is divided by f. Then the degree of h is at most $w-1$, which implies that $s^{w-1}h(\zeta) = s^{w-1}h(r/s) \in R$. If we denote this element by t, this implies that $\zeta^{nw} = st$ and then that $\zeta^{nw+1} = \zeta st = rt$ since $\zeta s = r$. ∎

5.2.12. Notation.

1. For $\zeta \in \Gamma(E)$, we say that $\sigma \in \Gamma(E)$ is a **polynomial in** ζ if $\sigma \in R[\zeta]$, the subring of $\Gamma(E)$ generated by $R \subseteq \Gamma(E)$ and ζ.
2. If $U \subseteq \mathrm{Spec}(R)$ we let $U^\uparrow$ denote the up closure of U so that $U^\uparrow = \{Q \mid \exists P \in U.\ P \subseteq Q\}$.

5.2.13. Lemma. *Let $\zeta \in \Gamma(E_R)$ be given, where R is (2,3)-closed. Then $\zeta \in R$ if there exists a patch-closed subset $U \subseteq \mathrm{Spec}(R)$ such that:*

1. *U is fractional for ζ with $\zeta(P) = r/s$ when $P \in U$.*
2. *ζ is grounded on $U^\uparrow \cap N(s)$.*
3. *Whenever $\sigma \in \Gamma(E)$ is a polynomial in ζ then $\sigma \in R$ if and only if σ is grounded on U.*

Proof. We will actually prove that $\sigma \in R$ whenever σ is a polynomial in ζ. Let $I = \bigcap\{P \mid P \in U\}$ and let $\overline{R} = R/I$. Note that $\overline{R}$ is semiprime that we can regard $\mathrm{Spec}(\overline{R})$ as the set of prime ideals $Q \in \mathrm{Spec}(R)$ for which $I \subseteq Q$. By Proposition 2.2.22, we can regard $\mathrm{Spec}(\overline{R})$ as $U^\uparrow$. It easily follows that $U^\uparrow$ is compact (in the patch topology and therefore patch-closed) as it is $\mathrm{Spec}(\overline{R})$ and the convergence of ultrafilters is preserved as we pass from $\mathrm{Spec}(\overline{R})$ to $\mathrm{Spec}(R)$.

Let σ be a given polynomial in ζ. Let d be the degree of this polynomial. Since $\zeta(P) = r/s$ for all $P \in U$, it follows that there exists $r_1, s_1 \in R$ with $\sigma(P) = r_1/s_1$ for all $P \in U$. Moreover, we can take $s_1 = s^d$. It follows that $N(s_1) = N(s)$. Let $W = U^\uparrow \cap N(s) = U^\uparrow \cap N(s_1)$

Clearly σ is grounded on W because ζ is grounded there. It follows by Proposition 3.9.3 that σ has a grounded representation $\mathscr{U} = \{U_1, \ldots, U_n\}$ on W. We proceed by induction on n, the cardinality of the representing set $\mathscr{U}$. If $n = 1$, then there exists

$a \in R$ such that σ and $\nu(a)$ have the same restriction to W. We may as well assume that $a = 0$ otherwise we can replace σ by $\sigma - a$. This means that $\overline{\sigma}$, the restriction of σ to $U^{\uparrow}$ satisfies the hypotheses of Lemma 5.2.11, applied to $\overline{R}$, so there exists a positive integer m and $t \in \overline{R}$ such that $\overline{\sigma}^m = s_1 t$ and $\overline{\sigma}^{m+1} = r_1 t$ on W_1. But this clearly implies that σ^m and σ^{m+1} are grounded on $U^{\uparrow}$ so, by the above hypothesis, we have σ^m and σ^{m+1} are in R. Since R is (2,3)-closed, this implies that $\sigma \in R$.

Finally, suppose we have proven that any polynomial σ in ζ is in R whenever it has a grounded representation on W of cardinality n. Let $\mathscr{U} = \{U_1, \ldots, U_{n+1}\}$ be a grounded representation for σ, of cardinality $n+1$. Then by Theorem 3.9.5, there exists $\mathscr{U}'$, a grounded representation of cardinality n for sufficiently large powers of $\sigma - b$, where $b \in R$. By our induction hypothesis, we see that these large powers of $\sigma - b$ are in R so, as R is (2,3)-closed, we have $\sigma - b \in R$ so $\sigma \in R$. ∎

We need the following technical definition.

5.2.14. Definition. *Let R and $\zeta \in \Gamma(E)$ and let $\mathscr{C}$ be a fractional cover for ζ. By an* **assignment** *for $\mathscr{C}$, we mean a choice, for each $U \in \mathscr{C}$, of a pair $r_U, s_U \in R$ so that $\zeta(P) = r_U/s_U$ for all $P \in U$ (note that this implies that $s_U \notin P$ for all $P \in U$).*

Suppose we are given an assignment for the fractional cover $\mathscr{C}$. Let $S = \{s_U \mid U \in \mathscr{C}\}$. The assignment is **gridded** *if whenever the prime ideals P, Q are in the same member of $\mathscr{C}$, then $S \cap P = S \cap Q$. Equivalently, if whenever $U, V \in \mathscr{C}$ are given, then either $U \subseteq N(s_V)$ or $U \subseteq Z(s_V)$.*

By a **gridded fractional cover for** ζ *or* **GFC for** ζ, *we mean a fractional cover together with a specified gridded assignment.*

5.2.15. Notation. Let $\mathscr{C}$ be a GFC. Then the set $S = \{s_U \mid U \in \mathscr{C}\}$, mentioned above, is called the **denominator set** for the GFC.

5.2.16. Proposition. *Every fractional cover can be refined to a GFC.*

Proof. Let $\mathscr{C}$ be a fractional cover for $\zeta \in \Gamma(E)$. Write $\mathscr{C} = \{U_1, U_2, \ldots, U_m\}$. For each $U_i \in \mathscr{C}$, choose (r_i, s_i) so that $\zeta(P) = r_i/s_i$ for all $P \in U_i$. Say that $G \subseteq \operatorname{Spec}(R)$ is a **grid set** if we can write $G = A_1 \cap A_2 \cap \ldots \cap A_m$ where each A_i is either $N(s_i)$ or $Z(s_i)$. Form a new cover consisting of all sets of the form $U_i \cap G$ where G is a grid set. Assign (r_i, s_i) to each set of the form $U_i \cap G$. Then this cover, together with this assignment, is easily seen to be a GFC for ζ. ∎

Obviously, this process could, in principle, enlarge a cover by m sets into one by $m2^m$ sets (actually somewhat less since $U_i \cap N(s_i) = \emptyset$).

5.2.17. Proposition. *If $\mathscr{C}$, together with a uniform assignment, is a GFC for ζ and if $f \in R[x]$ is a polynomial, then $\mathscr{C}$ is a GFC for $f(\zeta)$, when we modify the assignment in the obvious way as indicated in the proof.*

Proof. Suppose $U \in \mathscr{C}$ and write (r, s) for (r_U, s_U). Let n be the degree of the polynomial f. Then to make $\mathscr{C}$ into a GFC for $f(\zeta)$, we have to assign $(s^n f(r/s), s^n)$ to U. ∎

5.2.18. Remark. Suppose that $\zeta \in \Gamma(E)$, $f \in R[x]$, and $\mathscr{C}$, are a GFC for ζ. We will often say that $\mathscr{C}$ is also a GFC for $f(\zeta)$, without explicitly mentioning that we have modified the assignment used for ζ to give us an assignment for $f(\zeta)$.

5.2.19. Definition. *The element $\zeta \in \Gamma(E)$ has* **index** *k, denoted by $k = \operatorname{ind}(\zeta)$, when k is the least integer for which there is a GFC which contains exactly k elements at which ζ is ungrounded. Evidently, a grounded element has index 0.*

Let $\zeta \in \Gamma(E)$ have positive index and suppose that $\mathscr{C}$ is a GFC that realizes that index. Let $S = \{s_U \mid U \in \mathscr{C}\}$ be the denominator set. We say that ζ has **depth** *n* **on** *$U \in \mathscr{C}$ and write $n = \operatorname{dep}_U(\zeta)$ if for every $P \in U$, the set $S \cap P$ has exactly n elements.*

5.2.20. Lemma. *Let $\zeta \in \Gamma(E)$ be ungrounded at P. Write $\zeta(P) = r/s$. Then there exists a prime ideal Q with $P \subseteq Q$ and $s \in Q$.*

Proof. If not, then the image of s is non-zero in every prime ideal of R/P. This means that s has an inverse, s_1 modulo P, so $\zeta(P) = rs_1$, contradicting the fact that ζ is ungrounded at P. ∎

5.2.21. Theorem. *A semiprime ring is in the limit closure of all integrally closed domains if and only if it is (2,3)-closed.*

Proof. We note that the necessity of being (2,3)-closed is obvious because all integrally closed domains are (2,3)-closed and the class of all (2,3)-closed semiprime rings is easily seen to be closed under limits.

To prove the converse, let R be a (2,3)-closed, semiprime ring. Recall that $E = E_R$ is the canonical sheaf for R and we are identifying $K_{\mathrm{ic}}(R)$ with $\Gamma(E)$. We let $\zeta \in \Gamma(E)$ be given. We have to prove that $\zeta \in R$.

The argument is by induction on $\operatorname{ind}(\zeta)$. If $\operatorname{ind}(\zeta) = 0$, then ζ is grounded, and $\zeta \in R$ by Corollary 5.2.4.

Now assume that $\operatorname{ind}(\zeta) = k$ and that every element of $\Gamma(E)$ of index less than k belongs to R. Let $\mathscr{C}$ be a GFC that realizes this index for ζ. Choose $U \in \mathscr{C}$ on which ζ is ungrounded such that the depth d of U is as large as possible. It follows that if we find $V \in \mathscr{C}$ on which ζ has greater depth, then ζ is grounded on V.

Let $r, s \in R$ be such that $\zeta(P) = r/s$ for all $P \in U$. We claim that Lemma 5.2.13 applies, which completes the proof. Clearly if σ is any polynomial in ζ which is grounded on U, then σ is grounded whenever ζ is and, moreover, σ is grounded on U so σ has an index less than k so, by our induction hypothesis, $\sigma \in R$. It remains to show that ζ is grounded on $W = U^{\uparrow} \cap N(s)$.

We claim that ζ is grounded on W. If $Q \in W$, choose $V \in \mathscr{C}$ with $Q \in V$. It suffices to show that V has greater depth then U. But if S is the denominator set of $\mathscr{C}$, and if $s_1 \in S$ has the property that $s_1 \in P$ for all $P \in U$, then clearly $s_1 \in Q$ so $s_1 \in Q'$ for all $Q' \in V$. But $s = s_U$ is not in P for any $P \in U$ but is in Q and therefore is in Q' for all $Q' \in V$. This proves the claim which implies that ζ is grounded on V. ∎

5.2.22. Remark. The following are obvious consequences of the above theorem:

1. The functor K_{ic} which reflects semiprime rings to the limit closure of integrally closed domains, coincides with the reflection into the subcategory of (2,3)-closed domains.

2. Just as we can construct, in 4.2, the DL-reflection 4.2 by successive DL-extensions, we can construct $K_{\text{ic}}(R)$ as the closure of R under a succession of simple (2,3)-extensions.

3. We can also classify the elements of $K_{\text{ic}}(R)$ according to the "level" at which they appear, as in the definition below.

5.2.23. Definition. *For each ring R, we define $E_n(R) \subseteq K_{\text{ic}}(R)$ so that $E_0(R) = R$ and $\zeta \in E_{m+1}$ if there is an $r \in R$ and relatively prime positive integers n, k such that $(\zeta - r)^n$ and $(\zeta - r)^k$ are both in $E_m(R)$. Finally, we let $E(R) = \bigcup E_m(R)$.*

In view of the next result, we can define the **level** *of each $\zeta \in K_{\text{ic}}(R)$ as the smallest integer n for which $\zeta \in E_n$.*

An analysis of the argument in the proof above demonstrates:

5.2.24. Theorem. *$E(R) = K_{\text{ic}}(R)$.* ∎

5.3. Some examples. We start this section, with an example of a semiprime ring which is not (2,3)-closed and then give several examples of semiprime rings which are (2,3)-closed.

5.3.1. Example. Let $\mathbf{Z}[x]$ be the domain of all polynomials in the indeterminate x and with integer coefficients. Let $D \subseteq \mathbf{Z}[x]$ be the domain of all polynomials $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ for which a_1, the coefficient of the x-term, is zero. Then D is the test example in the sense that the semiprime ring R is (2,3)-closed if and only if every homomorphism $D \longrightarrow R$ extends to a homomorphism $\mathbf{Z}[x] \longrightarrow R$.

Proof. D is obviously not (2,3)-closed in $\mathbf{Z}[x]$ as $x \notin D$ but $x^2 \in D$ and $x^3 \in D$. It is also clear that $D \longrightarrow \mathbf{Z}[x]$ has the extension property because if $h : D \longrightarrow R$ is given, where R is (2,3)-closed, and if $h(x^2) = r$ and $h(x^3) = s$ then there is a unique $t \in R$ with $t^2 = h(x^2)$ and $t^3 = h(x^3)$ so the extension of h is the map $\mathbf{Z}[x] \longrightarrow R$ which takes x to t. It remains to show that whenever $r, s \in R$ satisfy $r^3 = s^2$ there exists a map $h : D \longrightarrow R$ with $h(x^2) = r$ and $h(x^3) = s$. Consider the ring $\mathbf{Z}[y, z]/(y^3 - z^2)$. Given $r, s \in R$ with the above properties, there exists $\mathbf{Z}[y, z]/(y^3 - z^2) \longrightarrow R$ which maps y to r and z to s. In particular, there exists a map $g : \mathbf{Z}[y, z]/(y^3 - z^2) \longrightarrow D$ which sends y to x^2 and z to x^3. We need to show that g is an isomorphism. It is clearly surjective, so it suffices to show it is injective. Suppose that $P(y, z) \in \ker(g)$. By replacing all terms in P involving z^2 with y^3, we can write find $P_0(y)$ and $P_1(y)$ such that $P(y, z)$ is equivalent to $P_0(y) + zP_1(y)$ $\pmod{y^3 - z^2}$. Then $g(P_0(y) + zP_1(y)) = P_0(x^2) + x^3P_1(x^2) = 0$. But $P_0(x^2)$ only involves even powers of x while $x^3P_1(x^2)$ only involves odd powers of x, so their sum is zero if and only if $P_0 = 0$ and $P_1 = 0$ which implies that $P(y, z) = 0 \pmod{y^3 - z^2}$. ∎

5.3.2. Remark. For a similar ring, see Example 6.3.1.

Here we give some examples of (2,3)-closed rings.

5.3.3. Example. The ring of global sections of any sheaf of (2,3)-closed domains is (2,3)-closed.

Proof. This follows from Theorem 3.5.3. ∎

5.3.4. Example. Let R be a (2,3)-closed ring and suppose $S \subseteq R$ is a multiplicatively closed set without zero divisors. Then $S^{-1}R$ is also (2,3)-closed.

Proof. Suppose that $(a/s)^3 = (b/t)^2$ with $s, t \in S$. This gives $t^2a^3 = s^3b^2$ which implies $s^3t^6a^3 = s^6t^4b^2$ or $(st^2a)^3 = (s^3t^2b)^2$. Thus there is a $c \in R$ with $st^2a = c^2$ and $s^3t^2b = c^3$ which implies that $a/s = (c/st)^2$ and $b/t = (c/st)^3$, as required. ∎

5.3.5. Example. If R is (2,3)-closed and X is a family of indeterminates, then $R[X]$ is (2,3)-closed.

Proof. Since models of an essentially algebraic category are closed under filtered colimits, it suffices to show it for a single variable. We will show that the class $\mathscr{R}$ of all semiprime rings R for which $R[x]$ satisfies the (2,3) property is $\mathscr{K}_{\mathrm{ic}}$. By [Eisenbud (1995), Exercise 4.17; see also the hints], whenever D is a domain integrally closed in its field F of fractions, $D[x]$ is integrally closed in $F[x]$. But $F[x]$ is a UFD and therefore integrally closed in its field of fractions, which is the rational function field $F(x)$. It follows that $D[x]$ is also integrally closed in $F(x)$, which is its field of fractions. Thus $\mathscr{R}$ contains all integrally closed domains. If we show that $\mathscr{R}$ is limit closed, it follows that $\mathscr{R} = \mathscr{K}_{\mathrm{ic}}$ since they are each the limit closure of the integrally closed domains. It is evident that $\mathscr{R}$ is closed under equalizers, so it suffices to show it closed under products. So suppose that $R = \prod R_i$ and that each $R_i \in \mathscr{R}$. Clearly $R[x] \subseteq \prod(R_i[x])$. The difference between the two rings is that in $R[x]$, every polynomial has a fixed degree, while in $\prod(R_i[x])$ the polynomials must, for each i, have a degree but there is not necessarily a uniform bound on them. Now suppose $a(x), b(x) \in R[x]$ are polynomials such that $a(x)^3 = b(x)^2$. Since each $R_i[x]$ is (2,3)-closed, it follows that there is a unique $c(x) = (c_i(x)) \in \prod(R_i[x])$ such that $a(x) = c(x)^2$ and $b(x) = c(x)^3$. Since each R_i is semiprime, it follows that $\deg(c_i) \le \deg(a)/2$ for all i, so that $c \in R[x]$. ∎

The following corollary is immediate, while showing this directly would seem to be non-trivial.

5.3.6. Corollary. *Suppose D is a (2,3)-closed domain with field F of fractions. If $p(x) \in F[x]$ is such that both $p(x)^2$ and $p(x)^3$ are in $D[x]$, then so is $p(x)$.* ∎

Recall that when $R \subseteq S$ we say that R is (2,3)-closed in S if whenever the square and cube of an element of S belong to R, so does the element. We have:

5.3.7. Corollary. *Suppose $R \subseteq S$ is (2,3)-closed. Then $R[x] \subseteq S[x]$ is also (2,3)-closed.*

Proof. From 2.3.1.9, we know that $K_{\mathrm{ic}}(R) \subseteq K_{\mathrm{ic}}(S)$. We claim that

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, hook] & S \arrow[d, hook] \\ K_{\mathrm{ic}}(R) \arrow[r, hook] & K_{\mathrm{ic}}(S) \end{tikzcd}$$

is a pullback. Suppose that $s \in S \cap K_{\mathrm{ic}}(R)$. Then we have from Theorem 5.2.24 that $s \in E_m$ for some positive integer m. But then there are relatively prime integers n and k an $r \in R$ such that $(s-r)^n \in R$ and $(s-r)^k$ are both in $E_{m-1}(R)$. If we make the inductive assumption that $S \cap E_{m-1}(R) = R$, it follows that $(s-r)^n$ and $(s-r)^k$ in R. The fact that R is (2,3)-closed in S implies, by 5.1.2 that $s - r \in R$ and hence $s \in R$.

It follows that

$$\begin{tikzcd} R[x] \arrow[r, hook] \arrow[d, hook] & S[x] \arrow[d, hook] \\ K_{\mathrm{ic}}(R)[x] \arrow[r, hook] & K_{\mathrm{ic}}(S)[x] \end{tikzcd}$$

is a pullback. If $p(x) \in S[x]$ is such that its square and cube lie in $R[x]$, then it lies in $K_{\mathrm{ic}}(R)[x]$ and hence in $R[x]$. ∎

5.3.8. Corollary. *Suppose R is (2,3)-closed in S and X is any set of variables. Then $R[X]$ is (2,3)-closed in $S[X]$.*

Proof. The previous corollary gives it for any finite X. But every element of $S[X]$ lies in $S[Y]$ for some finite subset $Y \subseteq X$ from which the conclusion is obvious. ∎

5.3.9. Example. For any topological space X, the ring $C(X)$ (respectively, $C(X, \mathbf{C})$) of continuous real-valued (respectively, complex-valued) functions on X is (2,3)-closed.

Proof. We will do this for $C(X, \mathbf{C})$; the real case can be easily done using the continuity of the real cube root function. Suppose $\zeta \in K_{\mathrm{ic}}(C(X, \mathbf{C}))$ is such that $\zeta^2 = g$ and $\zeta^3 = h$ both in $C(X, \mathbf{C})$. Define $f \in C(X, \mathbf{C})$ by

$$f(x) = \begin{cases} h(x)/g(x) & \text{if } g(x) \neq 0 \\ 0 & \text{if } g(x) = 0 \end{cases}$$

Clearly f is continuous at any point $x \in X$ at which $g(x) \neq 0$. So suppose $x \in X$ is such that $g(x) = 0$. From $g(x)^3 = h(x)^2$ we conclude that also $h(x) = 0$. Given $\epsilon > 0$, find a neighbourhood U of x such that $|h(y)| < \epsilon^3$ for $y \in U$. We claim that for $y \in U$, $|f(y)| < \epsilon$. This is evident if $g(y) = 0$. If $g(y) \neq 0$, we have that $|f(y)| = |h(y)|/|g(y)| = |h(y)|/|h(y)|^{2/3} = |h(y)|^{1/3} < \epsilon$. Thus f is continuous everywhere in X. At a point $x \in X$ at which $g(x) \neq 0$, we have $f(x)^2 = h(x)^2/g(x)^2 = g(x)^3/g(x)^2 = g(x)$ and $f(x)^3 = h(x)^3/g(x)^3 = h(x)^3/h(x)^2 = h(x)$. When $g(x) = h(x) = 0$, we also have $f(x)^2 = 0 = g(x)$ and $f(x)^3 = 0 = h(x)$ so that $f^2 = g = \zeta^2$ and $f^3 = h = \zeta^3$. In a semiprime ring, this means $f = \zeta$ as claimed. ∎

6. Perfect Rings

6.1. Generalities. Recall from 2.1.1.8 that a domain D is perfect if it is either of characteristic 0 or is of characteristic p and every element has a pth root. We denote by $\mathcal{A}_{\text{per}}$ the category of perfect domains and by $\mathscr{K}_{\text{per}}$ its limit closure. In this section, we will define what is meant by a "perfect ring" and show that a ring is in $\mathscr{K}_{\text{per}}$ if and only if it is a perfect ring (see Definition 6.2.1). We will prove that $\mathscr{K}_{\text{fld}} \cap \mathscr{K}_{\text{per}} = \mathscr{K}_{\text{pfld}}$ but show that $\mathscr{K}_{\text{ica}}$ is a proper subclass of $\mathscr{K}_{\text{ic}} \cap \mathscr{K}_{\text{per}}$.

6.1.1. Notation. For any domain D we let $\text{char}(D)$ denote the characteristic of D.

For technical reasons, we make the following definitions:

6.1.2. Definition. *Let p be an integer prime, then:*

1. *A domain is p-**perfect** if it either has characteristic other than p or is perfect.*

2. *A semiprime ring R satisfies the p-**condition** if, whenever $r, u_1, u_2, \dots, u_n, t \in R$ satisfy*
$$p(ru_1 - t)(ru_2 - t)\cdots(ru_n - t) = 0 \text{ and } p(r^{p+1} - t^p) = 0$$
there exists $u \in R$ such that
$$u^p = r \text{ and } p(ru - t) = 0$$

6.1.3. Notation. In what follows, we assume that R is a semiprime ring, that p is an integer prime, and that E_p is the sheaf associated with $\mathcal{A}_{p\text{-per}}$, the full subcategory of all p-perfect domains.

If D is a domain, then $G_{p\text{-per}}(D)$ denotes the G-operation associated with the limit closure of $\mathcal{A}_{p\text{-per}}$ Obviously, If $p \notin Q$ then $\text{char}(R/Q) \neq p$ so $G_{p\text{-per}}(R/Q) = R/Q$.

Our first goal is to show that R is in the limit closure of the p-perfect domains if and only if R satisfies the p-condition. Since every domain in $\mathcal{A}_{\text{ica}}$ is p-perfect, it follows, from Theorem 2.3.1 that we are in the $\mathcal{D}om$-invariant case. In particular, $\text{Spec}(R)$ has the domain topology and $\Gamma(E_p)$ is the reflection of R into the rings that satisfy the p-condition.

6.1.4. Lemma. *A domain D is p-perfect if and only if it satisfies the p-condition.*

Proof. We first show that if D is p-perfect then D satisfies the p-condition.

Case 1: $\text{char}(D) = p$. Since D is p-perfect and $\text{char}(D) = p$, we see that D is perfect. To prove that D satisfies the p-condition, assume that $r, u_1, u_2, \dots, u_n, t \in D$ satisfy:
$$p(ru_1 - t)(ru_2 - t)\cdots(ru_n - t) = 0 \text{ and } p(r^{p+1} - t^p) = 0$$
Since $\text{char}(D) = p$, these conditions are vacuous. Since D is perfect, there is a $u \in R$ such that $u^p = r$, while the condition $p(ru - t) = 0$ is vacuous.

Case 2: $\mathrm{char}(D) \neq p$. Assume that $r, u_1, u_2, \ldots, u_n, t \in D$ satisfy:

$$p(ru_1 - t)(ru_2 - t) \cdots (ru_n - t) = 0 \text{ and } p(r^{p+1} - t^p) = 0$$

Since $\mathrm{char}(D) \neq p$, we see that

$$(ru_1 - t)(ru_2 - t) \cdots (ru_n - t) = 0 \text{ and } r^{p+1} = t^p$$

We may assume that $r \neq 0$, otherwise we can choose $u = 0$. Since D is a domain, there exists i with $ru_i = t$. Let $u = u_i$. Then $u^p r^p = t^p = r^{p+1}$ and it follows that $u^p = r$.

Next we prove the converse, that if D satisfies the p-condition, then D is p-perfect. If $\mathrm{char}(D) \neq p$ then there is nothing to prove, so assume that $\mathrm{char}(D) = p$. Let $r \in D$ be given. We must find a pth root for r. But choose $n = 1$ and u_1 and t to be any elements of D. The conditions that $p(ru_1 - t) = 0$ and $p(r^{p+1} - t^p) = 0$ are vacuously true, so there exists $u \in D$ with $u^p = r$ while $p(ru - t) = 0$. It follows that u is the desired pth root. ∎

6.1.5. Lemma. *Let R be a semiprime ring, let $r, t \in R$ and let k be a positive integer. Then the following conditions on $u \in R$ are equivalent and uniquely determine u if such an element exists.*

1. *$u^k = r$ and $u^{k+1} = t$.*
2. *$ru = t$ and $u^k = r$.*
3. *$ru = t$ and $r^{k+1} = t^k$ and $N(r) \subseteq N(u)$.*

Proof.

1⇒2: Given 1, we have $ru = u^k u = u^{k+1} = t$

2⇒3: Given 2, it is clear that $r \in P$ implies $u \in P$. Also, $r^{k+1} = r^k u^k = t^k$.

3⇒1: Given 3, it suffices to show this mod every $P \in \mathrm{Spec}(R)$. If $r \in P$ then $t \in P$ (as $r^{k+1} = t^k$) and $P \in N(r) \subseteq N(u)$ so $u \in P$ and 1 trivially holds mod P. On the other hand, if $r \notin P$ then $r^k u^k = t^k = r^{k+1}$ and we can cancel r^k to see that $u^k = r$. Moreover, when $r \notin P$ we have $t \notin P$, as $r^{k+1} = u^k$ and from $t^k u^{k+1} = r^{k+1} u^{k+1} = t^{k+1}$ we can cancel the t^k to infer $u^{k+1} = t$.

Since condition 1 uniquely determines u, by Proposition 5.1.2, each of the conditions uniquely determines u when such a u exists. ∎

6.1.6. Lemma. *Let R be semiprime and let p be an integer prime. Let $r, u_1, \ldots, u_n, t \in R$ satisfy $p(ru_1 - t)(ru_2 - t) \cdots (ru_n - t) = 0$ and $p(r^{p+1} - t^p) = 0$. Then there exists at most one $u \in R$ for which $u^p = r$ and $p(ru - t) = 0$.*

Proof. We claim that the equations $u^p = r$ and $p(ru - t) = 0$ determine u uniquely modulo every prime ideal P of R. If $r \in P$ we see that $u \in P$. If $p \in P$ then u is determined modulo P as pth roots are unique in R/P. If $r \notin P$ and $p \notin P$, then we have $u^p = r$ and $ru = t$ modulo P. Thus Lemma 6.1.5.2 applies, with $k = p$, and the uniqueness of u, modulo P, follows as stated in that lemma. ∎

6.1.7. Lemma. *Let p be an integer prime. The class of all semiprime rings which satisfy the p-condition is closed under limits.*

Proof. It is immediate that this class is closed under arbitrary products and, in view of the above lemma, it is easy to prove closure under equalizers. ∎

6.1.8. Corollary. *Every ring in $\mathscr{K}_{\mathrm{per}}$ satisfies the p-condition for every prime p.* ∎

We will eventually prove the converse of the above, but we need some more lemmas. For technical reasons we need the following definition:

6.1.9. Definition. *Let R be a semiprime ring and let $k > 1$ be an integer. We say that R is k-**coherent** if, whenever $r, t \in R$ satisfy the following conditions:*

1. *$r^{k+1} = t^k$ and*
2. *for every prime ideal P of R, there exists $u_P \in R$ such that $ru_P = t \pmod P$,*

then there exists $u \in R$ such that $u^k = r$ and $u^{k+1} = t$.

6.1.10. Lemma. *For any $k > 1$, the semiprime ring R is k-coherent if and only if it is DL-closed.*

Proof. Assume that R is k-coherent and let E be the canonical sheaf for the limit closure of all domains (see Section 4). By Corollary 3.9.8, it suffices to show that for all $\zeta \in \Gamma(E)$, if ζ^k and ζ^{k+1} are in R, then $\zeta \in R$. Assume $\zeta^k = r \in R$ and $\zeta^{k+1} = t \in R$. Then, clearly, $r^{k+1} = t^k$. Also, for every prime ideal P, we can let $u_P \in R$ be any element whose image in R/P is $\zeta(P)$. Then, by k-coherence, there exists $u \in R$ with $u^k = r$ and $u^{k+1} = t$. By Proposition 5.1.2, applied to the ring $\Gamma(E)$, we see that $u = \zeta$.

Conversely, assume that R is DL-closed and that $r^{k+1} = t^k$ for some $r, t \in R$ and that for every prime ideal P, there exists $u_P \in R$ such that $ru_P = t \pmod{P}$). We may assume that $u_P = 0$ if $r \in P$. We can then define $\zeta \in \Gamma(E)$ by letting $\zeta(P)$ be the image of $u_P \in R/P$ for each prime P. To prove that ζ is continuous, note that $\zeta(P)$ is defined by conditions which, in view of Lemma 6.1.5.3, uniquely determine ζ as a local section in a neighbourhood of P. By uniqueness these local sections patch together and form a global section. Since R is DL-closed, there exists $u \in R$ with $\zeta = u$ so u is the desired element. ∎

6.1.11. Lemma. *Let R be a semiprime ring and let $r, t \in R$ be given. Then for every prime ideal P of R, there exists $u_P \in R$ such that $ru_P = t \pmod{P}$) if and only if there exist $u_1, u_2, \ldots, u_n \in R$ such that $(ru_1 - t)(ru_2 - t) \cdots (ru_n - t) = 0$.*

Proof. If there exist $u_1, u_2, \ldots, u_n \in R$ such that $(ru_1 - t)(ru_2 - t) \cdots (ru_n - t) = 0$ and if P is a prime ideal of R, then there obviously exists i with $ru_i - t \in P$ and we can choose $u_P = u_i$.

Conversely, if for every prime ideal P of R, there exists $u_P \in R$ such that $ru_P = t \pmod{P}$) then for each such P there is an open neighbourhood $N(ru_P - t)$ of P such that for all P' in the neighbourhood, we have $ru_P = t \pmod{P'}$. Cover $\mathrm{Spec}(R)$ by finitely many of these neighbourhoods. ∎

6.1.12. Proposition. *Let p be an integer prime. Then a semiprime ring which satisfies the p-condition is DL-closed.*

Proof. We claim that R is p-coherent. Suppose $r, t \in R$ satisfy $r^{p+1} = t^p$ and that, for every prime ideal P, there exists $u_P \in R$ such that $ru_P = t \pmod{P}$). Then there exist $u_1, u_2, \ldots, u_n \in R$ such that $(ru_1 - t)(ru_2 - t)\cdots(ru_n - t) = 0$. It is immediate that $p(r^{p+1} - t^p) = 0$ and that $p(ru_1 - t)(ru_2 - t)\cdots(ru_n - t) = 0$. By the p-condition, there exists $u \in R$ such that $u^p = r$ and $p(ru - t) = 0$. We need to show that $ru = t$, which suffices by Lemma 6.1.5.

It is enough to show that this is true modulo each prime ideal Q of R. But if $p \notin Q$, then from $p(ru - t) = 0$ we immediately see that $ru = t \pmod{Q}$. On the other hand, if $p \in Q$, then $(ru)^p = r^p u^p = r^p r = r^{p+1} = t^p$. It follows that ru and t are both pth roots of t^p so $ru = t \pmod{Q}$) as pth roots are unique in R/Q. ■

6.1.13. Remark. Whenever Q is a prime ideal of R with $p \notin Q$,we have $G_{p\text{-per}}(R/Q) = R/Q$ so, if $\zeta \in \Gamma(E_p)$, then $\zeta(Q) \in R/Q$.

6.1.14. Notation. Recall the following notational conventions:

1. Let $s \in R$ and $\zeta \in \Gamma(E_p)$ be given, We say that $\zeta(Q) = s \pmod{Q}$ if $\zeta(Q)$ coincides with the image of s in R/Q.

2. Let $\nu : R \longrightarrow \Gamma(E_p)$ be the canonical embedding. We often identify R with its image in $\Gamma(E_p)$. Thus given $\zeta \in \Gamma(E_p)$ we may say $\zeta \in R$ if $\zeta = \nu(r)$ for some $r \in R$ and, in this case, write $\zeta = r$. Note that if R is DL-closed, then we have $\zeta \in R$ if and only if $\zeta(P) \in R/P$ for all prime ideals P. For if $\zeta(P) \in R/P$ for all P, then ζ can be regarded as a global section of the canonical sheaf for $\mathscr{A}_{\mathrm{dom}}$ and, by Theorem 4.1.3 this implies that $\zeta \in R$.

6.1.15. Lemma. *Let the semiprime ring R satisfy the p-condition and let $\zeta \in \Gamma(E_p)$ be given. Then:*

1. *If $\zeta^p \in R$ and $\zeta^{p+1} \in R$ then $\zeta \in R$.*

2. *If $\zeta^w \in R$ for all sufficiently large integers w, then $\zeta \in R$.*

3. *If $\zeta^k \in R$ and $\zeta^\ell \in R$ for k, ℓ relatively prime positive integers, then $\zeta \in R$.*

Proof. Note that by the previous proposition, R is DL-closed. It suffices to prove 1 as the other conditions are equivalent using the argument used in Lemma 5.1.5. Let $\zeta^p = r \in R$ and $\zeta^{p+1} = t \in R$. Then clearly $r^{p+1} = t^p$. Note that if P is a prime ideal of R and $p \notin P$, then $\mathrm{char}(R/P) \neq p$ so $G_{p\text{-per}}(R/P) = R/P$ and there exists $u_P \in R$ with $\zeta(P) = u_P$. Thus in R/P, we have $u_P^p = r$ and $u_P^{p+1} = t \pmod{P}$. Since $\zeta(Q) = u_P$ for Q in a neighbourhood of P, we can cover the compact set of all prime ideals which do not contain p with finitely many such neighbourhoods so there are $u_1, u_2, \ldots, u_n \in R$ such that for each P with $p \notin P$, there exists i with $(ru_i - t) \in P$. It follows that $p(ru_1 - t)\cdots(ru_n - t) = 0$ (by checking that this holds at each prime ideal) and so, by the

p-condition, there exists a unique $u \in R$ such that $u^p = r$ and $p(u^{p+1} - t) = 0$. Since it is readily verified that ζ satisfies these same equations (checking them modulo each prime ideal) it follows from Lemma 6.1.6, applied to the ring $\Gamma(E_p)$, that $\zeta = u$. ∎

6.1.16. Definition. *The p-**spread** of $\zeta \in \Gamma(E)$ is the least cardinal of a grounded representation, $\mathscr{U}$, of ζ on $Z(p)$.*

6.1.17. Lemma. *For any $\zeta \in \Gamma(E_p)$ and any p-cover $U_1, U_2, \dots, U_m$ of ζ, there exist an element $s \in R$ and an integer w_0 such that whenever $w \geq w_0$, the sets $(U_1 \cup U_2), U_3, \dots, U_n$ are a p-cover of $(\zeta - s)^w$.*

Proof. This follows from Theorem 3.9.5.

6.1.18. Lemma. *Assume that the semiprime ring R satisfies the p-condition and that $\zeta \in \Gamma(E_p)$. Then $\zeta \in R$ if and only if $\zeta^p \in R$.*

Proof. Clearly if $\zeta \in R$, then $\zeta^p \in R$. To prove the converse, assume that $\zeta^p \in R$ and let $r \in R$ be such that $\zeta^p = r$. We will prove that $\zeta \in R$ by induction on $n = p\text{-spread}(\zeta)$.

Suppose that $n = 1$. Then the entire compact set of all prime ideals P with $p \notin P$ is uniformly grounded. Therefore there exists $u_1 \in R$ with $\zeta(P) = u_1 \pmod P$ whenever $p \notin P$.

Let $t = ru_1$. Then, trivially, we have $p(ru_1 - t) = 0$. We claim that $p(r^{p+1} - t^p) = 0$. It suffices to show that $p(r^{p+1} - t^p) \in P$ for every prime ideal P. But if $p \in P$ this is immediate. And if $p \notin P$ we have $\zeta(P) = u_1$ and $\zeta^p(P) = r = u_1^p$ so $t = u_1 r = u_1^{p+1}$ from which it follows that, modulo P, we have $r^{p+1} = u_1^{p(p+1)} = t^p$.

In view of the claim, it follows by the p-condition that there exists $u \in R$ with $u^p = r$ and $p(u^{p+1} - t) = 0$. We next claim that $\zeta = u$. It suffices to show that $\zeta(P) = u \pmod P$ for every prime ideal P. If $p \notin P$ then $r^{p+1} = t^p$ and $ru = t$ (both mod P). It also follows from the above argument that $\zeta^p(P) = r = u^p \pmod{P}$) and that:

$$\zeta^{p+1}(P) = \zeta(P)\zeta^p(P) = ru_1 = t = ru = u^{p+1}$$

therefore, modulo P, when $p \notin P$, we have shown that $\zeta^p(P) = u^p$ and $\zeta^{p+1}(P) = u^{p+1}$ so by Proposition 5.1.2, applied to the ring $\Gamma(E_p)$, it follows that $\zeta = u \pmod P$.

On the other hand, if P is a prime ideal with $p \in P$, we see that $\zeta(P)$ and u are both pth roots of r and, modulo P, such roots are unique as $p \in P$, Therefore $\zeta = u \pmod P$.

We continue the inductive argument by assuming that for every $\zeta \in \Gamma(E_p)$, if $\zeta^p \in R$ and $p\text{-spread}(\zeta) < m$ then $\zeta \in R$. We claim that if ζ has p spread m then $\zeta \in R$. But by Lemma 6.1.17 we have that for sufficiently large w, ζ^w has p-spread less than m and since $(\zeta^w)^p$ is clearly in R, we have, by the induction hypothesis, that $\zeta^w \in R$. It follows from Lemma 6.1.15 that $\zeta \in R$. ∎

6.1.19. Lemma. *If D is a domain then for every $x \in G_{p\text{-}per}(D)$, there exists a non-negative integer k with $x^{p^k} \in D$.*

Proof. We may assume $\mathrm{char}(D) = p$, as otherwise $G_{p\text{-per}}(D) = D$. Let $D \subseteq \overline{\mathbf{Q}}(D)$ embed D into the perfect closure of its field of fractions. Obviously $G_{p\text{-per}}(D)$ is the smallest subdomain of $\overline{\mathbf{Q}}(D)$ which contains D and is closed under the forming of pth roots. It suffices to show that the set of all $x \in \overline{\mathbf{Q}}(D)$ such that $x^{p^k} \in D$ for some $k \geq 0$ is closed under addition and multiplication. But this is trivial because if $x^{p^k} \in D$ and $y^{p^\ell} \in D$, then we may assume that $k = \ell$, as we can replace both k and ℓ by the larger of the two integers. ∎

6.1.20. Theorem. *A semiprime ring is in the limit closure of all p-perfect domains if and only if it satisfies the p-condition.*

Proof. The necessity of the condition follows from Corollary 6.1.8. To prove sufficiency, assume that R satisfies the p-condition. We must prove that every $\zeta \in \Gamma(E_p)$ belongs to R, which follows if there exists n such that $\zeta^{p^n} \in R$. By the above lemma, for each prime ideal P of R, there exists k_P with $\zeta^{p^{k_P}}(P) = r_P \in R$. Then the set of prime ideals Q with $\zeta^{p^{k_P}}(Q) = r_P$ is an open subset of $\mathrm{Spec}(R)$. Cover $\mathrm{Spec}(R)$ by finitely many such open sets. Thus there exist $k_1, k_2, \ldots, k_m$ such that for every prime ideal P of R there is some i with $\zeta^{p^{k_i}}(P) \in R$. Let n be the largest k_i, whence $\zeta^{p^n} \in R$.

The theorem now follows by induction on n. If $n = 1$ then the result follows from Lemma 6.1.18. Assume that for all $\sigma \in \Gamma(E_p)$, we have $\sigma \in R$ if $\sigma^{p^n} \in R$. Suppose that $\zeta^{p^{n+1}} \in R$. Letting $\sigma = \zeta^p$ we see that $\sigma^{p^n} \in R$ and so, by the induction hypothesis, $\sigma \in R$ and therefore $\zeta \in R$ by Lemma 6.1.18. ∎

6.2. Describing $\mathscr{K}_{\mathrm{per}}$. We will show that a semiprime ring is in the limit closure of the perfect domains if and only if it is perfect, where:

6.2.1. Definition. *A semiprime ring is* **perfect** *if it satisfies the p-condition for every prime integer p.*

6.2.2. Notation. We let E denote the canonical sheaf associated with the limit closure of the category of **all** perfect domains, while E_p is the sheaf for the limit closure of the category of p-perfect domains.

For each domain D we let $G_{p\text{-per}}(D)$ denote the G-operation associated with E_p and let $G_{\mathrm{per}}(D)$ denote the operation associated with $\mathscr{K}_{\mathrm{per}}$.

Clearly, for each domain D we have $G_{\mathrm{per}}(D)$ is given by:

1. If $\mathrm{char}(D) = p > 0$, then $G_{\mathrm{per}}(D) = G_{p\text{-per}}(D)$.

2. If $\mathrm{char}(D) = 0$, then $G_{\mathrm{per}}(D) = D$.

6.2.3. Lemma. *For each prime integer p, we can regard E_p as an open subsheaf of E.*

Proof. This is obvious since we can regard $G_{p\text{-per}}(D)$ as a subset of $G_{\mathrm{per}}(D)$. The fact that each E_p is an open subsheaf of E is straightforward. ∎

6.2.4. Lemma. *With the above notation, let $\zeta \in \Gamma(E)$ be given. Then there exists a finite set S of prime integers, such that for every prime ideal Q of R with* $\operatorname{char}(R/Q) \notin S$, *we have $\zeta(Q) \in R/Q$.*

Proof. If Q is a prime ideal of R with $\operatorname{char}(R/Q) = 0$, then there exists $r \in R$ with $\zeta(Q) = r \pmod{Q}$). Since global sections agree on open sets, Q has an open neighbourhood U_Q such that for all $Q' \in U_Q$, we have $\zeta(Q') = r \pmod{Q'}$).

Clearly the set of prime ideals Q with $\operatorname{char}(R/Q) = 0$ is covered by the open sets U_Q defined above. On the other hand, the prime ideals P with $\operatorname{char}(R/P) > 0$ are covered by the open sets $\{N(p)\}$ where p varies over the set of prime integers. So $\operatorname{Spec}(R)$ is covered by the sets $\{U_Q\}$ and $\{N(p)\}$. Let $U_{Q_1}, \ldots U_{Q_k}, N(p_1), \ldots, N(p_n)$ be a finite subcover. Then $S = \{p_1, \ldots, p_n\}$ has the desired property. ∎

6.2.5. Theorem. *A semiprime ring is in $\mathscr{K}_{\text{per}}$ if and only if it satisfies the p-condition for every prime integer p.*

Proof. Let R be a semiprime ring which satisfies the p-condition for every prime integer p. We need to show that $R = \Gamma(E)$. The converse direction is clear, as stated in Corollary 6.1.8.

Let $\zeta \in \Gamma(E)$ be given and let $S = \{p_1, p_2, \ldots, p_n\}$ be the set of prime integers such that if $\operatorname{char}(R/Q) \notin S$ then $\zeta(Q) \in R/Q$. We assume that the primes $p_1, p_2, \ldots, p_n$ are distinct. For each i, we let

$$q_i = p_1 \cdots p_{i-1} p_{i+1} \cdots p_n$$

be the product of all prime integer in S other than p_i. We note that, for each i, we have $q_i\zeta$ is in $\Gamma(E_{p_i})$ as $q_i\zeta(Q) \in G_{p_i\text{-per}}(R/Q)$ for all prime ideals Q of R. But since R satisfies the p_i-condition, we see that $R = \Gamma(E_{p_i})$ so $q_i\zeta \in R$. But there exist integers $\{c_i\}$ such that $c_1q_1 + c_2q_2 + \cdots + c_nq_n = 1$ since $\gcd\{q_1, q_2, \ldots, q_n\} = 1$. Therefore $\zeta = c_1q_1\zeta + \cdots + c_nq_n\zeta$ is in R. ∎

6.2.6. Example. If p is an integer prime and R is a semiprime ring in which $p = 0$, then R satisfies the p-condition if and only if every $r \in R$ has a pth root.

Proof. Most of the conditions in the p-condition are vacuous when $p = 0$, and the only non-vacuous condition is that given $r \in R$, there exists $u \in R$ with $u^p = r$ as in the proof of 6.1.4. ∎

6.2.7. Example. If p is an integer prime and R is a DL-closed semiprime ring in which p has an inverse, then R satisfies the p-condition.

Proof. Since R is DL-closed, it is p-coherent. It is easily verified that a p-coherent ring in which p is invertible satisfies the p-condition. ∎

6.2.8. Example. If p is an integer prime and R is a DL-closed semiprime ring in which p has an quasi-inverse, then R satisfies the p-condition if and only if each prime ideal P of R with $\mathrm{char}(R/P) = p$ has pth roots.

Proof. Let $\overline{p}$ be the quasi-inverse for which $p^2\overline{p} = p$ and $\overline{p}^2 p = \overline{p}$. Then $p\overline{p}$ is an idempotent and we can write $R = R_1 \times R_2$ where $p = 0$ in R_1 and p has a (genuine) inverse in R_2 and then apply the arguments of the two examples above. ∎

6.2.9. Theorem. *The semiprime ring R is in $\mathscr{K}_{\mathrm{pfld}}$ if and only if R is regular and perfect.*

Proof. If R is in the limit closure of perfect fields then, trivially, R is in the limit closure of perfect domains and in the limit closure of fields, so R is perfect and regular.

Conversely, assume that R is perfect and regular. Let E be the sheaf associated with $\mathscr{K}_{\mathrm{pfld}}$. Then for each prime ideal P of R we see that $G_{\mathrm{pfld}}(R/P)$ is $\overline{\mathbf{Q}}(R/P)$, the perfect closure of the field of fractions of R/P. But in a regular ring every prime ideal is maximal, so $G_{\mathrm{pfld}}(R/P)$ is the perfect closure of R/P. Thus E coincides with the canonical sheaf associated with $\mathscr{K}_{\mathrm{per}}$. Since R is perfect, it follows that $R = \Gamma(E)$ which implies $R \in \mathscr{K}_{\mathrm{pfld}}$. ∎

6.2.10. Corollary. $\mathscr{K}_{\mathrm{per}} \cap \mathscr{K}_{\mathrm{fld}} = \mathscr{K}_{\mathrm{pfld}}$. ∎

6.2.11. Corollary. $\mathscr{K}_{\mathrm{icp}} \cap \mathscr{K}_{\mathrm{fld}} = \mathscr{K}_{\mathrm{pfld}}$.

Proof. Since $\mathscr{A}_{\mathrm{icp}} \subseteq \mathscr{A}_{\mathrm{per}}$, it follows that $\mathscr{K}_{\mathrm{icp}} \subseteq \mathscr{K}_{\mathrm{per}}$ so $\mathscr{K}_{\mathrm{icp}} \cap \mathscr{K}_{\mathrm{fld}} \subseteq \mathscr{K}_{\mathrm{per}} \cap \mathscr{K}_{\mathrm{fld}} = \mathscr{K}_{\mathrm{pfld}}$. The reverse inclusion $\mathscr{K}_{\mathrm{pfld}} \subseteq \mathscr{K}_{\mathrm{per}} \cap \mathscr{K}_{\mathrm{fld}}$ is easy. ∎

6.3. Is $\mathscr{K}_{\mathrm{per}} \cap \mathscr{K}_{\mathrm{ic}} = \mathscr{K}_{\mathrm{ica}}$? Having proven the above corollaries, it seems natural to ask if the analogous result would describe $\mathscr{K}_{\mathrm{ica}}$. However, there are counter-examples and we present one here.

6.3.1. Definition. *Let $\mathbf{Z}[x]$ be the ring of polynomials in the indeterminate x and let $D \subseteq \mathbf{Z}[x]$ be the subring generated by $2x$ and x^2. Evidently, D is the ring of polynomials $a_0 + a_1x + \cdots + a_nx^n$ where each a_n is even whenever n is odd.*

6.3.2. Lemma. *With the above definition, $D \in \mathscr{K}_{\mathrm{per}} \cap \mathscr{K}_{\mathrm{ic}}$.*

Proof. Clearly D perfect because it is a domain of characteristic 0.

To prove that D is in $\mathscr{K}_{\mathrm{ic}}$ observe that the following diagram is a pullback, where $\mathbf{Z}_2[x^2] \hookrightarrow \mathbf{Z}_2[x]$ is the subring of all polynomials which have only even powers of x.

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & \mathbf{Z}[x] \arrow[d] \\ \mathbf{Z}_2[x^2] \arrow[r, hook] & \mathbf{Z}_2[x] \end{tikzcd}$$

Note that $\mathbf{Z}[x]$ and $\mathbf{Z}_2[x]$ are in $\mathscr{K}_{\mathrm{ic}}$ as they are UFDs. Also, $\mathbf{Z}_2[x]$ is isomorphic to $\mathbf{Z}_2[x^2]$ by the map that sends x to x^2. So $D \in \mathscr{K}_{\mathrm{ic}}$ as $\mathscr{K}_{\mathrm{ic}}$ is closed under pullbacks. ∎

6.3.3. Lemma. *If a and b are in the semiprime ring R and if $2a = 2b$ and $a^2 = b^2$ then $a = b$.*

Proof. Square $a - b$. ∎

6.3.4. Lemma. *The embedding $D \longrightarrow \mathbf{Z}[x]$ is epic in $\mathcal{SPR}$.*

Proof. Let $R \in \mathcal{SPR}$ and $g, h : \mathbf{Z}[x] \longrightarrow R$ be given such that g and h have the same restriction to D. Then $g(2x) = h(2x)$ and $g(x^2) = h(x^2)$ and, by the above lemma, we have $g(x) = h(x)$ which implies $g = h$. ∎

6.3.5. Lemma. *Let $R \in \mathcal{SPR}$ be given. A map from $f : D \longrightarrow R$ is determined by $f(2x) = r$ and $f(x^2) = s$ if and only if $r, s \in R$ satisfy $r^2 = 4s$. Moreover, the map f extends to $\mathbf{Z}[x]$ if and only if there exists $t \in R$ for which $2t = r$ and $t^2 = s$.*

Proof. If $f(2x) = r$ and $f(x^2) = s$ then it is clear that $r^2 = 4s$. Conversely, given such an $r, s \in R$, we define a map $f : D \longrightarrow R$ as follows. Each polynomial in D can be written uniquely in the form $(p, q) = p(x^2) + 2xq(x^2)$. The rules for adding and multiplying these canonical forms are: $(p_1, q_1) + (p_2, q_2) = (p_1 + p_2, q_1 + q_2)$ and $(p_1, q_1)(p_2, q_2) = (p_1p_2 + 4xq_1q_2, p_1q_2 + p_2q_1)$. It is clear that we can define $f(p, q) = p(s) + rq(s)$ which is a homomorphism if and only if $r^2 = 4s$.

Finally, if $t \in R$ satisfies $2t = r$ and $t^2 = s$, then f can be extended to a homomorphism $h : \mathbf{Z}[x] \longrightarrow R$ for which $h(x) = t$. ∎

6.3.6. Proposition. *The inclusion map $D \hookrightarrow \mathbf{Z}[x]$ is the reflection of D into $\mathscr{K}_{\mathrm{ica}}$.*

Proof. Since $\mathbf{Z}[x]$ is integrally closed and perfect it belongs to $\mathscr{K}_{\mathrm{icp}} = \mathscr{K}_{\mathrm{ica}}$. It suffices to show that the inclusion $D \hookrightarrow \mathbf{Z}[x]$ has the UEP (unique extension property) with respect to every $A \in \mathcal{A}_{\mathrm{icp}}$. In view of the above lemma, this boils down to showing that if $A \in \mathcal{A}_{\mathrm{icp}}$ and if $r, s \in A$ are such that $r^2 = 4s$ then there exists $t \in A$ such that $2t = r$ and $t^2 = s$. But suppose $\mathrm{char}(A) \neq 2$. Then the fraction $r/2$ is a square root of s so it must be in A, which is integrally closed, and we can choose $t = r/2$. On the other hand, if $\mathrm{char}(A) = 2$, then we must have $r = 0$ as $r^2 = 4s$. But since A is perfect, we see that s has a unique square root $t \in A$. And obviously $t^2 = s$ and $2t = 0 = r$. ∎

6.3.7. Corollary. *$D \in \mathscr{K}_{\mathrm{per}} \cap \mathscr{K}_{\mathrm{ic}}$ but $D \notin \mathscr{K}_{\mathrm{ica}}$.*

Proof. That $D \in \mathscr{K}_{\mathrm{per}} \cap \mathscr{K}_{\mathrm{ic}}$ is proven in 6.3.2. But $D \notin \mathscr{K}_{\mathrm{ica}}$ because the reflection of D into $\mathscr{K}_{\mathrm{ica}}$ is not an isomorphism. ∎

7. Limit closure of UFDs.

7.1. The reflector. In the preceding three sections we have characterized the limit closure of domains, of integrally closed domains, and of perfect domains in the category of commutative rings. In each case, the limit closure turned out to be the models of an essentially algebraic (or left exact) theory. When examining the limit closure of UFDs, we are surprised not only by how many domains are in it (for example every quadratic extension of $\mathbf{Z}$) but also by which domains are not in it (for example every non-principal ultraproduct of $\mathbf{Z}$). It is true that the limit closure of UFDs is contained in the (2,3)-rings because all UFDs are integrally closed (alternatively, if $b^3 = c^2$, we can directly find the prime factors of the element a for which $a^2 = b$ and $a^3 = c$).

We can dispose of the latter point by observing that if an element of a UFD has a square root then every prime divides it an even number of times. The converse is not necessarily true but if every prime divides an element an even number of times, then some associate of the element will be a square. For example, in the Gaussian integers $\mathbf{Z}[i]$ the element i is divisible by each prime an even number of times (0), but does not have a square root. Obviously the associate 1 does. Thus to show that a UFD is (2,3)-closed, it suffices to show that invertible elements do. But if $u^3 = v^2$ and u is invertible, then it is immediate that $(v/u)^2 = u$.

Next consider the domain $D_0 \subseteq k[x]$, with k a field, consisting all the polynomials $a_0 + a_2x^2 + a_3x^3 + \cdots$, that is with linear term 0. Then clearly x^6 has a square root in D, while x^2 doesn't. It can be shown that $D_0 \cong k[y,z]/(y^3 - z^2)$ from which we see that a necessary and sufficient condition that a domain D be a (2,3)-domain is that every map from $D_0 \longrightarrow D$ extend to a map $k[x] \longrightarrow D$.

We denote the limit closure of the UFDs by $\mathscr{K}_{\text{ufd}}$. We have,

7.1.1. Theorem. *The inclusion $\mathscr{K}_{\text{ufd}} \longrightarrow \mathcal{SPR}$ has a left adjoint.*

Proof. Apply Theorem 2.2.12. ∎

7.2. Quadratic extensions of $\mathbf{Z}$. We fix a square-free integer $n \in \mathbf{Z}$. Note that n can be positive or negative. We let $\omega = \sqrt{n}$. When $n \equiv 1 \pmod 4$, the ring $\mathbf{Z}[\omega]$ is not integrally closed; its integral closure is $\mathbf{Z}[\tau]$ for $\tau = (1+\omega)/2$.

It is known that for $n < 0$ there are only nine prime values of n, the largest being 163, for which $\mathbf{Z}[\omega]$ ($\mathbf{Z}[\tau]$ in the case $n \equiv 1 \pmod 4$) is a UFD. For $n > 0$, the ring seems to be a UFD infinitely often and a non-UFD infinitely often (the truths of these claims are not known).

In this section we will show, nonetheless that $\mathbf{Z}[\omega]$ is always in the limit closure of the UFDs.

For p prime, let $\mathbf{Q}_p \subseteq \mathbf{Q}$ consist of all rationals whose denominators are not divisible by p. It is clear that $\mathbf{Z}[\omega] = \bigcap_p \mathbf{Q}_p[\omega]$. We will show that except when $p = 2$ and $n \equiv 1 \pmod 4$, $\mathbf{Q}_p[\omega]$ is itself a UFD. In the missing case, we will show that $\mathbf{Q}_2[\tau]$ is a UFD, and that $\mathbf{Q}_2[\omega]$ is the equalizer of two maps from $\mathbf{Q}_2[\tau]$ to a field and hence also in the limit closure of UFDs.

We will therefore be studying the rings of the form $\mathbf{Q}_p[\theta]$ where $\theta = \omega$, unless $p = 2$ and $n \equiv 1 \pmod 4$, in which case $\theta = \tau$.

An element of $\mathbf{Q}_p[\omega]$ will be written $a + b\omega$ with $a, b \in \mathbf{Q}_p$. There is an automorphism on $\mathbf{Q}_p[\omega]$ that takes ω to $\overline{\omega} = -\omega$. Applied to $\mathbf{Q}_2[\tau]$, it takes τ to $\overline{\tau} = (1-\omega)/2 = 1 - \tau$. In either case the map that takes $a + b\omega$ to $a + b\overline{\omega}$ (respectively takes $a + b\tau$ to $a + \overline{\tau}$) is an automorphism and therefore the map $\mathcal{N} : \mathbf{Q}_p[\tau] \longrightarrow \mathbf{Q}_p$ defined by $\mathcal{N}(a + b\tau) = (a + b\tau)(a + b\overline{\tau})$ is multiplicative. The function $\mathcal{N}$ is called the **norm function**.

An element of $a \in \mathbf{Q}_p$ can be written $a = p^e b$ where e is a non-negative integer and b is invertible in $\mathbf{Q}_p$. We will call e the p-index of a and write $e = \mathrm{ind}_p(a)$. We will frequently omit the p when it is clear.

7.2.1. Remark. A statement such as "a divides b" implicitly refers to a ring R which contains a, b (and means that there exists $c \in R$ with $b = ca$). Unless we specifically mention otherwise, this will be interpreted to mean the smallest ring containing a and b. In particular, if a and b are both integers, then, unless a different ring is clearly indicated, the statement "a divides b" will be interpreted with respect to the ring of integers.

7.2.2. Proposition. *An element $\xi \in \mathbf{Q}_p[\theta]$ is invertible if and only $p \not| \mathcal{N}(\xi)$.*

Proof. If $\xi\xi^{-1} = 1$, then $\mathcal{N}(\xi)\mathcal{N}(\xi^{-1}) = \mathcal{N}(1) = 1$ so $\mathcal{N}(\xi)$ must be invertible, that is not divisible by p. Conversely, if $\mathcal{N}(\xi)$ is invertible, then $\xi \mid \mathcal{N}(\xi)$ is also invertible. ∎

The only ideals in $\mathbf{Q}_p$ are powers of p and so it makes sense to speak of two elements being congruent $\pmod{p^e}$. In particular, it makes sense in $\mathbf{Q}_2$ to talk of odd and even and of elements of being divisible by 4 or 8, etc.

7.2.3. Theorem. *$\mathbf{Q}_p[\theta]$ is a UFD for all primes p.*

This will be divided into seven cases. The treatments of four of those cases will be subsumed into the following proposition.

7.2.4. Proposition. *Suppose there is an element $\pi \in \mathbf{Q}_p[\theta]$ such that $p \mid \mathcal{N}(\pi)$ and for any $\xi \in \mathbf{Q}_p[\theta]$ whenever $p \mid \mathcal{N}(\xi)$ then $\pi \mid \xi$. Then π is prime in $\mathbf{Q}_p[\theta]$, every element of $\mathbf{Q}_p[\theta]$ is a power of π times an invertible element and therefore $\mathbf{Q}_p[\theta]$ is a UFD.*

Proof. If $\pi \mid \xi\eta$, then $p \mid \mathcal{N}(\pi) \mid \mathcal{N}(\xi)\mathcal{N}(\eta)$ and hence p must divide one the factors on the right and hence π must divide ξ or η so we see that π is prime.

Now suppose $\xi \in \mathbf{Q}_p[\theta]$. If ξ is not invertible, then $p \mid \mathcal{N}(\xi)$, whence $\pi \mid \xi$. Dividing out one factor of π divides the norm by $\mathcal{N}(\pi)$ so this process terminates in an invertible element after a finite number of steps. ∎

We now finish the proof of Theorem 7.2.3. The proof is divided into cases that depend on n and p.

Case that $p \mid n$: By Proposition 7.2.2, $a + b\omega$ is invertible in $\mathbf{Q}_p[\omega]$ if and only if $p \not| \mathcal{N}(a + b\omega) = a^2 - b^2 n$, which happens if and only if $p \not| a$. If $p \mid a$, the equations $(x + y\omega)\omega = a + b\omega$ leads to the equations $ny = a$ and $x = b$. Since n is square free, it is divisible by p but not by p^2 and hence $y = a/n \in \mathbf{Q}_p$. Thus every non-invertible element is divisible by ω. Since $p|\mathcal{N}(\omega) = -n$ the result follows from Proposition 7.2.4 with $\pi = \omega$.

Case that $p = 2$, and $n \equiv 3 \pmod{4}$: By Proposition 7.2.2, $a + b\omega \in \mathbf{Q}_2[\omega]$ is invertible if and only if $\mathcal{N}(a + b\omega)$ is odd if and only if a and b have opposite parity.

We claim that every non-invertible element of $\mathbf{Q}_2[\omega]$ is divisible by $1 + \omega$. If $a + b\omega$ is non-invertible, then a, b have the same parity and $a + b\omega = (1 + \omega)(x + y\omega)$ where $x = (bn - a)/(n - 1)$ and $y = (a - b)/(n - 1)$. Note that $n - 1 \equiv 2 \pmod{4}$ so that it is divisible by 2 but not by 4, hence both $x, y \in \mathbf{Q}_2$. Proposition 7.2.4 with $\pi = 1 - \omega$ gives the result.

Case that $p = 2$, $n \equiv 1 \pmod 4$: Let $m = (n-1)/4$. Then $\tau = (1+\sqrt{n})/2$ then $\mathcal{N}(\tau) = -m$. This case splits into three subcases depending on the residue of $m \pmod 4$.

Subcase that m is odd: In this case, it is clear that $\mathcal{N}(a + b\tau)$ is odd unless a and b are both even, whence $a + b\tau$ is divisible by 2. Thus an element is either invertible or divisible by 2 and Proposition 7.2.4 with $\pi = 2$ applies.

Subcase that $m \equiv 2 \pmod 4$: We begin by claiming that $\tau | 2$. Write $n = 8k + 1$ with k odd. Then we have that $\tau\overline{\tau} = -2k$ and k is odd so that 2 is divisible by both τ and $]tau$. Since $\mathcal{N}(a + b\tau) = a^2 - mb^2 + ab \equiv a^2 + ab \equiv a(a + b) \pmod 2$, we see that $a + b\tau$ is invertible if and only if a is odd and b is even. We further claim that if $a + b\tau$ is not invertible, then it is divisible by either τ or $\overline{\tau}$. If a and b are both even, then $\tau \mid 2 \mid a + b\tau$. If a is even and b is odd, then the equation $(x + y\tau)\tau = a + b\tau$ has the solution $y = -a/m$ and $y = b - m$. Note that $m \mid a$ since a is even and m is twice an odd number. If a and b are both odd, then $a + b\tau = a + b(1 - \overline{\tau}) = a + b - b\overline{\tau}$ and $a + b$ is even, b odd. Since the replacement of τ by $\overline{\tau}$ is an automorphism, the same argument shows that $\overline{\tau} \mid a + b + b\overline{\tau} = a + b\tau$.

We claim that τ and $\overline{\tau}$ are prime. We do this for τ. If $\tau \not| a + b\tau$, then a must be odd. Similarly, if $\tau \not| c + d\tau$, then $(a + b\tau)(c + d\tau) = ac + bdm + (ad + bc + bd)\tau$ and certainly $ac + bdm$ is odd and so τ does not divide the product. Since every division of an element by τ or $\overline{\tau}$ reduces the index of that element (the number of times it is divisible by 2) by 1, we conclude that every element is a product of a power of τ, a power of $\overline{\tau}$, and an invertible element.

Subcase that $m = (n-1)/4 \equiv 0 \pmod 4$: Write $n = 16k + 1$. Then $(1 + \tau)(1 + \overline{\tau}) = 1 + \tau + \overline{\tau} + \tau\overline{\tau} = 2(-2k + 1)$, which is twice an odd number so that both τ and $\overline{\tau}$ divide 2. As in the preceding subcase, we see that $a + b\tau$ is invertible if and only a is odd and b is even. If a and b are both even, then $a + b\tau$ is divisible by both $1 + \tau$ and $1 + \overline{\tau}$. Next we note that $\tau^2 = (1 + n + 2\sqrt{n})/4 = 4k + \tau$. Then

$$\frac{\tau}{1+\overline{\tau}} = \frac{\tau(1+\tau)}{(1+\tau)(1+\overline{\tau})} = \frac{\tau+\tau^2}{2-4k} = \frac{4k+2\tau}{2-4k} = \frac{2k+\tau}{1-2k}$$

and the denominator is odd, so that $1 + \overline{\tau} | \tau$. Now if a is even and b is odd then $\overline{\tau}$ divides $a + (b + 1)\tau$, as well as τ, so it divides $a + b\tau$. Finally, if a and b are both odd, then $1 + \tau$ divides $(a + 1) + (b + 1)\tau$ as well as $1 + \tau$ and hence divides $a + b\tau$. Thus every non-invertible element is divisible by at least one of $1 + \tau$ and $1 + \overline{\tau}$.

Note that when a is even and b is odd, then $(1 + \tau) \mid (a - 1 + b\tau)$ and does not divide 1 so that $(1 + \tau \not| a + b\tau)$ in that case. If also $(1 + \tau) \not| (c + d\tau)$, then

$$\begin{aligned}(a + b\tau)(c + d\tau) &= ac + (ad + bc)\tau + bd\tau^2 = ac + (ad + bc)\tau + bd(4k + \tau)\\ &= ac + 4kbd + (ad + bc + bd)\tau\end{aligned}$$

and that is also not divisible by $1+\tau$. Thus $1+\tau$ is prime and by symmetry, so is $1+\overline{\tau}$. The rest is routine.

Case that $p \neq 2$, $p \nmid n$ and n is a quadratic non-residue (QNR) of p: We begin by noting that if n is a QNR of p in $\mathbf{Z}$ it is also a QNR in $\mathbf{Q}_p$. For if there are $s, t \in \mathbf{Z}$ such that $p \nmid t$ and $(s/t)^2 \equiv n \pmod{p}$ and we choose $r \in \mathbf{Z}$ such that $rt \equiv 1 \pmod{p}$, then one readily sees that $(rs)^2 \equiv n \pmod{p}$.

If $\mathcal{N}(a+b\omega) = a^2 - b^2 n$ is divisible by p, then we must have $p \mid b$ and therefore $p \mid a$. Then $p \mid a+b\omega$. Thus Proposition 7.2.4 applies.

Case that $p \neq 2$, $p \nmid n$ and n is a quadratic residue (QR) of p: In this case, we can find a $k \in \mathbf{Z}$ so that $p \mid k^2 - n$. Replace k by $k+p$, if necessary, to force that $p^2 \nmid k^2 - n$. Now suppose that $p \mid \mathcal{N}(a+b\omega) = a^2 - b^2 n$. We have $0 \equiv a^2 - b^2 n \equiv a^2 - b^2 k^2 \equiv (a-bk)(a+bk) \pmod{p}$ and so $p \mid a+bk$ or $p \mid a-bk$. Choose $u = \pm 1$ so that $p \mid a - ubk$. We claim that $k+u\omega \mid a+b\omega$. In fact the equation $a+b\omega = (k+u\omega)(x+y\omega) = kx+uny+(ux+ky)\omega$ gives the equations $kx+uny = a$ and $ux+ky = b$ which have the solution $y = (a-ubk)/(n-k^2)$ and $x = (a-uny)/k$ which lie in $\mathbf{Q}_p$ since $p \mid a - ubk$ and $p^2 \nmid n - k^2$.

We can also conclude from this computation that $k+u\omega \nmid a+b\omega$ when $p \nmid a - ubk$. Suppose now that $k+u\omega \nmid a+b\omega$ and $k+u\omega \nmid c+d\omega$. Then we claim that $p \nmid (a+b\omega)(c+d\omega) = ac+bdn+(ad+bc)\omega$. In fact $ac+bdn-u(ad+bc)k \equiv ac-bdk^2-uadk-ubck \equiv (a-ubk)(c-udk) \not\equiv 0 \pmod{p}$. This proves that $k \pm \omega$ are the only primes and completes the proof for the last case. ∎

7.2.5. Theorem. *If $n \in \mathbf{Z}$ is square-free, then $\mathbf{Z}[\sqrt{n}]$ is in the limit closure of the UFDs.*

Proof. Let $\omega = \sqrt{n}$ as above. Clearly $\mathbf{Z}[\omega] = \bigcap \mathbf{Q}_p[\omega]$ the intersection taken over all primes. We know that when p is an odd prime, $\mathbf{Q}_p[\omega]$ is itself a UFD. In addition, $\mathbf{Q}_2[\omega]$ is a UFD if $n \equiv 3 \pmod{4}$. Thus it is sufficient to show that $\mathbf{Q}_2[\omega]$ is in the limit closure of the UFDs when $n \equiv 1 \pmod{4}$. In that case, we know that $\tau = (1+\omega)/2$ satisfies the equation $x^2 + x + m$ for $m = (n-1)/4$. Let R be the splitting field of x^2+x+m over the field of two elements. If m is even, $R = \mathbf{Z}/2\mathbf{Z}$ and the two roots are 0 and 1. If m is odd, the roots are the two new cube roots of 1 in the field of four elements. In either case, there are two maps $f, g : \mathbf{Q}_2[\tau] \longrightarrow R$, that take τ to one or the other root of the equation. Since $1+\omega = 2\tau$ is even, we have $f(1+\omega) = 0$, and $g(1+\omega) = 0$ so that $f(\omega) = g(\omega) = 1$ and so $Q_2[\omega]$ is the equalizer of f and g. ∎

8. Conclusions and open questions

The following summarizes the main results of this paper.

1. A semiprime ring is in $\mathscr{K}_{\mathrm{dom}}$ if and only if it is DL-closed.

2. A semiprime ring is in $\mathscr{K}_{\mathrm{fld}}$ if and only if it is regular.

3. A semiprime ring is in $\mathscr{K}_{\mathrm{qrat}}$ if and only if it is quasi-rational and DL-closed. Moreover, for some rings, the $\mathcal{A}_{\mathrm{qrat}}$-topology is strictly between the domain and patch topologies.
4. A semiprime ring is in $\mathscr{K}_{\mathrm{per}}$ if and only if it is perfect.
5. A semiprime ring is in $\mathscr{K}_{\mathrm{ic}}$ if and only if it is (2,3)-closed.
6. Every ring in $\mathscr{K}_{\mathrm{ica}}$ is perfect and (2,3)-closed, but there are perfect, semiprime, (2,3)-closed rings which are not in $\mathscr{K}_{\mathrm{ica}}$.
7. A semiprime ring is in $\mathscr{K}_{\mathrm{pfld}}$ if and only if it is perfect and regular.
8. In all of the above examples, the canonical sheaf representation property holds.

Open Questions. The following questions remain open:

1. Is there a concise description of $\mathscr{K}_{\mathrm{ica}}$? (See 6.3.1.)
2. If the conditions in 3.4.1 are satisfied, does the canonical sheaf representation property, 3.5, always hold?
3. If $\mathcal{A}$, $\mathscr{B}$, $\mathscr{K}$ satisfy the conditions in 3.4.1, must $\mathscr{K}$ be first order?
4. If $\mathscr{K}$ is a $\mathcal{D}om$-invariant subcategory, $R \in \mathscr{K}$ and $S \subseteq R$ a multiplicatively closed subset of non-zero divisors, does $S^{-1}R \in \mathscr{K}$? Compare 4.3.2 and 5.3.4.

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