A Functorial Excursion Between Algebraic Geometry and Linear Logic

Abstract—**The language of Algebraic Geometry combines two complementary and dependent levels of discourse: on the geometric side, schemes define spaces of the same cohesive nature as manifolds ; on the vectorial side, every scheme X comes equipped with a symmetric monoidal category of quasicoherent modules, which may be seen as generalised vector bundles on the scheme X. In this paper, we use the functor of points approach to Algebraic Geometry developed by Grothendieck in the 1970s to establish that every covariant presheaf X on the category of commutative rings — and in particular every scheme X — comes equipped "above it" with a symmetric monoidal closed category PshMod_X of presheaves of modules. This category PshMod_X defines moreover a model of intuitionistic linear logic, whose exponential modality is obtained by glueing together in an appropriate way the Sweedler dual construction on ring algebras. The purpose of this work is to establish on firm mathematical foundations the idea that linear logic should be understood as a logic of generalised vector bundles, in the same way as dependent type theory is understood today as a logic of spaces up to homotopy.**

I. INTRODUCTION

The first calculus ever designed in human history is probably elementary arithmetic with addition, subtraction and multiplication. Beautiful but still somewhat rudimentary, the core calculus becomes much more intricate and challenging when one extends it with division. The critical novelty of division with respect to the other operations is that, indeed, it is not a total function, because one needs to check that the denominator y is not equal to zero before performing the fraction x/y of a value x by the value y. Understood with programming languages in mind, elementary arithmetic extended with division thus provides the basic example of a language admitting "syntax errors" and meaningless expressions such as $3/0$ or $0/0$. In an extraordinarily fruitful insight, Descartes understood that elementary arithmetic is intrinsically related to geometry, and that every system of polynomial equations (constructed with addition, subtraction and multiplication) describes an **algebraic variety** defined as the set of solutions of the system of equations. Typically, the circle $\mathscr{C}$ of radius 1 with center positioned at the origin may be described as the set of coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}$ in the cartesian plane, satisfying the well-known quadric equation $x^2 + y^2 = 1$. Hence, the domain of definition $\mathscr{D}$ of the two-variable polynomial f with division

$$f \; : \; (x, y) \mapsto \tfrac{1}{x^2+y^2-1} \quad : \quad \mathscr{D} \longrightarrow \mathbb{R} \qquad (1)$$

is precisely defined as the complement set $\mathscr{D} = \mathbb{R}^2 \backslash \mathscr{C}$ of the circle $\mathscr{C}$ in the plane $\mathbb{R}^2$. It should be noted that the algebraic curve $\mathscr{C}$ defines a closed set in the usual euclidian topology as well as in the Zariski topology, and that the domain $\mathscr{D}$ of definition thus defines an open set in both topologies. There lies a basic lesson and important principle of continuity: every time an operation such as f obtained by dividing two polynomials can be performed on a given point x of the space, there exists a "sufficiently small" neighborhood U of the point x such that the operation f can be performed on every point $y \in U$ of that neighborhood.

a) Basic Principles of Algebraic Geometry: A number of exceptionally talented mathematicians were able to turn, generations after generations, the study of this basic and primitive calculus of polynomials with division into a sophisticated and flourishing field of investigation — called Algebraic Geometry — at the converging point of algebra, geometry and logic, see for instance [5], [7], [18], [8], [21]. Thanks to the visionary ideas by Grothendieck, who played a defining role in this specific turn, the prevailing point of view of Algebraic Geometry today is *operational* and *functorial* at the same time. Operational, because the notion of geometric space elaborated in the theory is not primary but secondary, and derived from the intuition that every space X should define a topological space (X, Ω_X) equipped for every open set U with a set $\mathscr{O}_X(U)$ of local operations, called **regular functions**. These regular functions are typically defined as polynomials with division, in the same fashion as (1). This set $\mathscr{O}_X(U)$ of regular functions is closed under addition, subtraction and multiplication, and thus defines a commutative ring. Functorial, because every space in Algebraic Geometry gives rise to a **ringed space** [10], [13] defined as a topological space (X, Ω_X) equipped with a contravariant functor

$$\mathscr{O}_X \; : \; \Omega_X^{op} \longrightarrow \mathbf{Ring}$$

from the category Ω_X of open sets of X ordered by inclusion to the category $\mathbf{Ring}$ of commutative rings. The purpose of the functorial action of $\mathscr{O}_X$ is to describe the **restriction functions**

$$\mathscr{O}_X(V \subseteq U) \; : \; \mathscr{O}_X(U) \longrightarrow \mathscr{O}_X(V).$$

which take an operation $f \in \mathscr{O}_X(U)$ defined on an open set U and restrict it to an operation

$$f|_V \; := \; \mathscr{O}_X(V \subseteq U)(f)$$

on an open subset $V \subseteq U$. This operational and functorial (rather than directly geometric) understanding of the notion of

space is supported by the fundamental observation that every commutative ring R defines a ringed space

$$\mathrm{Spec}\,R \quad = \quad (\mathrm{Spec}\,R, \mathscr{O}_{\mathrm{Spec}\,R}) \tag{2}$$

on the set of **prime ideals** $\mathfrak{p}$ of the commutative ring R. This ringed space $\mathrm{Spec}\,R$ is called the **affine scheme** associated to the commutative ring R. Every space X of the theory, called a **scheme**, is then defined as a patchwork of affine schemes $\mathrm{Spec}\,R_i$ carefully glued together using the functorial language of presheaf and sheaf theory.

b) Modules and Vector Bundles: One distinctive feature of Algebraic Geometry is that every scheme X comes equipped with a symmetric monoidal category $\mathbf{qcMod}_X$ of **quasicoherent sheaf of $\mathscr{O}_X$-modules**. By a presheaf of $\mathscr{O}_X$-modules M of a ringed space $(X, \mathscr{O}_X)$, one simply means a contravariant functor

$$M \quad : \quad \Omega_X^{op} \longrightarrow \mathbf{Ab} \tag{3}$$

to the category $\mathbf{Ab}$ of abelian groups, and such that each abelian group $M(U)$ defines a $\mathscr{O}_X(U)$-module in the usual algebraic sense. An important result of Algebraic Geometry called the **Serre-Swann theorem** states that a **vector bundle** on a suitable ringed space $(X, \mathscr{O}_X)$ can be equivalently encoded as its **sheaf of sections**, which defines a locally free and projective sheaf of $\mathscr{O}_X$-modules, see [30], [25]. Accordingly, a sheaf of $\mathscr{O}_X$-modules is called **quasicoherent** when it is locally presentable – in the expected sense that it is locally the cokernel of a morphism of free modules. The category $\mathbf{qcMod}_X$ extends the category of vector bundles in order to define an **abelian category** where **kernels** and **direct images** can be computed. Consequently, a sheaf of $\mathscr{O}_S$-modules A over a scheme S should be understood as some kind of very liberal notion of vector bundle over S.

This leads us to the foundational dichotomy between schemes and sheaves of modules which lies at the heart of contemporary Algebraic Geometry:

manifolds	$\sim$	schemes
vector bundles	$\sim$	sheaves of modules

c) Linear logic: Our main purpose in the paper is to associate to every scheme X a specific model of intuitionistic linear logic, where formulas are interpreted as generalised vector bundles, and where proofs are interpreted as suitable vector fields. In order to achieve that aim, we will construct a model of intuitionistic linear logic, which we find convenient to formulate directly as a **linear-non-linear adjunction**. Recall that in this formulation of the categorical semantics of intuitionistic linear logic, the formulas of the logic are interpreted as the objects of a symmetric monoidal closed category $(\mathbf{Linear}, \otimes, \multimap, I)$ equipped with an adjunction

$$\mathbf{Multiple} \underset{\mathbf{Lin}}{\overset{\mathbf{Exp}}{\underset{\top}{\rightleftarrows}}} \mathbf{Linear} \tag{4}$$

where (1) the category $\mathbf{Multiple}$ has finite products with cartesian product $\times$ and terminal object 1 and (2) the functor $\mathbf{Linear}$ is symmetric monoidal (in the strong sense) from $(\mathbf{Multiple}, \times, 1)$ to $(\mathbf{Linear}, \otimes, I)$. The resulting comonad

$$! \quad := \quad \mathbf{Lin} \circ \mathbf{Exp} \tag{5}$$

on the category $\mathbf{Linear}$ interprets the exponential modality of linear logic, whose effect is to relax the linearity constraint on the formulas of the form $!A$ which may be thus duplicated and erased. Note that the category $\mathbf{Multiple}$ itself is not required to be cartesian *closed*. The reason is that for every object K of the category $\mathbf{Multiple}$ and for every object L of the category $\mathbf{Linear}$, the object $K \Rightarrow L$ defined in the category $\mathbf{Linear}$ in the following way:

$$K \Rightarrow L \quad := \quad \mathbf{Lin}(K) \multimap L$$

comes equipped with a natural bijection:

$$\mathbf{Multiple}(K \times K', \mathbf{Exp}(L)) \quad \cong \quad \mathbf{Multiple}(K', \mathbf{Exp}(K \Rightarrow L)).$$

This form of currification is sufficient to interpret the simply-typed λ-calculus, in the category $\mathbf{Multiple}$, using the hierarchy of types of the form $\mathbf{Exp}(L)$ for L in the category $\mathbf{Linear}$, see [2], [22] for details. The reader will notice here important and fascinating connections with the properties of monoidal adjunctions in Algebraic Geometry, see for instance [19], [9].

d) The functor of points approach: In order to associate a model of linear logic (4) to every scheme X, one thus needs to define a symmetric monoidal closed category $\mathbf{Linear}$. In order to keep the construction simple, and to avoid using sheafification [18] and quasi-coherators [15], [17], [33], we take the radical step to define $\mathbf{Linear}$ as the category $\mathbf{PshMod}_X$ of presheaves of $\mathscr{O}_X$-modules, and module homomorphisms between them. The category

$$\mathbf{Linear} \quad = \quad \mathbf{PshMod}_X$$

is symmetric monoidal closed [12] and thus comes with a tensor product $\otimes_X$, a tensorial unit $\mathscr{O}_X$ (the structure sheaf) and an internal hom $\multimap_X$. Following a well-established tradition in linear logic, we then define $\mathbf{Multiple}$ as the cartesian category

$$\mathbf{Multiple} \quad = \quad \mathbf{PshCoAlg}_X$$

of **commutative comonoids** in the category $\mathbf{PshMod}_X$, which are called **commutative coalgebras** in that context. In order to work in a convenient setting, we will make great usage of the **functor of points approach** to Algebraic Geometry, developed by Grothendieck and his school in the 1970s [4], [12], [17]. This approach is based on the observation that the Spec construction (2) defines a functor

$$\mathrm{Spec} \quad : \quad \mathbf{Ring}^{op} \to \mathbf{Scheme} \tag{6}$$

into the category $\mathbf{Scheme}$ of schemes. The functor (6) induces in turn a **nerve functor**

$$\mathrm{nerve} \quad : \quad \mathbf{Scheme} \longrightarrow [\mathbf{Ring}, \mathbf{Set}]$$

which associates to every commutative ring R the **set of R-points in the scheme X**, simply defined as

$$\mathbf{Scheme}(\operatorname{Spec} R, X)$$

The nerve functor is fully faithful and enables one to see every scheme X as a specific covariant presheaf

$$\mathbf{nerve}(X) \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

over the category **Ring** of commutative rings. We find convenient to work in that functorial setting, and to extend our inquiry from schemes to **Ring-spaces**, defined as covariant presheaves over the category **Ring** of commutative rings. So, to every such **Ring**-space X, we will associate a specific model of intuitionistic linear logic, where formulas are interpreted in $\mathbf{PshMod}_X$ as presheaves of $\mathscr{O}_X$-module, understood as generalised vector bundles on the **Ring**-space X. The construction of the linear-non-linear adjunction (4) and of the exponential modality (5) is performed using the Sweedler dual construction, which reveals a fascinating duality between algebras and coalgebras [14], [1], [29], [26], [3].

e) Plan of the paper: We start by introducing the category **Ring** of commutative rings in §II and the category $\mathbf{Mod}_R$ of R-modules in §III. The categories **Mod** and $\mathbf{Mod}^\oplus$ of modules are formulated in §IV and §V. We then equip in §VI every category $\mathbf{Mod}_R$ with a tensor product noted $\otimes_R$, and deduce in §VII that **Mod** is symmetric monoidal as a ringed category. We then construct in §VIII and §IX the categories **Alg** and **CoAlg** of commutative algebras and coalgebras. The cofree commutative comonoid computed by Sweedler double dual is described in §X. We then introduce the functor of points approach in §XI and express in §XII the notion of presheaf modules in that language. The categories **PshMod** and $\mathbf{PshMod}^\oplus$ of presheaves of modules are introduced in §XIII and §XIV. Our main technical result comes in §XV with the observation that **PshMod** is symmetric monoidal closed above the cartesian closed category $[\mathbf{Ring}, \mathbf{Set}]$. From this, we deduce in §XVI that the category $\mathbf{PshMod}_X$ is symmetric monoidal closed. After introducing in §XVII the category **PshCoAlg** of presheaves of commutative coalgebras, we construct the linear-non-linear adjunction in §XVIII and conclude in §XIX.

II. THE CATEGORY **Ring** OF COMMUTATIVE RINGS

We suppose given a symmetric monoidal category $(\mathscr{A}, \otimes, 1)$ with reflexive coequalizers, preserved by the tensor product on each component. The basic example we have in mind is the category $\mathscr{A} = \mathbf{Ab}$ of abelian groups and group homomorphisms between them. For that reason, we choose to use the terminology of Algebraic Geometry, and define a **ring** as a **monoid object** (R, m, e) in the monoidal category $(\mathscr{A}, \otimes, 1)$. In other words, a ring (R, m, e) is a triple consisting of an

object R of the category $\mathscr{A}$ and two maps $m : R \otimes R \to R$ and $e : 1 \to R$ making the diagrams below commute:

A **commutative ring** is a ring (R, m, e) such that the diagram below commutes

where $\gamma_{A,B} : A \otimes B \to B \otimes A$ denotes the symmetry map at instance A, B of the symmetric monoidal category $\mathscr{A}$. Note that a commutative ring is just the same thing as a **commutative monoid object** (R, m, e) in the symmetric monoidal category $\mathscr{A}$. Given two rings R and S, a **ring homorphism**

$$u \quad : \quad (R, m_R, e_R) \longrightarrow (S, m_S, e_S)$$

is a map $u : R \to S$ of the category $\mathscr{A}$, making the diagrams below commute:

The category **Ring** has the commutative rings as objects, and the ring homomorphisms $u : R \to S$ between them as maps. It is worth mentioning that the category **Ring** has finite sums, defined by the tensor product $R, S \mapsto R \otimes S$ of the underlying category $\mathscr{A}$, together with the initial object defined as the monoidal unit 1 seen as a commutative monoid.

III. THE CATEGORY $\mathbf{Mod}_R$ OF R-MODULES

Given a commutative ring R defined in §II as a commutative monoid object in the category $\mathscr{A}$, an R**-module** (M, act) is a pair consisting of an object M and of a map

$$\operatorname{act} : R \otimes M \to M$$

in the category $\mathscr{A}$, making the diagrams below commute:

Equivalently, an R-module is an Eilenberg-Moore algebra for the monad $A \mapsto R \otimes A$ associated to the commutative ring R in the category $\mathscr{A}$. A R**-module homomorphism** $f : M \to N$

between two R-modules M and N is a map $f : M \to N$ making the diagram below commute:

$$
\begin{array}{ccc}
R \otimes M & \xrightarrow{\ R \otimes f\ } & R \otimes N \\
{\scriptstyle \mathrm{act}_M} \downarrow & & \downarrow {\scriptstyle \mathrm{act}_N} \\
M & \xrightarrow{\quad f \quad} & N
\end{array}
\qquad (7)
$$

We write $\mathbf{Mod}_R$ for the category of R-modules and R-module homomorphisms $f : M \to N$ between them.

IV. THE CATEGORY $\mathbf{Mod}$ OF MODULES AND MODULE HOMOMORPHISMS

In the same spirit, a **module** is defined as a pair (R, M) consisting of a commutative ring R and of an R-module M. Now, a **module homomorphism**,

$$
(u, f) : (R, M) \to (S, N)
$$

is a pair consisting of a ring homomorphism $u : R \to S$ and of a map $f : M \to N$ making the diagram below commute:

$$
\begin{array}{ccc}
R \otimes M & \xrightarrow{\ u \otimes f\ } & S \otimes N \\
{\scriptstyle \mathrm{act}_M} \downarrow & & \downarrow {\scriptstyle \mathrm{act}_N} \\
M & \xrightarrow{\quad f \quad} & N
\end{array}
\qquad (8)
$$

The category $\mathbf{Mod}$ has the modules (R, M) as objects, and the module homomorphisms $(u, f) : (R, M) \to (S, N)$ as morphisms. There is an obvious functor

$$
\pi \quad : \quad \mathbf{Mod} \longrightarrow \mathbf{Ring}
\qquad (9)
$$

which transports every module (R, M) to its underlying commutative ring R, and every module homomorphism $(u, f) : (R, M) \to (S, N)$ to its underlying ring homomorphism $u : R \to S$. For that reason, we often find convenient to write

$$
u \ : \ R \longrightarrow S \quad \models \quad f \ : \ M \longrightarrow N
$$

for a module homomorphism $(u, f) : (R, M) \to (S, N)$ as defined in (8). The notation is inspired by [23] and the intuition that every ring homomorphism $u : R \to S$ induces a "fiber" consisting of all the module homomorphisms of the form $(u, f) : (R, M) \to (S, N)$ living "above" and refining the ring homomorphism $u : R \to S$. Note moreover that the fiber of π of a commutative ring R coincides with the category $\mathbf{Mod}_R$ defined above.

As it is well-known, the functor $\pi : \mathbf{Mod} \to \mathbf{Ring}$ defines a Grothendieck fibration. The reason is that every pair consisting of a ring homomorphism $u : R \to S$ and of a S-module (N, act_N) induces an R-module noted

$$
\mathbf{res}_u N \quad = \quad (N, \mathrm{act}'_N)
$$

with same underlying object N as the original S-module, and with action map $\mathrm{act}'_N : R \otimes N \to N$ defined as the composite:

$$
\mathrm{act}'_N \quad = \quad R \otimes N \xrightarrow{\ u \otimes N\ } S \otimes N \xrightarrow{\ \mathrm{act}_N\ } N
$$

The S-module (N, act_N) comes moreover with a module homomorphism

$$
u \ : \ R \longrightarrow S \quad \models \quad \mathrm{id}_N \ : \ \mathbf{res}_u N \longrightarrow N
\qquad (10)
$$

which is weakly cartesian (or cartesian in the original sense by Grothendieck [11], Exposé VI, Def. 5.1) in the sense that every module homomorphism

$$
u \ : \ R \longrightarrow S \quad \models \quad f \ : \ M \longrightarrow N
$$

factors uniquely as

$$
M \xrightarrow{\ (\mathrm{id}_R, h)\ } \mathbf{res}_u N \xrightarrow{\ (u, \mathrm{id}_N)\ } N
$$

for a map $h : M \to \mathbf{res}_u N$ in the category $\mathbf{Mod}_R$. Note that the unique solution h is in fact equal to the original map $f : M \to N$, seen this time as a map between R-modules. Given a ring homomorphism $u : R \to S$, the existence of a weakly cartesian map of the form (10) for every S-module N ensures the existence of a functor

$$
\mathbf{res}_u \quad : \quad \mathbf{Mod}_S \longrightarrow \mathbf{Mod}_R
\qquad (11)
$$

called *restriction of scalar* along u. This added to the fact that the composite of two weakly cartesian maps is a weakly cartesian map, establishes that the functor $\pi : \mathbf{Mod} \to \mathbf{Ring}$ is a Grothendieck fibration.

At this stage, we use the fact that the category $\mathscr{A}$ has reflexive coequalizers, preserved by the tensor product componentwise, in order to show that the functor (11) has a left adjoint noted

$$
\mathbf{ext}_u \quad : \quad \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_S.
$$

The functor $\mathbf{ext}_u$ is constructed as follows. Every ring homomorphism $u : R \to S$ induces a $R \otimes S$-module noted $R \otimes_u S$ and defined as the reflexive coequalizer of the diagram:

$$
R \otimes R \otimes S \;
\begin{array}{c}
\xrightarrow{\ m_R \otimes S\ } \\[-2pt]
\xleftarrow{\ R \otimes e_R \otimes S\ } \\[-2pt]
\xrightarrow[(R \otimes m_S)\circ(R \otimes u \otimes S)]{}
\end{array}
\; R \otimes S
$$

computed in the category $\mathscr{A}$. Now, given three commutative rings R, S_1 and S_2, we define the functor

$$
\circledast_R \quad : \quad \mathbf{Mod}_{S_1 \otimes R} \times \mathbf{Mod}_{R \otimes S_2} \longrightarrow \mathbf{Mod}_{S_1 \otimes S_2}
\qquad (12)
$$

which transports a pair (M, N) consisting of a $S_1 \otimes R$-module M and a $R \otimes S_2$-module N to the $S_1 \otimes S_2$-module $M \otimes_R N$ defined as the reflexive coequalizer of the diagram:

$$
M \otimes R \otimes N \;
\begin{array}{c}
\xrightarrow{\ \mathrm{act}_M \otimes N\ } \\[-2pt]
\xleftarrow{\ M \otimes e_R \otimes N\ } \\[-2pt]
\xrightarrow[M \otimes \mathrm{act}_N]{}
\end{array}
\; M \otimes N
$$

computed in the category $\mathscr{A}$. Here, the two maps $\mathrm{act}_M : M \otimes R \to M$ and $\mathrm{act}_N : R \otimes N \to N$ are deduced by restriction of scalar from the $S_1 \otimes R$-module structure of M and the $R \otimes S_2$-module structure of N. The left adjoint functor $\mathbf{ext}_u$ is then defined in the following way

$$
\mathbf{ext}_u \quad : \quad M \quad \mapsto \quad M \circledast_R (R \otimes_u S)
$$

by applying the construction (12) to the R-module M and to the $R \otimes S$-module $R \otimes_u S$ defined earlier, in order to obtain the S-module $M \otimes_R (R \otimes_u S)$.

V. THE CATEGORY $\mathbf{Mod}^\ominus$ OF MODULES AND MODULE RETROMORPHISM

Here, we make the extra assumption that the category $\mathscr{A}$ is symmetric monoidal closed, with coreflexive equalizers. The internal hom-object in $\mathscr{A}$ is noted $[M, N]$. A **module retromorphism**

$$(u, f) : (S, N) \to (R, M)$$

is a pair (u, f) consisting of a ring homomorphism $u : R \to S$ and of a map $f : N \to M$ making the diagram below commute:

$$\begin{array}{ccccc} R \otimes M & \xleftarrow{R \otimes f} & R \otimes N & \xrightarrow{u \otimes N} & S \otimes N \\ {\scriptstyle \mathrm{act}_M} \downarrow & & & & \downarrow {\scriptstyle \mathrm{act}_N} \\ M & \xleftarrow{\hspace{2cm} f \hspace{2cm}} & & & N \end{array} \quad (13)$$

The category $\mathbf{Mod}^\ominus$ has ring modules as objects and module retromorphisms $(u, f) : (S, N) \to (R, M)$ as morphisms. There is an obvious functor

$$\pi \quad : \quad \mathbf{Mod}^\ominus \longrightarrow \mathbf{Ring} \quad (14)$$

which transports every module retromorphism (u, f) to the underlying ring homomorphism $u : R \to S$. Note that the functor π is a Grothendieck fibration, which coincides in fact with the opposite Grothendieck fibration of π defined in (9), see [31], [16]. It turns out that the functor π is in fact a bifibration. Given a ring homomorphism $u : R \to S$, the functor defined by *coextension of scalar* along u

$$\mathbf{coext}_u \quad : \quad \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_S$$

transports every R-module (M, act_M) to the S-module

$$[S, M]_u \quad (15)$$

defined as the coreflexive equalizer, in the category $\mathscr{A}$, of the diagram below:

$$[S, M] \; \substack{\xrightarrow{[u \otimes S, M] \circ [m_S, M]} \\ \xleftarrow{\quad [e_R \otimes S, M] \quad} \\ \xrightarrow{[R \otimes S, \mathrm{act}_M] \circ [R \otimes -, R \otimes -]}} \; [R \otimes S, M]$$

Note that the coreflexive equalizer $[S, M]_u$ provides an internal description, in the category $\mathscr{A}$, of the set of maps $f : S \to M$ making the diagram below commute:

$$\begin{array}{ccc} R \otimes M & \xleftarrow{R \otimes f} & R \otimes S \\ & & \downarrow {\scriptstyle u \otimes S} \\ {\scriptstyle \mathrm{act}_M} \downarrow & & S \otimes S \\ & & \downarrow {\scriptstyle m_S} \\ M & \xleftarrow{\quad f \quad} & S \end{array} \quad (16)$$

or equivalently, as the set of R-module homomorphisms $f : \mathbf{res}_u S \to M$. In summary, putting together the results and constructions of §III, §IV and §V, every ring homomorphism $u : R \to S$ induces three functors

$$\mathbf{Mod}_R \; \substack{\xrightarrow{\mathbf{coext}_u} \\ \xleftarrow{\;\mathbf{res}_u\;} \\ \xrightarrow{\mathbf{ext}_u}} \; \mathbf{Mod}_S$$

organized into a sequence of adjunctions

$$\mathbf{ext}_u \dashv \mathbf{res}_u \dashv \mathbf{coext}_u$$

where extension of scalar $\mathbf{ext}_u$ is left adjoint, and coextension of scalar $\mathbf{coext}_u$, right adjoint to restriction of scalar $\mathbf{res}_u$.

VI. THE CATEGORY $\mathbf{Mod}_R$ IS SYMMETRIC MONOIDAL CLOSED

A well-known fact of algebra is that the category $\mathbf{Mod}_R$ of R-modules is symmetric monoidal closed. Its tensor product

$$\otimes_R \quad : \quad \mathbf{Mod}_R \times \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R$$

transports every pair of R-modules M, N into the reflexive coequalizer of the diagram below

$$M \otimes R \otimes N \; \substack{\xrightarrow{\mathrm{act}_M \otimes N} \\ \xleftarrow{\; M \otimes e_R \otimes N \;} \\ \xrightarrow{M \otimes \mathrm{act}_N}} \; M \otimes N$$

computed in the category $\mathscr{A}$, and its tensorial unit is the commutative ring $1_R = R$ itself, seen as an R-module. The internal hom of the category $\mathbf{Mod}_R$ noted

$$[-, -]_R \quad : \quad \mathbf{Mod}_R^{op} \times \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R$$

transports every pair of R-modules M, N to the coreflexive equalizer of the diagram below:

$$[M, N] \; \substack{\xrightarrow{[\mathrm{act}_M, M]} \\ \xleftarrow{\;[e_R \otimes M, N]\;} \\ \xrightarrow{[R \otimes M, \mathrm{act}_N] \circ [R \otimes -, R \otimes -]}} \; [R \otimes M, N]$$

Note that the definitions of the tensor product $\otimes_R$ and of the internal hom $[-, -]_R$ are mild variations of the constructions (12) and (15) described previously in §IV and §V.

VII. THE CATEGORY $\mathbf{Mod}$ AS A SYMMETRIC MONOIDAL RINGED CATEGORY

The category $\mathbf{Mod}$ defined in §IV comes with a functor $\pi : \mathbf{Mod} \to \mathbf{Ring}$ whose fibers are precisely the categories $\mathbf{Mod}_R$ of R-modules. For that reason, the family of tensor products $\otimes_R$ described in §VI induces a fibrewise (or vertical) tensor product on $\mathbf{Mod}$ above $\mathbf{Ring}$, which may be conveniently described in the following way. Consider the category $\mathbf{Mod} \times_{\mathbf{Ring}} \mathbf{Mod}$ defined by the pullback diagram below:

$$\begin{array}{ccc} \mathbf{Mod} \times_{\mathbf{Ring}} \mathbf{Mod} & \longrightarrow & \mathbf{Mod} \\ \downarrow & & \downarrow {\scriptstyle \pi} \\ \mathbf{Mod} & \xrightarrow{\quad \pi \quad} & \mathbf{Ring} \end{array}$$

The fibrewise tensor product

$$\otimes_{\mathbf{Mod}} \quad : \quad \mathbf{Mod} \times_{\mathbf{Ring}} \mathbf{Mod} \longrightarrow \mathbf{Mod} \qquad (17)$$

transports every pair of modules (R, M) and (R, N) on the same commutative monoid R to the R-module $(R, M \otimes_R N)$, and every pair of module homomorphisms

$$u \; : \; R \longrightarrow S \quad \models \quad h_1 \; : \; M_1 \longrightarrow N_1$$
$$u \; : \; R \longrightarrow S \quad \models \quad h_2 \; : \; M_2 \longrightarrow N_2$$

above the same ring homomorphism $u : R \to S$ to the module homomorphism

$$u \; : \; R \to S \; \models \; h_1 \otimes_u h_2 \; : \; M_1 \otimes_R M_2 \to N_1 \otimes_S N_2$$

where $h_1 \otimes_u h_2$ is defined as the unique map making the diagram below commute:

$$
\begin{array}{ccc}
M_1 \otimes R \otimes M_2 & \xrightarrow{h_1 \otimes u \otimes h_2} & N_1 \otimes S \otimes N_2 \\
\scriptstyle{\mathrm{act}_{M_1} \otimes M_2} \Big\downarrow \Big\downarrow \scriptstyle{M_1 \otimes \mathrm{act}_{M_2}} & & \scriptstyle{\mathrm{act}_{N_1} \otimes N_2} \Big\downarrow \Big\downarrow \scriptstyle{N_1 \otimes \mathrm{act}_{N_2}} \\
M_1 \otimes M_2 & \xrightarrow{h_1 \otimes h_2} & N_1 \otimes N_2 \\
\scriptstyle{quotient\,map} \Big\downarrow & & \Big\downarrow \scriptstyle{quotient\,map} \\
M_1 \otimes_R M_2 & \xrightarrow{h_1 \otimes_u h_2} & N_1 \otimes_S N_2
\end{array}
$$

The fibrewise unit is defined as the functor

$$1_{\mathbf{Mod}} \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Mod} \qquad (18)$$

which transports every commutative ring R into itself, seen as an R-module. The categorical situation may be understood in the following way. A **ringed category** is defined as a pair $(\mathscr{C}, \pi)$ consisting of a category $\mathscr{C}$ and of a functor $\pi : \mathscr{C} \to \mathbf{Ring}$ to the category of commutative rings. The slice 2-category $\mathbf{Cat}/\mathbf{Ring}$ has ringed categories as objects, fibrewise functors and natural tranformations as 1-cells and 2-cells. The 2-category $\mathbf{Cat}/\mathbf{Ring}$ is cartesian, with cartesian product defined by the expected pullback above $\mathbf{Ring}$. Using that setting, one observes that the fibrewise tensor product (17) and tensor unit (18) define a symmetric pseudomonoid structure on the ringed category $(\mathbf{Mod}, \pi)$ in the 2-category $\mathbf{Cat}/\mathbf{Ring}$.

VIII. THE CATEGORY $\mathbf{Alg}$ OF COMMUTATIVE ALGEBRAS

Given a commutative ring R in the category $\mathscr{A}$, a **commutative R-algebra** A is defined as a commutative monoid in the symmetric monoidal category $(\mathbf{Mod}_R, \otimes_R, 1_R)$. A **commutative algebra** is defined as a pair (R, A) consisting of a commutative ring R and of a commutative R-algebra A. An algebra map $(u, f) : (R, A) \to (S, B)$ is a pair (u, f) consisting of a ring homomorphism $u : R \to S$ and of a module homomorphism $f : M \to N$ making the diagrams commute:

$$
\begin{array}{ccc}
A \otimes_R A & \xrightarrow{f \otimes_u f} & B \otimes_S B \\
\scriptstyle{m_A} \Big\downarrow & & \Big\downarrow \scriptstyle{m_B} \\
A & \xrightarrow{f} & B
\end{array}
\qquad
\begin{array}{ccc}
1_R & \xrightarrow{u} & 1_S \\
\scriptstyle{e_A} \Big\downarrow & & \Big\downarrow \scriptstyle{e_B} \\
A & \xrightarrow{f} & B
\end{array}
$$

We suppose that for every commutative monoid R, the symmetric monoidal category $\mathbf{Mod}_R$ has a free commutative monoids. This means that the forgetful functor

$$\mathbf{Forget}_R \quad : \quad \mathbf{Alg}_R \longrightarrow \mathbf{Mod}_R$$

has a left adjoint, which we note

$$\mathbf{Sym}_R \quad : \quad \mathbf{Mod}_R \longrightarrow \mathbf{Alg}_R$$

By glueing together the fibers $\mathbf{Mod}_R$ and $\mathbf{Alg}_R$, we obtain in this way an adjunction

$$
\mathbf{Mod} \underset{\mathbf{Sym}}{\overset{\mathbf{Forget}}{\rightleftarrows}} \top \mathbf{Alg}
$$

which is moreover vertical (or fibrewise) above $\mathbf{Ring}$.

IX. THE CATEGORY $\mathbf{CoAlg}$ OF COMMUTATIVE COALGEBRAS

Suppose given a commutative ring R. A **commutative R-coalgebra** K is defined as a commutative comonoid in the symmetric monoidal category $(\mathbf{Mod}_R, \otimes_R, 1_R)$. In the same way as in §VIII, we suppose that for every commutative monoid R, the symmetric monoidal category $\mathbf{Mod}_R$ has a cofree commutative comonoids. This means that the forgetful functor

$$\mathbf{Forget}_R \quad : \quad \mathbf{CoAlg}_R \longrightarrow \mathbf{Mod}_R$$

has a right adjoint, which we note

$$\mathbf{CoFree}_R \quad : \quad \mathbf{Mod}_R \longrightarrow \mathbf{CoAlg}_R$$

By glueing together the fibers $\mathbf{Mod}_R$ and $\mathbf{CoAlg}_R$, we obtain in this way an adjunction

$$
\mathbf{CoAlg} \underset{\mathbf{Forget}}{\overset{\mathbf{CoFree}}{\rightleftarrows}} \top \mathbf{Mod} \qquad (19)
$$

which is moreover vertical (or fibrewise) above $\mathbf{Ring}$.

X. THE COFREE CONSTRUCTION FOR $\mathscr{A} = \mathbf{Ab}$

One remarkable aspect of the adjunction (19) in §IX is that it can be defined at a purely formal level, without the need for an explicit description of the cofree construction $\mathbf{CoFree}_R$ performed in each fiber $\mathbf{Mod}_R$. This is important because Murfet has given [26], [3] an explicit description of the construction in the case when $R = k$ of an algebraically closed field of characteristic 0, but the finite dual construction by Sweedler [32], [14], [1], [28], [29] remains somewhat mysterious in the case of a general commutative ring R. Let us give a brief description of the construction here. First of all, the free commutative algebra $\mathbf{Sym}_R$ can be computed as follows with enough colimits in the category $\mathbf{Mod}_R$:

$$\mathbf{Sym}_R \quad : \quad M \quad \mapsto \quad \bigoplus_{n \in \mathbb{N}} M \otimes_R^{sym} \cdots \otimes_R^{sym} M$$

where $M \otimes_R^{sym} \cdots \otimes_R^{sym} M$ denotes the symmetrized tensor product of M with itself, taken n times. In the case when $R = k$ is a field, the Sweedler construction transports every k-vector space V into the commutative k-algebra defined as

$$\mathbf{CoFree}_k(V) \quad = \quad (\mathbf{Sym}_R(V^*))^{\circ}$$

where the vector space V^* denotes the dual of the vector space V, and $A \mapsto A^{\circ}$ denotes **Sweedler's finite dual construction**, which transports every R-algebra A (not necessarily commutative) to a R-coalgebra A°. In the general case of a commutative ring R, one needs to apply instead the formal construction designed in [28], [29] and adapted to the construction of the cofree commutative R-coalgebra $\mathbf{CoFree}_R(M)$ generated by an R-module M in $\mathbf{Mod}_R$.

XI. FUNCTORS OF POINTS AND **Ring**-SPACES

We work with covariant presheaves X, Y on the category **Ring** of commutative rings, which we call **Ring-spaces**. To every such **Ring**-space

$$X \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

we associate its Grothendieck category $\mathbf{Points}(X)$ whose objects are the pairs (R, x) with $x \in X(R)$ and whose maps $(R, x) \rightarrow (S, y)$ are ring homomorphisms $u : R \rightarrow S$ transporting the element $x \in X(R)$ to the element $y \in X(S)$, in the sense that

$$X(u)(x) \quad = \quad y.$$

The category $\mathbf{Points}(X)$ comes with an obvious functor

$$\pi_X \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{Ring}$$

This functor, called the functor of point of X, defines in fact a discrete Grothendieck opfibration above the category **Ring**. A map $f : X \rightarrow Y$ between **Ring**-spaces may be equivalently defined as a functor

$$f \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{Points}(Y)$$

making the diagram below commute:

$$\mathbf{Points}(X) \xrightarrow{\quad f \quad} \mathbf{Points}(Y)$$
$$\pi_X \searrow \quad \mathbf{Ring} \quad \swarrow \pi_Y$$

Note that the functor f is itself necessarily a discrete opfibration: this follows from the fact that discrete fibrations define a right orthogonality class of a factorization system on **Cat**, with cofinal functors as elements of the left orthogonality class. Note also that the **Ring**-space

$$\mathrm{Spec}\,\mathbb{Z} \quad : \quad R \quad \mapsto \quad \{*_R\} \qquad (20)$$

is the terminal object of the category $[\mathbf{Ring}, \mathbf{Set}]$, and that its Grothendieck category is isomorphic to category **Ring** itself.

XII. PRESHEAVES OF MODULES

We suppose given a **Ring**-space

$$X \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

The following definition is adapted from [4], [17].

Definition 1: A **presheaf of modules** M **on the Ring-space** X or more simply, **an** $\mathscr{O}_X$**-module** M, consists of the following data:

- for each point $(R, x) \in \mathbf{Points}(X)$, a module M_x over the commutative ring R,
- for each map $u : (R, x) \rightarrow (S, y)$ in $\mathbf{Points}(X)$, a module homomorphism

$$u : R \longrightarrow S \quad \models \quad \theta(u, x) : M_x \longrightarrow N_y$$

living over the ring homomorphism $u : R \rightarrow S$.

The map θ is moreover required to satisfy the following functorial properties: first of all, the identity on the point (R, x) in the category $\mathbf{Points}(X)$ is transported to the identity map on the associated R-module

$$\mathrm{id}_R \quad \models \quad \theta(\mathrm{id}_{(R,x)}) = \mathrm{id}_{M_x}$$

and given two maps

$$(u, x) : (R, x) \rightarrow (S, y) \qquad (v, y) : (S, y) \rightarrow (T, z)$$

in the category $\mathbf{Points}(X)$, one has:

$$v \circ u \quad \models \quad \theta((v, y) \circ (u, x)) = \theta(v, y) \circ \theta(u, x)$$

where composition is computed in the discrete opfibration $\mathbf{Points}(X) \rightarrow \mathbf{Ring}$.

In the sequel, we will use the following equivalent formulation of presheaves of modules:

Proposition 1: An $\mathscr{O}_X$-module M is the same thing as a functor

$$M \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{Mod}$$

making the diagram below commute:

$$\mathbf{Points}(X) \xrightarrow{\quad M \quad} \mathbf{Mod}$$
$$\pi_X \searrow \quad \mathbf{Ring} \quad \swarrow \pi$$

Note that this specific formulation of presheaves of modules is the one used by Kontsevich and Rosenberg in their work on noncommutative geometry [17]. See also the discussion on the nLab entry [27]. It should be noted moreover that every **Ring**-space X comes equipped with a specific presheaf of module, called the **structure presheaf of modules** of X, and defined as the composite

$$\mathscr{O}_X \quad : \quad \mathbf{Points}(X) \xrightarrow{\ \pi_X\ } \mathbf{Ring} \xrightarrow{\ 1_{\mathbf{Mod}}\ } \mathbf{Mod} \qquad (21)$$

where the functor $\mathscr{O} : \mathbf{Ring} \rightarrow \mathbf{Mod}$ denotes the section of $\pi : \mathbf{Mod} \rightarrow \mathbf{Ring}$ which transports every commutative ring R to itself, seen as an R-module.

We construct the category **PshMod** of **presheaves of modules** and **forward morphisms**, in the following way. A **presheaf of modules** (X, M) is defined as a pair consisting of a **Ring**-space

$$X \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

together with a presheaf $\mathscr{O}_X$-module M, as formulated in §XII. A **forward morphism** between presheaf of modules

$$(f, \varphi) \quad : \quad (X, M) \longrightarrow (Y, N)$$

is defined as a morphism (= natural transformation) of **Ring**-spaces $f : X \to Y$ together with a natural transformation

$$(22)$$

The natural transformation is moreover required to be vertical (or fibrewise) above **Ring**, in the sense that the natural transformation obtained by composing

$$\pi \circ M \xrightarrow{\ \pi \circ \varphi\ } \pi \circ N \circ \mathbf{Points}(f)$$

is equal to the identity natural transformation. Equivalently, and expressed in a somewhat more fundamental way, one asks that the natural transformation

$$\pi_X \xrightarrow{\ id\ } \pi \circ M \xrightarrow{\ \pi \circ \varphi\ } \pi \circ N \circ \mathbf{Points}(f) \xrightarrow{\ id\ } \pi_Y \circ \mathbf{Points}(f)$$

obtained by composing the three natural transformations depicted below

coincides with the identity natural transformation from π_X to $\pi_Y \circ f$. There is an obvious functor

$$\mathbf{p} \quad : \quad \mathbf{PshMod} \longrightarrow [\mathbf{Ring}, \mathbf{Set}] \qquad (23)$$

which transports every presheaf of modules (X, M) to its underlying **Ring**-space X, and every forward morphism $(f, \varphi) : (X, M) \to (Y, N)$ to its underlying morphism $f : X \to Y$ between **Ring**-spaces. We thus find convenient to write

$$f \ : \ X \longrightarrow Y \quad \models \quad \varphi \ : \ M \longrightarrow N$$

for a forward morphism $(f, \varphi) : (X, M) \to (Y, N)$ between presheaves of modules.

The functor $\mathbf{p}$ is a Grothendieck fibration because every morphism $f : X \to Y$ between **Ring**-spaces X and Y induces a functor

$$f^* \quad : \quad \mathbf{PshMod}_Y \longrightarrow \mathbf{PshMod}_X \qquad (24)$$

which transports every $\mathscr{O}_Y$-module N into the $\mathscr{O}_X$-module $N \circ \mathbf{Points}(f)$ obtained by precomposition with the discrete fibration $\mathbf{Points}(f)$, as depicted below:

In fact, it turns out that the functor $\mathbf{p}$ is also a Grothendieck bifibration, but for less immediate reasons. In order to establish the property, we make the extra assumption that the category **Ring** as well as every category $\mathbf{Mod}_R$ associated to a commutative ring R has small colimits. Note that the property holds in the case of the category $\mathscr{A} = \mathbf{Ab}$ of abelian groups.

Proposition 2: For every morphism $f : X \to Y$ between **Ring**-spaces, there exists a functor

$$f_! \quad : \quad \mathbf{PshMod}_X \longrightarrow \mathbf{PshMod}_Y$$

left adjoint to the functor f^*.
A proof of the statement based on a purely 2-categorical construction of the $\mathscr{O}_Y$-module $f_!(M)$ appears in the Appendix, §A. It is worth noting that the $\mathscr{O}_Y$-module $f_!(M)$ can be also described more directly with an explicit formula:

$$f_!(M) \ : \ y \in Y(R) \mapsto \bigoplus_{\{x \in X(R), fx = y\}} M_x \in \mathbf{Mod}_R.$$

The adjunction $f_! \dashv f^*$ gives rise to a sequence of natural bijections, formulated in the type-theoretic fashion of [23]:

$$\frac{\dfrac{\mathrm{id}_X : X \to X \models M \to f^*(N)}{f : X \to Y \models M \to N}}{\mathrm{id}_Y : Y \to Y \models f_!(M) \to N}$$

XIV. THE CATEGORY **PshMod**$^\ominus$ OF PRESHEAF OF
MODULES AND BACKWARD MORPHISMS

We construct the category **PshMod**$^\ominus$ of presheaves of modules and **backward morphisms**, in the following way. A **backward morphism** between presheaves of modules

$$(f, \psi) \quad : \quad (X, M) \longrightarrow (Y, N)$$

is defined as a morphism $f : X \to Y$ between **Ring**-spaces together with a natural transformation

$$\psi \quad : \quad N \circ f \Longrightarrow M \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{Points}(Y)$$

which is moreover vertical in the sense that the diagram below commutes:

$$
\begin{array}{ccc}
\mathbf{Points}(X) & \xrightarrow{\quad f \quad} & \mathbf{Points}(Y) \\
\end{array}
$$

with M, ψ, N to $\mathbf{Mod}$, π_X, π_Y, and π to $\mathbf{Ring}$.

The category $\mathbf{PshMod}^{\ominus}$ has presheaves of modules as objects, and backward morphism as morphisms. There is an obvious functor

$$
\mathbf{p}^{\ominus} \quad : \quad \mathbf{PshMod}^{\ominus} \longrightarrow [\mathbf{Ring}, \mathbf{Set}]
$$

We thus find convenient to write

$$
f : X \longrightarrow Y \quad \models^{op} \quad \psi \quad : \quad M \longrightarrow N
$$

for such a backward morphism $(f, \psi) : (X, M) \to (Y, N)$ between presheaves of modules. As the opposite of the fibration $\mathbf{p}$, the functor $\mathbf{p}^{\ominus}$ is also a Grothendieck fibration with the functor

$$
(f^*)^{op} \quad : \quad \mathbf{PshMod}_Y^{op} \longrightarrow \mathbf{PshMod}_X^{op}
$$

as pullback functor. In order to establish the following property, we make the extra assumption that the category $\mathbf{Ring}$ as well as every category $\mathbf{Mod}_R$ associated to a commutative ring R has small limits.

Proposition 3: For every morphism $f : X \to Y$ between $\mathbf{Ring}$-spaces, there exists a functor called *direct image*

$$
f_* \quad : \quad \mathbf{PshMod}_X \longrightarrow \mathbf{PshMod}_Y.
$$

right adjoint to the functor f^*.
Note that, accordingly, the functor $(f_*)^{op}$ is left adjoint to the functor $(f^*)^{op}$. Quite interestingly, the proof of the statement works just as in the case of Prop. 2, and relies on a purely 2-categorical construction of the $\mathscr{O}_Y$-module $f_*(M)$, dual to the construction of the $\mathscr{O}_Y$-module $f_!(M)$, see the Appendix, §B for details. The adjunction $f^* \dashv f_*$ gives rise to a sequence of natural bijections, formulated in the type-theoretic fashion of [23]:

$$
\frac{\mathrm{id}_X : X \to X \models^{op} M \to f^*(N)}{\dfrac{f : X \to Y \models^{op} M \to N}{\mathrm{id}_Y : Y \to Y \models^{op} f_*(M) \to N}}
$$

In summary, we obtain that every morphism $f : X \to Y$ between $\mathbf{Ring}$-spaces X and Y induces three functors

$$
\mathbf{PshMod}_X \quad
\begin{array}{c}
\xrightarrow{\ f_* \ } \\
\xleftarrow{\ f^* \ } \\
\xrightarrow{\ f_! \ }
\end{array}
\quad \mathbf{PshMod}_Y \qquad (25)
$$

organized into a sequence of adjunctions

$$
f_! \dashv f^* \dashv f_*.
$$

XV. THE CATEGORY $\mathbf{PshMod}$ IS SYMMETRIC MONOIDAL CLOSED ABOVE $[\mathbf{Ring}, \mathbf{Set}]$

We establish in this section one of the main conceptual and technical contributions of the paper, directly inspired by the work by Mellies and Zeilberger on refinement type systems [23], [24]. As a presheaf category, the category $[\mathbf{Ring}, \mathbf{Set}]$ of $\mathbf{Ring}$-spaces is cartesian closed. We exhibit here a symmetric monoidal closed structure on the category $\mathbf{PshMod}$ of presheaf of modules, designed in such a way that the functor $\mathbf{p}$ is symmetric monoidal closed. The construction is new in Algebraic Geometry, as far as we know. The result is important, as it establishes the general presheaves of modules (instead of the more traditional notion of quasi-coherent sheaf of modules) as an appropriate foundation for a connection between Algebraic Geometry and formal logic. Suppose given a pair of $\mathbf{Ring}$-spaces

$$
X, Y \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}
$$

and a pair of presheaves of modules M and N over them:

$$
M \in \mathbf{PshMod}_X \qquad N \in \mathbf{PshMod}_Y.
$$

Recall that the cartesian product $X \times Y$ of $\mathbf{Ring}$-spaces is defined pointwise:

$$
X \times Y \quad : \quad R \mapsto X(R) \times Y(R).
$$

The tensor product

$$
M \otimes N \in \mathbf{PshMod}_{X \times Y}
$$

is defined using the isomorphism:

$$
\mathbf{Points}(X \times Y) \;\cong\; \mathbf{Points}(X) \times_{\mathbf{Ring}} \mathbf{Points}(Y)
$$

as the presheaf of modules

$$
\mathbf{Points}(X \times Y) \xrightarrow{\ (M,N)\ } \mathbf{Mod} \times_{\mathbf{Ring}} \mathbf{Mod} \xrightarrow{\ \otimes\ } \mathbf{Mod}
$$

where the functor (M, N) is defined by universality. The unit of the tensor product just defined is the structure presheaf of modules

$$
(\mathrm{Spec}\,\mathbb{Z}, \mathscr{O}_{\mathrm{Spec}\,\mathbb{Z}}) \quad : \quad (R, *_R) \mapsto R \in \mathbf{Mod}_R
$$

on the terminal object $\mathrm{Spec}\,\mathbb{Z}$ of the category $[\mathbf{Ring}, \mathbf{Set}]$, where $*_R$ denotes the unique element of the singleton set $\mathrm{Spec}\,\mathbb{Z}(R)$, see (20).

Before explaining the definition of the internal hom $M \multimap N$ on presheaves of modules in (27), we recall that the internal hom $X \Rightarrow Y$ in $[\mathbf{Ring}, \mathbf{Set}]$ is defined as the covariant presheaf which associates to every commutative ring R the set

$$
X \Rightarrow Y \quad : \quad R \mapsto ([\mathbf{Ring}, \mathbf{Set}]/\mathbf{y}_R)(\mathbf{y}_R \times X, \mathbf{y}_R \times Y)
$$

of natural transformations making the diagram commute:

$$
\begin{array}{ccc}
\mathbf{y}_R \times X & \xrightarrow{\quad f \quad} & \mathbf{y}_R \times Y \\
& \searrow{\scriptstyle \pi_{R,X}} \quad \swarrow{\scriptstyle \pi_{R,Y}} & \\
& \mathbf{y}_R &
\end{array}
\qquad (26)
$$

Here, $\mathbf{y}_R \in [\mathbf{Ring}, \mathbf{Set}]$ denotes the Yoneda presheaf generated by the commutative ring R, in the following way:

$$\mathbf{y}_R \quad : \quad S \mapsto \mathbf{Ring}(R, S) \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

while $\pi_{R,X}$ and $\pi_{R,Y}$ denote the first projections. The main contribution of the section comes now, with the following construction. The presheaf of modules

$$M \multimap N \in \mathbf{PshMod}_{(X \Rightarrow Y)} \tag{27}$$

is constructed in the following way. To every element $f \in (X \Rightarrow Y)(R)$, we associate the R-module

$$(M \multimap N)_f$$

consisting of all natural transformations φ making the diagram commute:

$$(28)$$

Such a natural transformation φ is a family of module homomorphisms

$$\mathrm{id}_S \quad : \quad S \longrightarrow S \quad \models \quad \varphi_{x,u} \quad : \quad M_x \longrightarrow N_{f(x,u)}$$

for $u : R \to S$ and $x \in X(S)$, natural in u and x in the sense that for every ring homomorphism $v : S \to S'$ with $X(v)(x) = x'$, the diagram should commute:

The R-module of such natural transformations can be computed in the category $\mathbf{Mod}_R$ using the following end formula:

$$\int_{(u:R \to S, x \in X(S)) \in \mathbf{Points}(\mathbf{y}_R \times X)} \mathbf{res}_u \left([M_x, N_{f(x,u)}]_S \right)$$

One establishes that

Theorem 1: The tensor product $M, N \mapsto M \otimes N$ and the implication $M, N \mapsto M \multimap N$ equip the category $\mathbf{PshMod}$ with the structure of a symmetric monoidal category. This structure is moreover transported by the functor $\mathbf{p}$ in (23) to the cartesian closed structure of the presheaf category $[\mathbf{Ring}, \mathbf{Set}]$.

The reader will find in the Appendix, §C, the central argument for the proof of Thm. 1.

$$
\frac{
\frac{
\frac{
\frac{
\frac{
\frac{
\frac{
\mathrm{id}_X : X \to X \models (M \otimes_X N) \to P
}{
\mathrm{id}_X : X \to X \models \Delta^*(M \otimes N) \to P
}
}{
\mathrm{id}_X : X \to X \models^{op} P \to \Delta^*(M \otimes N)
}
}{
\Delta : X \to X \times X \models^{op} P \to M \otimes N
}
}{
\mathrm{id}_{X \times X} : X \times X \to X \times X \models^{op} \Delta_*(P) \to M \otimes N
}
}{
\mathrm{id}_{X \times X} : X \times X \to X \times X \models M \otimes N \to \Delta_*(P)
}
}{
curry : X \to X \Rightarrow (X \times X) \models N \to M \multimap \Delta_*(P)
}
}{
\mathrm{id}_X : X \to X \models N \to curry^*(M \multimap \Delta_*(P))
}
$$
$$
\mathrm{id}_X : X \to X \models N \to (M \multimap_X P)
$$

Fig. 1. Sequence of natural bijections establishing that $(M \otimes_X -)$ is left adjoint to $(M \multimap_X -)$ for $M \in \mathbf{PshMod}_X$.

XVI. The category $\mathbf{PshMod}_X$ is symmetric monoidal closed

We illustrate the benefits of Thm. 1 by establishing, in the spirit of [23], [24], that the category $\mathbf{PshMod}_X$ associated to a given $\mathbf{Ring}$-space

$$X \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

is symmetric monoidal closed. The tensor product $M \otimes_X N$ of a pair of $\mathscr{O}_X$-modules M, N is defined as

$$M \otimes_X N \quad := \quad \Delta^*(M \otimes N)$$

where we use the notation

$$\Delta \quad : \quad X \longrightarrow X \times X$$

to denote the diagonal map coming from the cartesian structure of the category $[\mathbf{Ring}, \mathbf{Set}]$ of covariant presheaves. The tensorial unit is defined as the structure presheaf of modules $\mathscr{O}_X$ defined in (21). The internal hom $M \multimap_X N$ of a pair of $\mathscr{O}_X$-modules M, N is defined as

$$M \multimap_X N \quad := \quad curry^*(M \multimap \Delta_*(N))$$

where

$$curry \quad : \quad X \longrightarrow X \Rightarrow (X \times X)$$

is the map obtained by currifying the identity map

$$\mathrm{id}_{X \times X} \quad : \quad X \times X \longrightarrow X \times X$$

on the second component X. One obtains that

Proposition 4: The category $\mathbf{PshMod}_X$ equipped with $\otimes_X$ and $\multimap_X$ defines a symmetric monoidal closed category.

The proof that $(M \otimes_X -)$ is left adjoint to $(M \multimap_X -)$ can be decomposed in a sequence of elementary natural bijections, as described in Fig. 1.

Moreover, given a morphism $X \to Y$ in $[\mathbf{Ring}, \mathbf{Set}]$ and two $\mathscr{O}_Y$-modules M and N, the fact that $\Delta_Y \circ f = (f \times f) \circ \Delta_X$ and the isomorphism

$$(f \times f)^*(M \otimes N) \cong f^*(M) \otimes f^*(N)$$

imply that

$$f^* \; : \; \mathbf{PshMod}_Y \longrightarrow \mathbf{PshMod}_X$$

defines a **strongly monoidal functor**, in the sense that there exists a family of isomorphisms in $\mathbf{PshMod}_X$

$$m_{X,M,Y,N} \; : \; f^*(M) \otimes_X f^*(N) \xrightarrow{\;\sim\;} f^*(M \otimes_Y N)$$

$$m_{X,Y} \; : \; \mathscr{O}_X \xrightarrow{\;\sim\;} f^*(\mathscr{O}_Y)$$

making the expected coherence diagrams commute. From this follows that its right adjoint functor f_* as well as the adjunction $f^* \dashv f_*$ are lax symmetric monoidal ; and that its left adjoint functor $f_!$ as well as the adjunction $f_! \dashv f^*$ are oplax symmetric monoidal. In particular, the two functors f_* and $f_!$ come equipped with natural families of morphisms:

$$f_*(M) \otimes_N f_*(Y) \longrightarrow f_*(M \otimes_X N) \qquad \mathscr{O}_Y \longrightarrow f_*(\mathscr{O}_X)$$

$$f_!(M \otimes_X N) \longrightarrow f_!(M) \otimes_Y f_!(N) \qquad f_!(\mathscr{O}_X) \longrightarrow \mathscr{O}_Y$$

parametrized by $\mathscr{O}_X$-modules M and N. See for instance the discussion in [22], Section 5.15.

XVII. The category $\mathbf{PshCoAlg}$ of presheaves of commutative coalgebras

A **presheaf (X, K) of commutative coalgebras** is defined as a pair consisting of a $\mathbf{Ring}$-space $X : \mathbf{Ring} \to \mathbf{Set}$ and of a functor

$$K \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{CoAlg} \qquad (29)$$

making the diagram below commute:

$$
\begin{array}{ccc}
\mathbf{CoAlg} & \xrightarrow{\;\;\mathbf{Forget}\;\;} & \mathbf{Mod} \\
{\scriptstyle K} \uparrow & & \downarrow {\scriptstyle \pi} \\
\mathbf{Points}(X) & \xrightarrow{\;\;\pi_X\;\;} & \mathbf{Ring}
\end{array}
$$

A morphism $(f, \varphi) : (X, K) \to (Y, L)$ between two such presheaves of commutative coalgebras is defined as a pair consisting of a map $f : X \to Y$ between presheaves and of a natural transformation $\varphi : K \to L \circ f$ making the diagram below commute:

The resulting category $\mathbf{PshCoAlg}$ of presheaves of commutative coalgebras comes equipped with an obvious forgetful functor

$$\mathbf{Lin} \quad : \quad \mathbf{PshCoAlg} \longrightarrow \mathbf{PshMod} \qquad (30)$$

and thus with a composite functor

$$\mathbf{q} = \mathbf{p} \circ \mathbf{Lin} \quad : \quad \mathbf{PshCoAlg} \longrightarrow [\mathbf{Ring}, \mathbf{Set}]$$

We write $\mathbf{PshCoAlg}_X$ for the fiber category of the functor $\mathbf{q}$ above a given $\mathbf{Ring}$-space $X \in [\mathbf{Ring}, \mathbf{Set}]$. By construction, $\mathbf{PshCoAlg}_X$ is the category of **commutative $\mathscr{O}_X$-coalgebras**, defined as the presheaves of commutative coalgebras of the form (X, K) ; and of morphisms of the form $(\mathrm{id}_X, \varphi) : (X, K) \to (X, L)$ between them. An important observation for the construction of the model of linear logic which comes next is that

Proposition 5: The category $\mathbf{PshCoAlg}_X$ coincides with the category of commutative comonoids in the symmetric monoidal category $(\mathbf{PshMod}_X, \otimes_X, \mathscr{O}_X)$.

Another important property to notice at this stage is that the functor $\mathbf{q}$ is a bifibration. This essentially comes from the fact that the adjunction $f_! \dashv f^*$ on presheaves of modules described in (25) is in fact an oplax monoidal adjunction (see the discussion in §XVI) and thus lifts to an adjunction

$$\mathbf{PshCoAlg}_X \;\; \underset{f_!}{\overset{f^*}{\longleftrightarrow}} \;\; \mathbf{PshCoAlg}_Y$$

between the categories of commutative coalgebras.

XVIII. The linear-non-linear adjunction on $\mathbf{PshMod}_X$

We have established in §XVI that the category $\mathbf{PshMod}_X$ is symmetric monoidal closed for every $\mathbf{Ring}$-space X. In order to obtain a model of intuitionistic linear logic, we construct below a linear-non-linear adjunction

The functor $\mathbf{Lin}$ defined in (30) is a functor of categories fibered above $\mathbf{Ring}$, and the functor $\mathbf{Lin}_X$ is thus obtained by restricting it to the fiber of X:

$$\mathbf{Lin}_X \quad : \quad \mathbf{PshCoAlg}_X \longrightarrow \mathbf{PshMod}_X$$

By a general and well-known property of categories of commutative comonoids, together with Prop. 5, the category $\mathbf{PshCoAlg}_X$ is cartesian, with the cartesian product

$$K \times L \quad = \quad K \otimes_X L$$

and the terminal object defined as the structure presheaf of modules $\mathscr{O}_X$ and tensorial unit of $\mathbf{PshMod}_X$, see for instance [22], Section 6.5, for a discussion. From this follows that the functor $\mathbf{Lin}_X$ is strong symmetric monoidal.

The functor $\mathbf{Exp}_X$ is defined in the following way: it transports an $\mathscr{O}_X$-module M defined by a functor

$$\mathbf{Points}(X) \xrightarrow{\;\;M\;\;} \mathbf{Mod}$$

to the commutative $\mathscr{O}_X$-algebra defined by postcomposition

$$\mathbf{Points}(X) \xrightarrow{\ M\ } \mathbf{Mod} \xrightarrow{\ \mathbf{CoFree}\ } \mathbf{CoAlg}$$

We claim that the functor $\mathbf{Exp}_X$ is right adjoint to the forgetful functor $\mathbf{Lin}_X$. Indeed, given a commutative $\mathscr{O}_X$-coalgebra K and an $\mathscr{O}_X$-module M, there is a family of bijections

$$\mathbf{PshMod}_X(\mathbf{Lin}_X(K), M) \ \cong \ \mathbf{PshCoAlg}_X(K, \mathbf{Exp}_X(M))$$

natural in K and M, derived from the fact that there is a one-to-one relationship between the $\mathscr{O}_X$-module homomorphisms

$$\varphi \ : \ \mathbf{Lin}_X(K) \longrightarrow M$$

defined as the (fibrewise) natural transformations of the form

and the commutative $\mathscr{O}_X$-coalgebra morphisms

$$\psi \ : \ K \longrightarrow \mathbf{Exp}_X(M)$$

defined as the (fibrewise) natural transformations of the form

The bijection itself comes from the fact that the functor $\mathbf{CoFree}$ is right adjoint to $\mathbf{Forget}$ in the 2-category $\mathbf{Cat}/\mathbf{Ring}$ of ringed categories, see for instance [22], section 5.11, for a discussion. We conclude with the main result of the article:

Theorem 2: The adjunction (31) is a linear-non-linear adjunction, and thus defines for every $\mathbf{Ring}$-space X a model of intuitionistic linear logic on the symmetric monoidal closed category $\mathbf{PshMod}_X$ of presheaves of $\mathscr{O}_X$-modules.

XIX. CONCLUSION AND FUTURE WORKS

Our main technical contribution in this work is to resolve an old open question in the field of mathematical logic, which is to construct a model of linear logic — including the exponential modality — in the functorial language of Algebraic Geometry. By performing this construction in the present paper, we hope to integrate linear logic as a basic and very natural component in the current process of geometrization of type theory. The guiding idea here is that linear logic should be seen as the logic of generalised vector bundles, in the same way as Martin-Löf type theory with identity is seen today as the logic of spaces up to homotopy, formulated in the language of ∞-topos theory. One would thus obtain the following dictionary:

dependent types	$\sim$	spaces up to homotopy
linear types	$\sim$	vector bundles

The idea was already implicit in Ehrhard's differential linear logic [6] and it is thus very good news to see this foundational intuition confirmed by our construction. In a nice and inspiring series of recent works, Murfet and his student Clift [26], [3] have established that $\mathbf{Mod}_R$ equipped with the Sweedler exponential modality defines a model of *differential* linear logic whenever the commutative ring $R = k$ is an algebraically closed field of characteristic 0. One important question which we leave for future work is to understand whether the vector bundle semantics of linear logic just constructed in $\mathbf{PshMod}_X$ extends as it stands (or as a slight variant) to a model of *differential* linear logic. Another important research direction in good harmony with homotopy type theory will be to shift to Derived Algebraic Geometry in the style of Toën and Lurie [34], [20] by building our constructions on the symmetric monoidal category $\mathscr{A} = \mathbf{dgAb}$ of differential graded abelian groups, with its category $\mathbf{Ring} = \mathbf{dgAlg}$ of commutative differential graded algebras considered up to quasi-isomorphisms. We leave that important and fascinating question for future work.

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APPENDIX A
PROOF OF PROP. 2

We use the fact that the functor $\pi : \mathbf{Mod} \to \mathbf{Ring}$ is a bifibration, and more specifically a Grothendieck opfibration. From this follows that the category $\mathbf{Mod}$ has small colimits, which are moreover preserved by the functor π. Now, suppose that we are in the following situation
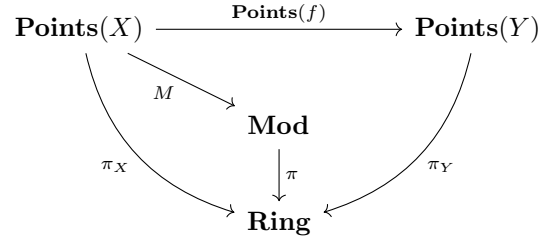

We start by computing the left Kan extension of the functor M along the discrete fibration

$$\mathbf{Points}(f) \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{Points}(Y)$$

One obtains in this way a functor $M' : \mathbf{Points}(Y) \to \mathbf{Mod}$ and a natural transformation λ which exhibits M' as the left Kan extension of M along $\mathbf{Points}(f)$, as depicted below:
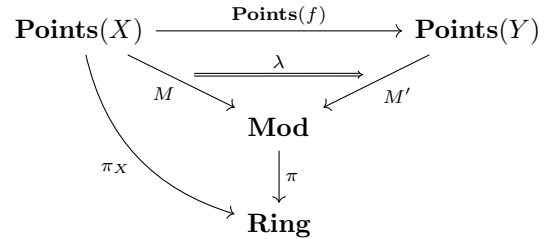
The left Kan extension is a pointwise Kan extension, computed by small colimits. As already mentioned, the functor π preserves small colimits, and thus pointwise left Kan extensions. This establishes that the natural transformation $\pi \circ \lambda$ exhibits the functor $\pi \circ N'$ as the left Kan extension of the composite functor $\pi \circ M$ along $\mathbf{Points}(f)$. The universality property of left Kan extensions ensures the existence of a unique natural transformation $\mu : \pi \circ N' \Rightarrow \pi_Y$ as depicted below
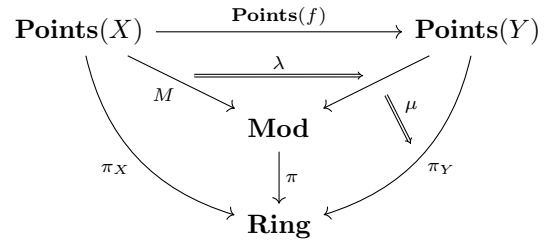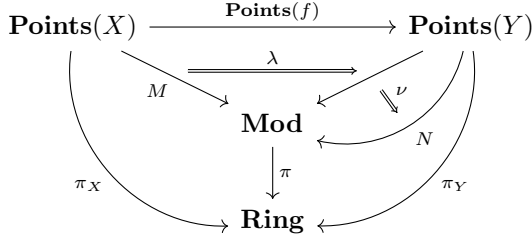

such that the composite

$$\pi \circ M \xrightarrow{\pi \circ \lambda} \pi \circ M' \circ \mathbf{Points}(f) \xrightarrow{\mu \circ \mathbf{Points}(f)} \pi_Y \circ \mathbf{Points}(f)$$

is equal to the identity natural transformation on the functor $\pi_X = \pi_Y \circ \mathbf{Points}(f)$. The functor π is a Grothendieck bifi-

bration. From this follows that the associated postcomposition functor

$$\mathbf{Cat}(\mathbf{Points}(Y), \mathbf{Mod}) \longrightarrow \mathbf{Cat}(\mathbf{Points}(Y), \mathbf{Ring})$$

is also a Grothendieck bifibration. In particular, there exists for that reason a functor $N : \mathbf{Points}(Y) \to \mathbf{Ring}$ and a natural transformation $\nu : M' \to N$ which is cocartesian above the natural transformation $\mu : \pi \circ M' \to \pi \circ N$, as depicted in the diagram below:

$$
\begin{array}{c}
\mathbf{Points}(X) \xrightarrow{\ \mathbf{Points}(f)\ } \mathbf{Points}(Y) \\
\lambda \quad \nu \\
M \searrow \quad \swarrow \quad N \\
\mathbf{Mod} \\
\pi_X \quad \pi \downarrow \quad \pi_Y \\
\mathbf{Ring}
\end{array}
$$

In particular, the natural transformation

$$\varphi = \nu \circ \lambda \quad : \quad M \Longrightarrow N \circ \mathbf{Points}(f)$$

obtained by composing ν and λ is vertical, in the sense that $\pi \circ \varphi$ is equal to the identity, and $N \circ \pi = \pi_Y$. The $\mathscr{O}_Y$-module $f_!(M)$ associated to the $\mathscr{O}_X$-module N is simply defined as

$$N \quad : \quad \mathbf{Points}(Y) \longrightarrow \mathbf{Ring}.$$

We obtain in this way a functor

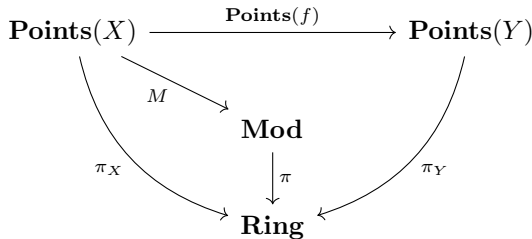$$f_! \quad : \quad \mathbf{PshMod}_X \longrightarrow \mathbf{PshMod}_Y$$

left adjoint to the inverse image functor

$$f^* \quad : \quad \mathbf{PshMod}_Y \longrightarrow \mathbf{PshMod}_X$$

formulated in (24).
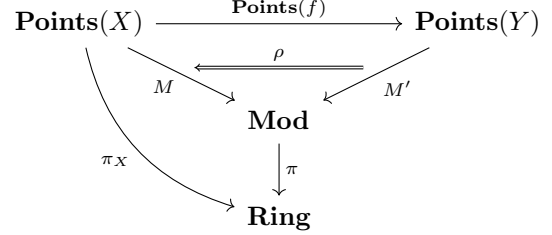
APPENDIX B
PROOF OF PROP. 3

We proceed exactly as in the proof of Prop. 2 in the previous section, except that the orientation of the natural transformations is reversed. We thus use the fact in the proof that the functor $\pi : \mathbf{Mod} \to \mathbf{Ring}$ is a bifibration, and more specifically a Grothendieck fibration. From this follows that the category $\mathbf{Mod}$ has small limits, which are moreover preserved by the functor π. Now, suppose that we are in the following situation
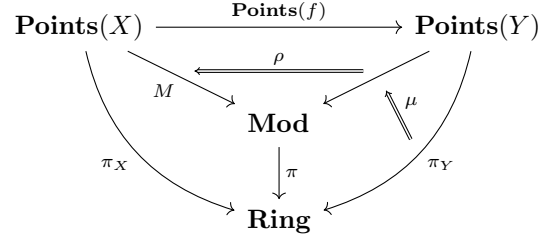
$$
\begin{array}{c}
\mathbf{Points}(X) \xrightarrow{\ \mathbf{Points}(f)\ } \mathbf{Points}(Y) \\
M \searrow \\
\mathbf{Mod} \\
\pi_X \quad \pi \downarrow \quad \pi_Y \\
\mathbf{Ring}
\end{array}
$$

We start by computing the right Kan extension of the functor M along the discrete fibration

$$\mathbf{Points}(f) \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{Points}(Y)$$

One obtains in this way a functor $M' : \mathbf{Points}(Y) \to \mathbf{Mod}$ and a natural transformation ρ which exhibits M' as the right Kan extension of M along $\mathbf{Points}(f)$, as depicted below:

$$
\begin{array}{c}
\mathbf{Points}(X) \xrightarrow{\ \mathbf{Points}(f)\ } \mathbf{Points}(Y) \\
\rho \\
M \searrow \quad \swarrow M' \\
\mathbf{Mod} \\
\pi_X \quad \pi \downarrow \\
\mathbf{Ring}
\end{array}
$$

The right Kan extension is a pointwise Kan extension, computed by small limits. The functor π preserves small limits, and thus pointwise right Kan extensions. This establishes that the natural transformation $\pi \circ \rho$ exhibits the functor $\pi \circ N'$ as the right Kan extension of the composite functor $\pi \circ M$ along $\mathbf{Points}(f)$. The universality property of right Kan extensions ensures the existence of a unique natural transformation $\mu : \pi_Y \Rightarrow \pi \circ N'$ as depicted below

$$
\begin{array}{c}
\mathbf{Points}(X) \xrightarrow{\ \mathbf{Points}(f)\ } \mathbf{Points}(Y) \\
\rho \\
M \searrow \quad \swarrow \quad \mu \\
\mathbf{Mod} \\
\pi_X \quad \pi \downarrow \quad \pi_Y \\
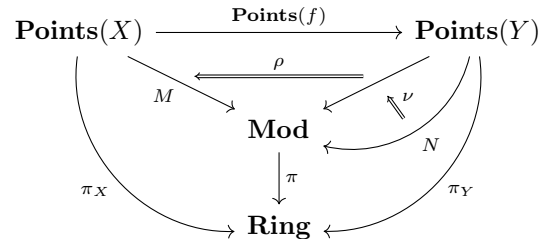\mathbf{Ring}
\end{array}
$$

such that the composite

$$\pi_Y \circ \mathbf{Points}(f) \xrightarrow{\mu \circ \mathbf{Points}(f)} \pi \circ M' \circ \mathbf{Points}(f) \xrightarrow{\pi \circ \rho} \pi \circ M$$

is equal to the identity natural transformation on the functor $\pi_X = \pi_Y \circ \mathbf{Points}(f)$. The functor π is a Grothendieck bifibration. From this follows that the associated postcomposition functor

$$\mathbf{Cat}(\mathbf{Points}(Y), \mathbf{Mod}) \longrightarrow \mathbf{Cat}(\mathbf{Points}(Y), \mathbf{Ring})$$

is also a Grothendieck bifibration. In particular, there exists for that reason a functor $N : \mathbf{Points}(Y) \to \mathbf{Ring}$ and a natural transformation $\nu : N \to M'$ which is cartesian above the natural transformation $\mu : \pi \circ N \to \pi \circ M'$, as depicted in the diagram below:

$$
\begin{array}{c}
\mathbf{Points}(X) \xrightarrow{\ \mathbf{Points}(f)\ } \mathbf{Points}(Y) \\
\rho \\
M \searrow \quad \swarrow \quad \nu \\
\mathbf{Mod} \quad N \\
\pi_X \quad \pi \downarrow \quad \pi_Y \\
\mathbf{Ring}
\end{array}
$$

In particular, the natural transformation

$$\psi = \rho \circ \nu \quad : \quad N \circ \mathbf{Points}(f) \Longrightarrow M$$

obtained by composing ρ and ν is vertical, in the sense that $\pi \circ \psi$ is equal to the identity, and $N \circ \pi = \pi_Y$. The $\mathscr{O}_Y$-module $f_*(M)$ associated to the $\mathscr{O}_X$-module N is simply defined as

$$N \quad : \quad \mathbf{Points}(Y) \longrightarrow \mathbf{Ring}.$$

We obtain in this way a functor

$$f_* \quad : \quad \mathbf{PshMod}_X \longrightarrow \mathbf{PshMod}_Y$$

right adjoint to the inverse image functor

$$f^* \quad : \quad \mathbf{PshMod}_Y \longrightarrow \mathbf{PshMod}_X$$

formulated in (24).

We suppose given a triple of **Ring**-spaces X, Y, Z together with a morphism

$$f \quad : \quad X \times Y \longrightarrow Z$$

The category $[\mathbf{Ring}, \mathbf{Set}]$ is cartesian closed, and the morphism f thus gives rise by currification to a morphism noted

$$g \quad : \quad Y \longrightarrow X \Rightarrow Z$$

We establish now a bijection between the set

$$\mathbf{PshMod}_f(M \otimes N, P)$$

of forward module morphisms of the form

$$f \; : \; X \times Y \longrightarrow Z \quad \models \quad \varphi \; : \; M \otimes N \longrightarrow P$$

and the set

$$\mathbf{PshMod}_g(N, M \multimap P)$$

of forward module morphisms of the form

$$g \; : \; Y \longrightarrow X \Rightarrow Z \quad \models \quad \psi \; : \; N \longrightarrow M \multimap P$$

The bijection is established by a series of elementary bijection applied to the enriched end formulas:

$$\mathbf{PshMod}_g(N, M \multimap P)$$

$$\cong \int_{(R,y)\in\mathbf{Points}(Y)} \left[N_y \, , \, \int_{(u:R\to S, x\in X(S))\in\mathbf{Points}(\mathbf{y}_R \times X)} \mathbf{res}_u \left(\, [M_x, P_{f(x,Y(u)(y))}]_S \, \right) \right]_R$$

$$\cong \int_{(R,y)\in\mathbf{Points}(Y)} \int_{(u:R\to S, x\in X(S))\in\mathbf{Points}(\mathbf{y}_R \times X)} \left[N_y \, , \, \mathbf{res}_u \left(\, [M_x, P_{f(x,Y(u)(y))}]_S \, \right) \right]_R$$

$$\cong \int_{(R,y)\in\mathbf{Points}(Y)} \int_{(u:R\to S, x\in X(S))\in\mathbf{Points}(\mathbf{y}_R \times X)} \mathbf{res}_u \left[\, \mathbf{ext}_u N_y \, , \, [M_x, P_{f(x,Y(u)(y))}]_S \, \right]_S$$

$$\cong \int_{(y\in Y(R), u:R\to S, x\in X(S))} \mathbf{res}_u[\mathbf{ext}_u N_y, [M_x, P_{f(x,Y(u)(y))}]_S]_S$$

$$\cong \int_{(S,y)\in\mathbf{Points}(Y),(S,x)\in X(S)} [N_y, [M_x, P_{f(x,y)}]_S]_S$$

$$\cong \int_{(S,x,y)\in\mathbf{Points}(X\times Y)} [M_x \otimes_S N_y, P_{f(x,y)}]_S$$

$$\cong \quad \mathbf{PshMod}_f(M \otimes N, P)$$