Approximately Stable Matchings with General Constraints*

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ABSTRACT

This paper focuses on two-sided matching where one side (a hospital or firm) is matched to the other side (a doctor or worker) so as to maximize a cardinal objective under general feasibility constraints. In a standard model, even though multiple doctors can be matched to a single hospital, a hospital has a *responsive preference* and a *maximum quota*. However, in practical applications, a hospital has some complicated cardinal preference and constraints. With such preferences (e.g., submodular) and constraints (e.g., knapsack or matroid intersection), stable matchings may fail to exist. This paper first determines the complexity of checking and computing stable matchings based on preference class and constraint class. Second, we establish a framework to analyze this problem on *packing* problems, and the framework enables us to access the wealth of online packing algorithms so that we construct *approximately stable* algorithms as a variant of generalized deferred acceptance algorithm. We further provide some inapproximability results.

KEYWORDS

Mechanism design; Stable matchings; Approximation algorithms

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1 INTRODUCTION

This paper studies a two-sided, one-to-many matching model when one side (a hospital or firm) is allocated members from the other side (a doctor or worker), covering constraints to satisfy practical or social demands and prohibiting infeasible allocation (matching). The theory of two-sided matching has been extensively developed, as illustrated by the comprehensive surveys [29, 33]. Matching with constraints has been prominent across computer science and economics since the seminal work by [21]. In many applications, various constraints are often imposed on an outcome, e.g., *type-specific quotas* on hospitals to assign several different types (skills) of doctors [1], *budget constraints* on hospitals to limit the total amount of wages [2, 23, 24]. The current paper exactly covers these complicated constraints and further generalizes them. Specifically, we consider general constraints of upper bounds known as *independence system constraints*, i.e., any subset of a feasible set of doctors is also feasible. We assume that each constraint is represented by a *capacity* (maximum quota), an *intersection of multiple matroids*, or a *multi-dimensional knapsack*. It should be noted that every independence system constraint can be represented by an intersection of multiple matroids and a multi-dimensional knapsack. In addition, we assume that each hospital's preference is represented by a utility function. We consider three important classes of cardinal utilities: *cardinality*, *additive*, and *submodular*. [1]

With such preferences and constraints, stable matchings may fail to exist. Determining whether a given matching market has a stable matching is hard in general. It is known to be Σ_2^{P}-complete when hospitals' utilities are additive, and the constraints are given as (1-dimensional) knapsack [17]. Note that the existence of stable matchings is guaranteed when the utilities are additive and the constraints are matroid [26] or the utilities are cardinality (matched size) and the constraints are knapsack [23].

There are several possibilities to circumvent the nonexistence problem. One modifies the notion of stability and proposes a variant of the Deferred Acceptance (DA) algorithm [2, 16, 21]. Another restricts hospitals' priorities to ensure the existence of a stable matching, e.g., lexicographic priorities [8]. Alternatively, Kawase and Iwasaki [23] and Nguyen et al. [31] focused on *near-feasible* stable matchings that approximately satisfy each budget of the hospitals.

This paper focuses on *approximately stable* matchings where the participants are only willing to change the assignments for a multiplicative improvement of a certain amount [3]. This idea can be interpreted as one in which a hospital in a blocking pair changes its match as soon as its utility after the change increases by any (arbitrarily small) amount. Arkin et al. [3] examined a stable roommate problem, which is a non-bipartite one-to-one matching problem, while we examine a bipartite one-to-many matching problem. It is reasonable for a hospital to change its assignments only in favor of a significant improvement; even though the grass may be greener on the other side, crossing the fence takes effort.

Our results

The following is the contribution of this paper. First, we analyze the problem of checking the stability on (offline) *packing problems* so that we understand the features and obtain the complexity results, which vary according to hospitals' utilities and imposed constraints. In particular, Theorem 3.1 proves that given a matching, checking whether it is stable or not is equivalent to solving a packing problem. Once we know the complexity of a packing problem in a given setting (utilities and constraints), we obtain that of the stability checking problem associated with the setting. Our results are summarized in Table 1 (a). For example, the stability checking problem

*A full version of this paper is available at [25].

[1] A function $u\colon 2^D \to \mathbb{R}_+$ is called cardinality, additive, and submodular if $u(S) = |S|$, $u(S) = \sum_{d\in S} u(\{d\})$, and $u(S) + u(T) \geq u(S\cup T) + u(S\cap T)$ $(\forall S, T \subseteq D)$, respectively.

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is solvable in polynomial time if the utilities are additive and the constraints are 2-matroid intersections. Also, if the utilities are submodular, the stability checking problem is coNP-complete even for the capacity constraints.

Second, Table 1 (b) summarizes our trichotomy results characterizing the complexity of determining whether a given market has a stable matching or not. The problem is polynomially solvable in very restricted classes of utilities and constraints, while it is NP-complete or Σ_2^{P}-complete in the other settings. Here, Σ_2^{P} (also known as $\mathrm{NP}^{\mathrm{NP}}$) is the class of problems solvable in polynomial time by a nondeterministic Turing machine with an oracle for some NP-complete problem. To prove the NP-hardness, we give a reduction from **DisjointMatching**, which is NP-complete [13]. In addition, we prove the Σ_2^{P}-hardness by reductions from the **∃∀3DM** or **∃∀SubsetSum**, which are Σ_2^{P}-complete [5, 30].

Finally, we introduce a framework that leads us to construct algorithms that find approximately stable matchings as a variant of generalized deferred acceptance (GDA). Theorem 5.1 enables GDA-based algorithms to access the proficiency of *online* packing algorithms. Intuitively, each hospital utilizes an *online* packing algorithm while running a GDA procedure. We show that if there exists an α-competitive algorithm for a class of online packing problems, then we can construct an algorithm that always yields an α-stable matching for the markets in a corresponding class. Table 1 (c) summarizes the upper and lower bounds of the approximation ratios that we obtained. Note that Theorem 6.1 provides a basis to derive some novel lower bounds. Here, for the knapsack constraints, we assume that the weight of each element on every dimension is at most a $(1 - \epsilon)$ fraction of the total capacity.

Let us note that recently there have been a certain amount of studies on two-sided matchings in the AI community, although this literature has been established mainly in fields across algorithms and economics. Drummond and Boutilier [10, 11] examined preference elicitation procedures for two-sided matching. In the context of *mechanism design*, Hosseini et al. [18] considered a mechanism for a situation where agents' preferences dynamically change.

Table 1: Summary of results ($k \geq 3$ and $\rho \geq 2$)

(a) Complexity of checking the stability

Constraints \ Hosp. Utils	Cardinality	Additive	Submodular
Capacity	P	P	coNP-c
Matroid	P	P	coNP-c
2-mat. int.	P	P	coNP-c
k-mat. int.	coNP-c	coNP-c	coNP-c
1-dim. knap.	P	coNP-c [†]	coNP-c
ρ-dim. knap.	coNP-c	coNP-c	coNP-c

(b) Complexity of checking existence

Constraints \ Hosp. Utils	Cardinality	Additive	Submodular
Capacity	P [*]	P [*]	Σ_2^{P}-c [b]
Matroid	P [*]	P [*]	Σ_2^{P}-c [b]
2-mat. int.	NP-c [a]	NP-c [a]	Σ_2^{P}-c [b]
k-mat. int.	Σ_2^{P}-c [c]	Σ_2^{P}-c [c]	Σ_2^{P}-c [b,c]
1-dim. knap.	P [‡]	Σ_2^{P}-c [†]	Σ_2^{P}-c [†,b]
ρ-dim. knap.	Σ_2^{P}-c [d]	Σ_2^{P}-c [†,d]	Σ_2^{P}-c [†,b,d]

(c) Approximation ratios (Upper Bound / Lower Bound)

Constraints \ Hosp. Utils	Cardinality	Additive	Submodular
Capacity	1 [*] / 1	1 [*] / 1	4 [g] / 1.28 [l]
Matroid	1 [*] / 1	1 [*] / 1	4 [g] / 1.28 [l]
2-matroid int.	2 [e] / 2 [k]	$(\sqrt{2}+1)^2$ [f] / 2 [k]	8 [g] / 2 [k]
k-matroid int.	k [e] / 2 [k]	$(\sqrt{k}+\sqrt{k-1})^2$ [f] / k [m]	$4k$ [g] / k [m]
1-dim. knap.	1 [‡] / 1	$\frac{1}{\epsilon}$ [‡] / $\frac{1}{\epsilon}$ [‡]	$O(\frac{1}{\epsilon^2})$ [j] / $\frac{1}{\epsilon}$ [‡]
ρ-dim. knap.	ρ [h] / 2 [k]	$\frac{\rho}{\epsilon}$ [i] / $\frac{\rho}{2\epsilon}$ [n]	$O(\frac{\rho}{\epsilon^2})$ [j] / $\frac{\rho}{2\epsilon}$ [n]

[*] [26]; [†] [17]; [‡] [24]; [a] Thm. 4.4; [b] Thm. 4.1; [c] Thm. 4.2; [d] Thm. 4.3; [e] Cor. 5.2; [f] Cor. 5.3; [g] Cor. 5.4; [h] Cor. 5.6; [i] Cor. 5.7; [j] Cor. 5.8; [k] Ex. 2.2; [l] Ex. 2.3; [m] Thm. 6.2; [n] Thm. 6.3.

2 MODEL

This section describes our model of two-sided matching markets. A market is a tuple $(D, H, \succ_D, u_H, \mathcal{I}_H)$ where each component is defined as follows. There are a finite set of doctors $D = \{d_1, \ldots, d_n\}$ and a finite set of hospitals $H = \{h_1, \ldots, h_m\}$. We denote by $\succ_D = (\succ_d)_{d \in D}$ the doctors' preference profile where $\succ_d$ is the strict relation of $d \in D$ over $H \cup \{\emptyset\}$; $s \succ_d t$ means that d strictly prefers s to t, where $\emptyset$ denotes being unmatched. Let $u_H = (u_h)_{h \in H}$ denote the hospitals' cardinal preference profile where u_h is the *utility function* $u_h \colon 2^D \to \mathbb{R}_+$. We assume that u_h is *normalized* (i.e., $u_h(\emptyset) = 0$) and *monotone* (i.e., $u_h(D'') \leq u_h(D')$ for any $D'' \subseteq D' \subseteq D$). Let $\mathcal{I}_H = (\mathcal{I}_h)_{h \in H}$ denote the *feasibility constraints* for hospitals where $\mathcal{I}_h \subseteq 2^D$ for each $h \in H$. We assume that $(D, \mathcal{I}_h)$ is an *independence system* for each $h \in H$, i.e., (I1) $\emptyset \in \mathcal{I}_h$ and (I2) $S \subseteq T \in \mathcal{I}_h$ implies $S \in \mathcal{I}_h$. Here, $\mathcal{I}_h$ is called an *independence family*. We say that $h \in H$ is *acceptable* to $d \in D$ if $h \succ_d \emptyset$. In addition, $D' \subseteq D$ is said to be *feasible* to $h \in H$ if $D' \in \mathcal{I}_h$.

Note that a pair of utility u_h and constraint $\mathcal{I}_h$ can be represented by a single (non-monotone) utility function $\hat{u}_h \colon 2^D \to \mathbb{R} \cup \{-\infty\}$ such that $\hat{u}_h(X) = u_h(X)$ if $X \in \mathcal{I}_h$ and $\hat{u}(X) = -\infty$ otherwise. However, we treat it separately to define classes of markets clearly.

For a set of utility functions $\mathcal{U}$ and a set of independence families Γ, a market $(D, H, \succ_D, u_H, \mathcal{I}_H)$ is $(\mathcal{U}, \Gamma)$-market if $u_h \in \mathcal{U}$ and $\mathcal{I}_h \in \Gamma$ for all $h \in H$. Namely, the $(\mathcal{U}, \Gamma)$-markets are those in which the utilities and the feasibility constraints are restricted to be in $\mathcal{U}$ and Γ, respectively. We analyze the properties of the $(\mathcal{U}, \Gamma)$-markets based on utility class $\mathcal{U}$ and constraint class Γ.

A *matching* is a set of pairs $\mu \subseteq \{(d, h) \in D \times H : h \succ_d \emptyset\}$ such that each doctor appears in at most one pair of μ; that is, we have $|\{(d', h') \in \mu : d' = d\}| \leq 1$ for any $d \in D$. For $d \in D$, $h \in H$, and a matching $\mu \subseteq D \times H$, we define $\mu(h) := \{d' : (d', h) \in \mu\}$ $(\subseteq D)$ and $\mu(d) := s$ $(\in H \cup \{\emptyset\})$ where $s = \emptyset$ if $\{(d', h') \in \mu : d' = d\} = \emptyset$ and $\{(d, s)\} = \{(d', h') \in \mu : d' = d\}$ otherwise.

We call matching μ *feasible* if $\mu(h) \in \mathcal{I}_h$ for all $h \in H$. Given a matching μ and a real $\alpha \geq 1$, a set of doctors $D' \in \mathcal{I}_h$ is an *α-blocking coalition* for hospital h if (i) $h \succeq_d \mu(d)$ for any $d \in D'$ and (ii) $u_h(D') > \alpha \cdot u_h(\mu(h))$.[2] We then obtain a stability concept.

Definition 2.1 (Arkin et al. [3]). A feasible matching μ is *α-stable* if there exists no α-blocking coalition.

Note that 1-stability is equivalent to the standard stability concept. As we will see in Examples 2.2 and 2.3, 1-stable matching may not exist in general. Intuitively, α-stability means that the hospitals are only willing to change the assignments for a multiplicative improvement of α. This idea regards the value of α as a switching cost for the hospitals.

2.1 Classes of Utilities and Constraints

Here, we formally describe three important classes of utility functions: *cardinality*, *additive*, and *submodular*, which capture wide varieties of applications. We assume that utility functions are monotone and nonnegative throughout this paper. First, a utility function $u\colon 2^D \to \mathbb{R}_+$ is called *cardinality* if $u(D') = |D'|$ for all $D' \subseteq D$. Let us denote $\mathcal{U}_{\text{card}}$ as the set of cardinality utility functions. Second, it is called *additive* (or modular) if $u(D') = \sum_{d \in D'} u(d)$ holds for all $D' \subseteq D$ (where we denote $u(\{d\})$ by $u(d)$ for simplicity). Third, it is called *submodular* if $u(D') + u(D'') \geq u(D' \cup D'') + u(D' \cap D'')$ holds for all $D', D'' \subseteq D$ (see [14] for more details). As well as for the cardinality functions, we define the set of additive and submodular utilities as $\mathcal{U}_{\text{add}}$ and $\mathcal{U}_{\text{sub}}$, respectively. Here, we have $\mathcal{U}_{\text{card}} \subsetneq \mathcal{U}_{\text{add}} \subsetneq \mathcal{U}_{\text{sub}}$.

Next, we formally define three classes of constraints: *capacity*, *matroid intersection*, and *multidimensional knapsack*. An independence system $(D, \mathcal{I})$ represents a *capacity* constraint of rank r if $\mathcal{I} = \{D' \subseteq D : |D'| \leq r\}$. We define the set of independence families that represent rank r capacities as $\Gamma^{(r)}_{\text{cap}}$. Also, we denote Γ_{cap} as $\bigcup_{r \in \mathbb{Z}_+} \Gamma^{(r)}_{\text{cap}}$. This class represents a standard matching model with maximum quotas.

An independence system $(D, \mathcal{I})$ is called *matroid* if, for $D', D'' \in \mathcal{I}$, $|D'| < |D''|$ implies the existence of $d \in D'' \setminus D'$ such that $D' \cup \{d\} \in \mathcal{I}$. Moreover, it is called *k-matroid intersection* if there exist k matroids $(D, \mathcal{I}^1), \ldots, (D, \mathcal{I}^k)$ such that $\mathcal{I} = \bigcap_{i \in [k]} \mathcal{I}^i$, where $[k]$ denotes set $\{1, \ldots, k\}$. We denote the set of independence families of the k-matroid intersection as $\Gamma^{(k)}_{\text{mat}}$.

Note that we have $\Gamma_{\text{cap}} \subsetneq \Gamma^{(1)}_{\text{mat}} \subsetneq \Gamma^{(2)}_{\text{mat}} \subsetneq \cdots$. We assume that each independence system $(D, \mathcal{I})$ in $\Gamma^{(k)}_{\text{mat}}$ is represented by $\bigcap_{i \in [k]} \mathcal{I}^i$ with matroids $(D, \mathcal{I}^i)$ $(i \in [k])$, and every $\mathcal{I}^i$ $(i \in [k])$ is given by a compact representation. For more details on matroids, see, e.g., [32].

Furthermore, for a natural number ρ and a positive real ϵ, the set of *ρ-dimensional knapsack with ϵ-slack* $\Gamma^{(\rho,\epsilon)}_{\text{knap}}$ is defined as the set of independence families $\mathcal{I}$ that can be represented as

$$\mathcal{I} = \left\{D' \subseteq D : \textstyle\sum_{d \in D'} w(d, i) \leq 1 \text{ for all } i \in [\rho]\right\}$$

with weights $w(d, i) \in [0, 1 - \epsilon]$ for each $d \in D$ and $i \in [\rho]$, where $[\rho] = \{1, \ldots, \rho\}$. We assume that independence systems $(D, \mathcal{I})$ in $\Gamma^{(\rho,\epsilon)}_{\text{knap}}$ are given by weights.

Note that every independence system can be represented by a matroid intersection and a multidimensional knapsack. The representability of matroid intersection is not stronger than that of multidimensional knapsack, and vice versa. Formally, $\Gamma^{(1)}_{\text{mat}} \not\subseteq \Gamma^{(\rho,\epsilon)}_{\text{knap}}$ and $\Gamma^{(1,\epsilon)}_{\text{knap}} \not\subseteq \Gamma^{(k)}_{\text{mat}}$ for any positive integers ρ and k and any nonnegative real $\epsilon < 1$.

For an independence system $(D, \mathcal{I})$ and a subset $A \subseteq D$, the *restriction* of $(D, \mathcal{I})$ to A is defined as $\mathcal{I}|A := \{X : A \supseteq X \in \mathcal{I}\}$. In this paper, we only consider a constraint class Γ that is closed under the restriction, i.e., $\mathcal{I} \in \Gamma$ implies $\mathcal{I}|D' \in \Gamma$ for all $D' \subseteq D$. We remark that $\Gamma^{(r)}_{\text{cap}}$, $\Gamma^{(k)}_{\text{mat}}$, and $\Gamma^{(\rho,\epsilon)}_{\text{knap}}$ satisfy the condition.

2.2 Applications

This section illustrates several existing and critical situations raised in the literature of matchings with constraints and describes how our constraint representation (the feasible subsets of doctors) is reduced to such situations.

Type-specific quotas One of the simplest examples of the feasibility family is *type-specific quotas*, in which doctors are partitioned based on their types, and each hospital has type-specific quotas in addition to its capacity [1]. Fix hospital h and suppose that $(D_t)_{t \in T}$ is the partition of doctors with types T, i.e., $\bigcup_{t \in T} D_t = D$ and $D_t \cap D_{t'} = \emptyset$ for all $t, t' \in T$ with $t \neq t'$. Let $q \in \mathbb{Z}_+$ be the capacity of h, and let $q_t \in \mathbb{Z}_+$ be the quota for type $t \in T$. Then D' $(\subseteq D)$ belongs to $\mathcal{I}_h$ if and only if $|D'| \leq q$ and $|D' \cap D_t| \leq q_t$ for every $t \in T$. In this case, $(D, \mathcal{I}_h)$ is a matroid, and if the utilities are additive, a 1-stable matching always exists and can be found efficiently. If the utilities are submodular, the matching no longer exists (see Example 2.3). However, we reveal that a 4-stable matching always exists and is efficiently found (Theorem 5.4).

Overlapping types Here, we generalize the type-specific quotas to those where each doctor can simultaneously belong to multiple types [28]. Fix a hospital h, let T be the set of types and let $D_t \subseteq D$ be the set of type t $(\in T)$ doctors. Here, $D_t \cap D_{t'}$ may not be empty even if $t \neq t'$. In addition, let $q \in \mathbb{Z}_+$ be the capacity and $q_t \in \mathbb{Z}_+$ be the quota for type $t \in T$. Then D' $(\subseteq D)$ belongs to $\mathcal{I}_h$ if and only if $|D'| \leq q$ and $|D' \cap D_t| \leq q_t$ for every $t \in T$. In this case, $(D, \mathcal{I}_h)$ is a $|T|$-matroid intersection. Kurata et al. [28] treated quotas as soft constraints that can be violated and found a quasi-stable matching in a different manner. To the best of our knowledge, we are the first to treat them as hard constraints that should be precisely satisfied and for finding an approximately stable matching.

Budget constraints Under budget constraints, one side (a firm or hospital) can make monetary transfers (offer wages) to the other (a worker or doctor), and each hospital has a *fixed budget*; that is, the total amount of wages allocated by each hospital to doctors is constrained [2, 23, 24]. Let $w^h(d)$ be the offered wage from hospital h to doctor d, and let b_h be its budget. Then constraint $\mathcal{I}_h$ is defined as $\mathcal{I} = \{D' \subseteq D : \sum_{d \in D'} w^h(d) \leq b^h\}$ and becomes a 1-dimensional knapsack. In fact, we have $\mathcal{I}_h \in \Gamma^{(1,\epsilon)}_{\text{knap}}$ with $\epsilon = 1 - \max_{d \in D} w^h(d)/b^h$. Kawase and Iwasaki [24] considered

[2] Although the second condition can also be defined in an additive manner: $u(X'') > u(X'_h) + \alpha$, it is inappropriate since it does not satisfy scale invariance. For example, when a market has no α-stable matching in the additive sense, a market with the hospitals' utilities that are multiplied by 100 has no 100α-stable matching.

up to the additive utility case and we disentangled the submodular utility case.

Refugee match (multiple resource constraints) In refugee resettlement, different refugee families require such various services as school seats, hospital beds, slots in language classes, and employment training programs [9]. Suppose that the set of services is Σ and the capacity of h (local areas in this context) is b_s^h for each $s \in \Sigma$. In addition, each doctor (refugee family) d needs $w^h(d,s)$ units of service $s \in \Sigma$. Then, the feasibility constraint of hospital h is defined: $\mathcal{I}_h = \{D' \subseteq D : \sum_{d \in D'} w^h(d,s) \leq b_s^h \ (\forall s \in \Sigma)\}$. In this case, the constraint is a $|\Sigma|$-dimensional knapsack. In fact, we have $\mathcal{I}_h \in \Gamma_{\text{knap}}^{(|\Sigma|,\epsilon_h)}$ with $\epsilon_h = 1 - \max_{s\in\Sigma} \max_{d\in D} \frac{w^h(d,s)}{b_s^h}$.

Separating conflicting groups In some applications, the authority should not allocate different types of individuals to the same place. For example, in refugee resettlement, refugees from conflicting religious or ethnic groups should be separated [22]. Formally, there exists a partition of the doctors $D = \bigcup_{t\in T} D_t$ with an index set T such that D' $(\subseteq D)$ is in $\mathcal{I}_h$ if and only if $D' \subseteq D_t$ for some $t \in T$. Then, we have $\mathcal{I}_h \in \Gamma_{\text{mat}}^{(k)}$ where $k = \max_{t\in T} |D_t|$. These constraints are different from those for overlapping types, although both are represented by k-matroid intersection constraints.

2.3 Markets without Stable Matchings

Let us show a market may not have 1-stable matchings, even when the utilities are cardinality.

Example 2.2. Consider a market with four doctors $D = \{d_1, d_2, d_3, d_4\}$ and two hospitals $H = \{h_1, h_2\}$. The preferences of the doctors are $h_1 \succ_{d_i} h_2 \succ_{d_i} \emptyset$ for $i = 1,2$ and $h_2 \succ_{d_i} h_1 \succ_{d_i} \emptyset$ for $i = 3,4$. Suppose that each hospital has a cardinality utility. The feasibility families are $\mathcal{I}_{h_1} = 2^{\{d_1,d_3\}} \cup 2^{\{d_2,d_4\}}$ and $\mathcal{I}_{h_2} = 2^{\{d_1,d_4\}} \cup 2^{\{d_2,d_3\}}$. Then, it is straightforward to see that, for $1 \leq \alpha < 2$, this market has no α-stable matching.

Let us remark that the independence systems $(D, \mathcal{I}_{h_1})$ and $(D, \mathcal{I}_{h_2})$ can be represented by a 2-matroid intersection and a 2-dimensional knapsack with ϵ $(< 1/2)$. For example, $\mathcal{I}_{h_1}$ is in $\Gamma_{\text{mat}}^{(2)}$ because $\mathcal{I}_{h_1} = \mathcal{I}^1 \cap \mathcal{I}^2$ for

$$\mathcal{I}^1 = \left\{\hat{D} \subseteq D : \begin{smallmatrix} |\hat{D}\cap\{d_1,d_2\}|\leq 1, \\ |\hat{D}\cap\{d_3,d_4\}|\leq 1 \end{smallmatrix}\right\} \text{ and } \mathcal{I}^2 = \left\{\hat{D} \subseteq D : \begin{smallmatrix} |\hat{D}\cap\{d_1,d_4\}|\leq 1, \\ |\hat{D}\cap\{d_2,d_3\}|\leq 1 \end{smallmatrix}\right\}.$$

Further, $\mathcal{I}_{h_1}$ is in $\Gamma_{\text{knap}}^{(2,\epsilon)}$ because it is represented by the following weights:

$$\begin{array}{llll} w(d_1,1) = 1-\epsilon, & w(d_2,1) = 1/2, & w(d_3,1) = 0, & w(d_4,1) = 1/2, \\ w(d_1,2) = 0, & w(d_2,2) = 1/2, & w(d_3,2) = 1-\epsilon, & w(d_4,2) = 1/2. \end{array}$$

Moreover, if hospitals' utilities are submodular, a market fails to have 1-stable matchings even under capacity constraints.

Example 2.3. Consider a market with four doctors $D = \{d_1, d_2, d_3, d_4\}$ and two hospitals $H = \{h_1, h_2\}$. The preferences of the doctors are $h_1 \succ_{d_i} \emptyset \succ_{d_i} h_2$ for $i = 1,2$, $h_2 \succ_{d_3} h_1 \succ_{d_3} \emptyset$, and $h_1 \succ_{d_4} h_2 \succ_{d_4} \emptyset$. Suppose the $\mathcal{I}_{h_1}$ and $\mathcal{I}_{h_2}$ are capacity constraints of rank 2 and rank 1, respectively. Let u_{h_1} be a submodular utility such that $u_{h_1}(D') = \sum_{e\in\bigcup_{d_i\in D'} A_i} w(e)$ where $A_1 = \{a_1, a_3\}$, $A_2 = \{a_2, a_4\}$, $A_3 = \{a_3, a_4, a_5\}$, $A_4 = \{a_1, a_2\}$, $w(a_1) = w(a_2) = w(a_5) = 4$, and $w(a_3) = w(a_4) = \sqrt{17} - 1$. Here, u_{h_1} is clearly submodular since it is a weighted-coverage function. Let u_{h_2} be an additive utility such that $u_{h_2}(d_3) = 1$ and $u_{h_2}(d_4) = 2$. Then, it is straightforward to see that, there exists no $(1+\sqrt{17})/4$ (≈ 1.28)-stable matching in this market.

3 CHECKING THE STABILITY OF A GIVEN MATCHING

In this section, we discuss the computational complexity of checking the α-stability of a given matching. We are going to prove that the problem is equivalent to computing an offline *packing problem* which finds a set $X \in \mathcal{I}$ that maximizes a given utility $u(X)$. Note that u and $\mathcal{I}$ are given from $\mathcal{U}$ and Γ, respectively. Formally, we call it the $(\mathcal{U}, \Gamma)$-packing problem, which corresponds with a $(\mathcal{U}, \Gamma)$-market.

Theorem 3.1. *Fix a set of utility functions $\mathcal{U}$ and a set of independence families Γ. If the $(\mathcal{U}, \Gamma)$-packing problem is solvable in polynomial time, then the α-stability of a given matching in a $(\mathcal{U}, \Gamma)$-market can be checked in polynomial time for any α (≥ 1). Moreover, the problem of deciding whether a given solution is α-approximate to an instance of the $(\mathcal{U}, \Gamma)$-packing problem is polynomial-time reducible to the problem of deciding whether a given matching is α-stable in a $(\mathcal{U}, \Gamma)$-market.*

Proof. We first prove the former part. Let $(D, H, \succ_D, u_H, \mathcal{I}_H)$ be a $(\mathcal{U}, \Gamma)$-market and let $\mu \subseteq D \times H$ be a matching. Then μ is α-stable if and only if

$$\alpha \cdot u(\mu(h)) \geq \max\{u(D') : D' \in \mathcal{I}_h|D_h\}$$

for all $h \in H$, where $D_h := \{d \in D : h \succeq_d \mu(d)\}$ and $\mathcal{I}_h|D_h$ is the restriction $\mathcal{I}_h$ to D_h. The right-hand side value is computed in polynomial time if the corresponding $(\mathcal{U}, \Gamma)$-packing problem is solvable in polynomial time. Thus, the α-stability is checked efficiently.

Next, we give a reduction to prove the latter part. For an instance $(D, u, \mathcal{I})$ of the $(\mathcal{U}, \Gamma)$-packing problem, let us consider a $(\mathcal{U}, \Gamma)$-market with doctors D and one hospital $H = \{h^*\}$. Suppose that $h^* \succ_d \emptyset$ for all $d \in D$, $u_{h^*} = u$, and $\mathcal{I}_{h^*} = \mathcal{I}$. We reduce the $(\mathcal{U}, \Gamma)$-packing instance $(D, u, \mathcal{I})$ to the $(\mathcal{U}, \Gamma)$-market $(D, H, \succ_D, u_H, \mathcal{I}_H)$. Then, a matching μ is α-stable if and only if $u_{h^*}(\mu(h^*))$ is an α-approximation of $\max\{u(D') : D' \in \mathcal{I}_{h^*}\}$. Thus the claim holds. □

The theorem enables us to access the proficiency of packing problems. For example, since the $(\mathcal{U}_{\text{add}}, \Gamma_{\text{mat}}^{(2)})$-packing problem (i.e., the weighted matroid intersection problem) is solvable in polynomial time [12], one can efficiently check the α-stability of a given matching in a $(\mathcal{U}_{\text{add}}, \Gamma_{\text{mat}}^{(2)})$-market. The book of Garey and Johnson [15] presents the coNP-completeness of several packing problems. They notify us that checking the stability of a matching is coNP-complete in the corresponding markets, summarized in Table 1 (a).

4 HARDNESS OF COMPUTING A STABLE MATCHING

In this section, we discuss the negative side of computing an α-stable matching. Kojima et al. [26] reveals that we can efficiently find a 1-stable matching for any $(\mathcal{U}_{\text{add}}, \Gamma_{\text{mat}}^{(1)})$-market and Kawase and

Iwasaki [23] proves the same for $(\mathcal{U}_{\text{card}}, \Gamma^{(1,0)}_{\text{knap}})$-market. In general, the existence problems we consider belong to Σ_2^{P}, since *yes*-instance can be verified by checking the stability of a guessed 1-stable matching with the NP-oracle.

We can say that it is NP-hard to find (or determine the nonexistence of) an α-stable matching in a $(\mathcal{U}, \Gamma)$-market if it is NP-hard to find an α-approximate solution to a $(\mathcal{U}, \Gamma)$-packing instance, by applying the similar argument in Theorem 3.1. Furthermore, we can conclude that the existence problem for the hard cases are all Σ_2^{P}-complete. Note that the Σ_2^{P}-completeness for the $(\mathcal{U}_{\text{add}}, \Gamma^{(1)}_{\text{knap}})$-markets has been shown by Hamada et al. [17]. We prove the hardness for the other cases by reductions from the **∃∀3DM** or **∃∀SubsetSum**, which are Σ_2^{P}-complete [5, 30].

∃∀3DM We are given three disjoint sets X_1, X_2, X_3 of the same cardinality, and two disjoint subsets $S^\exists, S^\forall \subseteq X_1 \times X_2 \times X_3$. Our task is to determine whether there exists $T^\exists \subseteq S^\exists$ so that, for any $T^\forall \subseteq S^\forall$, $T^\exists \cup T^\forall$ is not a perfect matching, i.e., $|\{M \in T^\exists \cup T^\forall \,:\, e \in M\}| \neq 1$ for some $e \in X_1 \cup X_2 \cup X_3$.

∃∀SubsetSum We are given two disjoint sets $S^\exists, S^\forall$ with weights $a\colon S^\exists \cup S^\forall \to \mathbb{Z}_+$, and an integer q. Our task is to determine whether there exists $T^\exists \subseteq S^\exists$ so that $\sum_{e \in T^\exists \cup T^\forall} a(e) \neq q$ for any $T^\forall \subseteq S^\forall$.

We show Theorems 4.1 and 4.2 by reductions from **∃∀3DM** and Theorem 4.3 by a reduction from **∃∀SubsetSum**.

Theorem 4.1. *It is Σ_2^{P}-hard to decide whether a given $(\mathcal{U}_{\text{sub}}, \Gamma_{\text{cap}})$-market has a 1-stable matching.*

Theorem 4.2. *It is Σ_2^{P}-hard to decide whether a given $(\mathcal{U}_{\text{card}}, \Gamma^{(3)}_{\text{mat}})$-market has a 1-stable matching.*

Theorem 4.3. *It is Σ_2^{P}-hard to decide whether a given $(\mathcal{U}_{\text{card}}, \Gamma^{(2)}_{\text{knap}})$-market has a 1-stable matching.*

Proof. Here we only provide a proof for Theorem 4.1. Proofs for Theorems 4.2 and 4.3 can be obtained in similar ways.

We give a reduction from **∃∀3DM**. Suppose that disjoint sets $S^\exists, S^\forall \subseteq X_1 \times X_2 \times X_3$ are given as an instance of **∃∀3DM**.

We construct a $(\mathcal{U}_{\text{sub}}, \Gamma_{\text{cap}})$-market that has a 1-stable matching if and only if the given instance is a *yes*-instance. Consider a market $(D, H, \succ_D, u_H, \mathcal{I}_H)$ with $D = \{d_1, d_2, d_3, d_4\} \cup \{d^e\}_{e \in S^\exists \cup S^\forall}$ and $H = \{h^*, h_1, h_2\} \cup \{h^e\}_{e \in S^\exists}$. The doctors' preferences over the acceptable hospitals are given as:

- $d^e : h^e, h^*$ $(e \in S^\exists)$,
- $d^e : h^*$ $(e \in S^\forall)$,
- $d_1 : h^*, h_1$,
- $d_2 : h_1$,
- $d_3 : h_2, h_1$,
- $d_4 : h_1, h_2$.

Here, and henceforth, preference lists are ordered from left to right in decreasing order of preference. The feasibility constraint is the capacity constraint of rank 1 for h_2 and h^e $(e \in S^\forall)$, rank 2 for h_1, and rank $|X_1|$ $(= |X_2| = |X_3|)$ for h^*. Suppose that u_{h_1} and u_{h_2} are the same as Example 2.3 (the utilities of unmatchable doctors are considered to be zero), and u_{h^e} is identically zero $(\forall e \in S^\exists)$. In addition, for $X \subseteq D$, we define

$$u_{h^*}(X) = \left|\bigcup_{d^{(x_1,x_2,x_3)} \in X \setminus \{d_1,d_2,d_3,d_4\}} \{x_1, x_2, x_3\}\right| + 2|X \cap \{d_1\}|.$$

This is a weighted-coverage function and hence submodular.

Consider the case when the instance is a *yes*-instance. We show that there exists a 1-stable matching in this case. Let $\hat{T}^\exists \subseteq S^\exists$ be a certificate of the instance. Without loss of generality, we may assume that $|\hat{T}^\exists| \leq |X_1| - 1$, $|\hat{T}^\exists| + |S^\forall| \geq |X_1|$, and $\hat{T}^\exists$ is a matching. Let us define

$$\hat{T}^\forall \in \arg\max\left\{u_{h^*}\left(\{d^e\}_{e \in \hat{T}^\exists \cup T^\forall}\right) \,:\, |\hat{T}^\exists| + |T^\forall| = |X_1|,\ T^\forall \subseteq S^\forall\right\}.$$

Then, $\hat{T}^\exists \cup \hat{T}^\forall$ is not a matching, and hence there exists $e^* \in \hat{T}^\forall$ such that

$$u_{h^*}\left(\{d^e\}_{e \in \hat{T}^\exists} \cup \{d^e\}_{e \in \hat{T}^\forall}\right) \leq u_{h^*}\left(\{d^e\}_{e \in \hat{T}^\exists} \cup \{d^e\}_{e \in \hat{T}^\forall \setminus \{e^*\}}\right) + 2.$$

Thus, the matching

$$\begin{aligned}\mu^* = &\{(d^e, h^*)\}_{e \in \hat{T}^\exists} \cup \{(d^e, h^*)\}_{e \in \hat{T}^\forall \setminus \{e^*\}} \\ &\cup \{(d^e, h^e)\}_{e \in S^\exists \setminus \hat{T}^\exists} \cup \{(d_1, h^*), (d_2, h_1), (d_3, h_2), (d_4, h_1)\}\end{aligned}$$

is 1-stable.

Conversely, consider the case when the instance is a *no*-instance. We show that there exists no 1-stable matching in this case. Suppose to the contrary that μ is a 1-stable matching. Then, μ must contain (d_1, h^*) since the submarket induced by $\{d_1, d_2, d_3, d_4\}$ and $\{h_1, h_2\}$ is equivalent to Example 2.3 (which has no 1-stable matching). Hence, $u_{h^*}(\mu(h^*)) \leq 3|X_1| - 1$. Let $\tilde{T}^\exists = \{e \in S^\exists \,:\, d^e \in \mu(h^*)\}$. Since the instance is a *no*-instance, there exists $\tilde{T}^\forall \subseteq S^\forall$ such that $|\tilde{T}^\exists| + |\tilde{T}^\forall| = |X_1|$ and $\tilde{T}^\exists \cup \tilde{T}^\forall$ is a matching. Thus, we have $u_{h^*}\left(\{d^e\}_{e \in \tilde{T}^\exists} \cup \{d^e\}_{e \in \tilde{T}^\forall}\right) = 3|X_1|$, which implies that $\tilde{T}^\exists \cup \tilde{T}^\forall$ is a 1-blocking coalition for h^*. □

Now the remaining cases to be treated are $(\mathcal{U}_{\text{card}}, \Gamma^{(2)}_{\text{mat}})$- and $(\mathcal{U}_{\text{add}}, \Gamma^{(2)}_{\text{mat}})$-markets. For a $(\mathcal{U}_{\text{add}}, \Gamma^{(2)}_{\text{mat}})$-market, although the α-stability (especially 1-stability) of a given matching can be checked in polynomial time by Theorem 3.1, the existence problem becomes NP-complete, even if utilities are restricted to cardinality. We prove the NP-hardness by a reduction from **DisjointMatching**, which is NP-complete [13].

DisjointMatching We are given two bipartite graphs, $(S, T; A_1)$ and $(S, T; A_2)$ with $|S| = |T|$, and our task is to determine whether perfect matchings $M_1 \subseteq A_1$ and $M_2 \subseteq A_2$ exist such that $M_1 \cap M_2 = \emptyset$.

Theorem 4.4. *It is NP-complete to decide whether there exists a 1-stable matching in a given $(\mathcal{U}_{\text{card}}, \Gamma^{(2)}_{\text{mat}})$-market.*

Proof. The problem is in NP by Theorem 3.1 since the $(\mathcal{U}_{\text{add}}, \Gamma^{(2)}_{\text{mat}})$-packing problem is solvable in polynomial time [12].

To prove NP-hardness of the problem, we give a reduction from **DisjointMatching**. Let $(S, T; A_1)$ and $(S, T; A_2)$ be the bipartite graphs of a given disjoint matching instance and let $A_1 \cup A_2 = \{a_1, \ldots, a_\ell\}$. Without loss of generality, we assume $|A_1 \cup A_2| \geq 2|S|$.

We construct a market that has a 1-stable matching if and only if the given instance has disjoint perfect matchings. Consider a market $(D, H, \succ_D, u_H, \mathcal{I}_H)$ with 4ℓ doctors $D = \bigcup_{k=1}^{\ell} \{d_1^k, d_2^k, d_3^k, d_4^k\}$ and $2\ell + 3$ hospitals $H = \{h_1, h_2, h_3\} \cup \bigcup_{k=1}^{\ell} \{h_1^k, h_2^k\}$. The doctors' preferences over the acceptable hospitals are given as:

- $d_1^k : h_1, h_2, h_3, h_k^1, h_k^2$ $(k \in [\ell])$,
- $d_2^k : h_1^k, h_2^k$ $(k \in [\ell])$,
- $d_3^k : h_2^k, h_1^k$ $(k \in [\ell])$,

- $d_4^k : h_2^k, h_1^k$ $(k \in [\ell])$.

Suppose that each hospital has the cardinality utility. We equate each doctor $d_1^k \in D$ with edge $a_k \in A_1 \cup A_2$. Then, the feasibility constraint for each hospital is defined:

- $\mathcal{I}_{h_1} = \{D' \subseteq A_1 : D' \text{ is a matching in } (S, T; A_1)\}$,
- $\mathcal{I}_{h_2} = \{D' \subseteq A_2 : D' \text{ is a matching in } (S, T; A_2)\}$,
- $\mathcal{I}_{h_3} = \{D' \subseteq A_1 \cup A_2 : |D'| \le |A_1 \cup A_2| - 2|S|\}$,
- $\mathcal{I}_{h_1^k} = 2^{\{d_1^k, d_3^k\}} \cup 2^{\{d_2^k, d_4^k\}}$ $(k \in [\ell])$,
- $\mathcal{I}_{h_2^k} = 2^{\{d_1^k, d_4^k\}} \cup 2^{\{d_2^k, d_3^k\}}$ $(k \in [\ell])$.

Note that each feasible family can be represented as an intersection of two matroids.

Consider the case when the instance has disjoint perfect matchings. Let $M_1 \subseteq A_1$ and $M_2 \subseteq A_2$ be the matchings. Then

$$\mu = \{(d, h_1)\}_{d \in M_1} \cup \{(d, h_2)\}_{d \in M_2} \cup \{(d, h_3)\}_{d \in (A_1 \cup A_2) \setminus (M_1 \cup M_2)} \cup \bigcup_{k=1}^{\ell} \{(d_2^k, h_1^k), (d_3^k, h_2^k), (d_4^k, h_1^k)\}$$

is a 1-stable matching.

Conversely, consider the case when the instance has no disjoint perfect matchings. Let μ be a feasible matching. Then, there exists a doctor d_1^k such that $\mu(d_1^k) \notin \{h_1, h_2, h_3\}$. In this case, μ is not 1-stable since doctors $\{d_1^k, d_2^k, d_3^k, d_4^k\}$ and hospitals $\{h_1^k, h_2^k\}$ form the same market as in Example 2.2. □

5 APPROXIMABILITY OF STABLE MATCHINGS

To deal with the nonexistence or the hardness of stable matchings, we focus on an approximately stable matching. This section pays attention to an online version of packing problems, i.e., *online packing problems* (with cancellation) and incorporates the proficiency into a variant of GDA in such a manner that choice functions of hospitals are replaced with an online packing algorithm. We establish a framework so that the bounds of the algorithms become consistent with how much stability is violated. Note that Kawase and Iwasaki [24] apply a similar idea for $(\mathcal{U}_{\text{add}}, \Gamma_{\text{knap}}^{(1,\epsilon)})$-markets.

In what follows, we consider algorithms that take a market as input and yield an approximately stable matching as output. An *algorithm* is called *α-stable* if it always produces an α-stable matching for a certain α.

5.1 Online Packing Problem

Let us briefly introduce an online packing problem, which is a generalization of several online problems such as an online removable knapsack problem [19]. Its instance consists of a set of elements $D = \{d_1, \ldots, d_n\}$, a utility function $u \colon 2^D \to \mathbb{R}_+$, and a feasibility family $\mathcal{I} \subseteq 2^D$. We assume that u is monotone and $\mathcal{I}$ is an independence family. Elements in D are given to an online algorithm one by one in an unknown order. When an element is presented, the algorithm must accept or reject it immediately without knowledge about the ordering of future elements. Although accepted elements can be canceled, the elements that are once rejected (or canceled) can never be recovered. The set of selected elements must be feasible in each round. Suppose that elements are given according to order σ, which is a bijection from $[n]$ to D. We denote by $\text{Alg}(\sigma(1), \ldots, \sigma(i))$ the set of selected elements at the end of the ith round, in which $\sigma(i) \in D$ arrives. We denote it as $\text{Alg}(\sigma, i)$ for brevity. Then we have $\text{Alg}(\sigma, 0) = \emptyset$, and $\text{Alg}(\sigma, i-1) \cup \{\sigma(i)\} \supseteq \text{Alg}(\sigma, i) \in \mathcal{I}$ holds for every $i \in [n]$. In addition, for $i \in [n]$ and the two orders of elements σ and τ, equality $\text{Alg}(\sigma, i) = \text{Alg}(\tau, i)$ holds if $\sigma(j) = \tau(j)$ for any $j \in [i]$. Our task is to maximize value $u(\text{Alg}(\sigma, i))$ for unknown order σ and i $(\in [n])$.

The performance of an online algorithm is measured by the *competitive ratio*. Denote

$$\text{OPT}(\sigma, i) \in \arg\max\{u(S) : S \in \mathcal{I}|\{\sigma(1), \ldots, \sigma(i)\}\}.$$

Online algorithm Alg is called *α-competitive* $(\alpha \ge 1)$ for $(D, u, \mathcal{I})$ if

$$\alpha \cdot u(\text{Alg}(\sigma, i)) \ge u(\text{OPT}(\sigma, i))$$

for any σ and i. Also, Alg is called α-competitive for an online $(\mathcal{U}, \Gamma)$-packing problem if it is α-competitive for any instance $(D, u, \mathcal{I})$ with $u \in \mathcal{U}$ and $\mathcal{I} \in \Gamma$.

Algorithm 1: Generalized DA algorithm

```
input: D, H, (≻_d)_{d∈D}, (Alg_h)_{h∈H}    output: matching μ
1  μ ← ∅, R_d ← H (∀d ∈ D);
2  L ← {d ∈ D : max_{≻_d}(R_d ∪ {∅}) ≠ ∅};
3  a^h ← () for all h ∈ H;
4  while L ≠ ∅ do
5      pick d ∈ L arbitrarily and let h ← max_{≻_d} R_d;
6      append d to the end of a^h;
7      μ ← {(d', h') ∈ μ : h' ≠ h} ∪ Alg_h(a^h);
8      R_d ← R_d \ {h};
9      L ← {d ∈ D : max_{≻_d}(R_d ∪ {μ(d)}) ≠ μ(d)};
10 return μ;
```

5.2 Generalized Deferred Acceptance Algorithm

We use a modified version of the generalized DA algorithm, which is formally described in Algorithm 1. In GDA, each doctor is initialized to be unmatched. Then an unmatched doctor makes a proposal to her most preferred hospital h that has not rejected her yet. Let $\boldsymbol{a}^h$ be the ordered list of proposed doctors to h. Then it chooses a set of doctors according to the output of online algorithm $\text{Alg}_h(\boldsymbol{a}^h)$. The proposal procedure continues as long as an unmatched doctor has a non-rejected acceptable hospital.

The next theorem guarantees that if Alg_h is α-competitive for each $h \in H$, then Algorithm 1 is α-stable.[3]

Theorem 5.1. *Algorithm 1 outputs an α-stable matching if* Alg_h *is α-competitive for each $h \in H$.*

Proof. We prove that the output μ is an α-stable matching by contradiction. Suppose that $D' \subseteq D$ is an α-blocking coalition for h. Then, we have $u_h(D') > \alpha \cdot u_h(\mu(h)) = \alpha \cdot u_h(\text{Alg}_h(\boldsymbol{a}^h))$ and $h \succeq_d \mu(d)$ for all $d \in D'$. By the definition of the algorithm, $h \succeq_d \mu(d)$ implies that d is in $\boldsymbol{a}^h$. Hence, we have $\alpha \cdot u_h(\text{Alg}_h(\boldsymbol{a}^h)) \ge u_h(\text{OPT}_h(\boldsymbol{a}^h)) \ge u_h(D')$ since Alg is α-competitive. This is a contradiction. □

[3]Although the output of Algorithm 1 depends on the order of doctors selected in Line 5, this claim holds regardless of the order.

This theorem assures that if there exists an online packing algorithm in a setting, we can construct a stable algorithm in the corresponding market with it.

Let us first apply a greedy algorithm to a matching problem for k-matroid intersection constraints: Start from the empty solution and add an element to the current solution if and only if its addition preserves feasibility. This greedy algorithm can be implemented in the online setting as shown in Algorithm 2. If the utilities are cardinality, it is a k-competitive algorithm [20, 27]. By Theorem 5.1, we obtain the following corollary.

Corollary 5.2. *For the $(\mathcal{U}_{\mathrm{card}}, \Gamma^{(k)}_{\mathrm{mat}})$-markets, there exists a k-stable algorithm.*[4]

Algorithm 2:

input: $\sigma(1), \ldots, \sigma(i)$ **output:** $\mathrm{Alg}(\sigma(1), \ldots, \sigma(i))$
if $i = 0$ **then return** $\emptyset$;
let $Y \leftarrow \mathrm{Alg}(\sigma(1), \ldots, \sigma(i-1)) \cup \{\sigma(i)\}$;
if $Y \in \mathcal{I}$ **then return** Y;
else return $\mathrm{Alg}(\sigma(1), \ldots, \sigma(i-1))$;

However, we require more sophisticated packing algorithms in the online setting to handle the additive or submodular utility case.

For the additive case, a $(\sqrt{k} + \sqrt{k-1})^2$-competitive algorithm was given by Ashwinkumar [4]. For the submodular case, a $4k$-competitive algorithm was given by Chakrabarti and Kale [6]. Thus, we obtain the following corollaries:

Corollary 5.3. *For the $(\mathcal{U}_{\mathrm{add}}, \Gamma^{(k)}_{\mathrm{mat}})$-markets, there exists a $(\sqrt{k} + \sqrt{k-1})^2$-stable algorithm.*

Corollary 5.4. *For the $(\mathcal{U}_{\mathrm{sub}}, \Gamma^{(k)}_{\mathrm{mat}})$-markets, there exists a $4k$-stable matching algorithm.*

Next, let us consider ρ-dimensional knapsack constraints with ϵ-slack. To the best of our knowledge, no suitable online packing algorithm has been proposed for the cardinality or additive utility case. We develop a simple greedy algorithm, which is formally given as Algorithm 3. Intuitively, the algorithm chooses doctors according to decreasing order of utility per largest size $u(d)/\max_{i \in [\rho]} w(d, i)$. It is not difficult to see that the algorithm is ρ-competitive for the cardinality utility case and ρ/ϵ-competitive for the additive utility case.

Theorem 5.5. *Algorithm 3 is ρ- and ρ/ϵ-competitive for the online $(\mathcal{U}_{\mathrm{card}}, \Gamma^{(\rho,\epsilon)}_{\mathrm{knap}})$- and $(\mathcal{U}_{\mathrm{add}}, \Gamma^{(\rho,\epsilon)}_{\mathrm{knap}})$-packing problems, respectively.*

Accordingly, in conjunction with Theorem 5.1, we obtain the following corollaries.

Corollary 5.6. *For the $(\mathcal{U}_{\mathrm{card}}, \Gamma^{(\rho,\epsilon)}_{\mathrm{knap}})$-markets, there exists a ρ-stable algorithm.*

[4]This claim can be generalized to the case when the set of independence families is the k-*system*, which is an extension of k-matroid intersection. An independence system $(D, \mathcal{I})$ is called a k-system if for all $S \subseteq D$, the ratio of the cardinality of the largest to the smallest maximal independent subset of S is at most k.

Algorithm 3:

input: $\sigma(1), \ldots, \sigma(i)$ **output:** $\mathrm{Alg}(\sigma(1), \ldots, \sigma(i))$
if $i = 0$ **then return** $\emptyset$;
let $Y \leftarrow \mathrm{Alg}(\sigma(1), \ldots, \sigma(i-1)) \cup \{\sigma(i)\}$;
while $\sum_{d \in Y} \max_{i \in [\rho]} w(d, i) > 1$ **do**
    $Y \leftarrow Y \setminus \{a\}$ with $a \in \arg\min_{d \in Y} \frac{u(d)}{\max_{i \in [\rho]} w(d,i)}$;
return Y;

Corollary 5.7. *For the $(\mathcal{U}_{\mathrm{add}}, \Gamma^{(\rho,\epsilon)}_{\mathrm{knap}})$-markets, there exists a $\frac{\rho}{\epsilon}$-stable algorithm.*

Unfortunately, it is not easy to generalize the greedy algorithm to the submodular utility case. However, we can borrow an $O(\rho/\epsilon^2)$-competitive algorithm for the online $(\mathcal{U}_{\mathrm{sub}}, \Gamma^{(\rho,\epsilon)}_{\mathrm{knap}})$-packing problem [7] and provide the following corollary.

Corollary 5.8. *For the $(\mathcal{U}_{\mathrm{sub}}, \Gamma^{(\rho,\epsilon)}_{\mathrm{knap}})$-markets, there exists an $O(\rho/\epsilon^2)$-stable algorithm.*

6 INAPPROXIMABILITY OF STABLE MATCHINGS

In this section, we show some inapproximability results, i.e., lower bounds. As we saw in Example 2.2, for any $\epsilon \in [0, 1/2)$ and $\alpha \in [1, 2)$, there exists a $(\mathcal{U}_{\mathrm{card}}, \Gamma^{(2)}_{\mathrm{mat}} \cap \Gamma^{(2,\epsilon)}_{\mathrm{knap}})$-market not admitting α-stable matchings. Also, Kawase and Iwasaki [24] derived the lower bound for the case of additive utilities and 1-dimensional knapsack with ϵ-slack constraints. There exists a market that has no α-stable matching with $\alpha < 1/\epsilon$ if $\epsilon \in [0, 1/2)$.

To fill up the remaining shown in Table 1 (c), we now provide a basis for deriving lower bounds. Unfortunately, we cannot directly apply lower bounds for online packing problems because hospitals are not offered by doctors in an arbitrary order. Hence, we consider online packing problems with a restriction on the input orderings. Roughly speaking, we partition elements D to $D_1, \ldots, D_s$ and only consider input sequences such that, if an online algorithm rejects an element in D_t for each $t \in [s]$, then a new element in D_t is given to the algorithm. In addition, we require that the $(\mathcal{U}, \Gamma)$-markets allow hospitals to have any additive utilities and the rank one capacity.

Theorem 6.1. *Suppose that $\mathcal{U}$ is a set of utilities and Γ is a set of independence families such that $\mathcal{U}_{\mathrm{add}} \subseteq \mathcal{U}$ and $\Gamma^{(1)}_{\mathrm{cap}} \subseteq \Gamma$. Let $(\hat{D}, \hat{u}, \hat{\mathcal{I}})$ be a $(\mathcal{U}, \Gamma)$-packing instance, let $D_1, \ldots, D_s$ be a partition of $\hat{D}$ with $D_t = \{d^t_1, \ldots, d^t_{r_t}\}$ for each $t \in [s]$ (where $r_t = |D_t|$), and let $q_1, \ldots, q_s$ be positive integers such that $q_t \le r_t$ for each $t \in [s]$. We define*

$$\mathrm{cl}(D') = \bigcup_{t \in [s]} \left\{ d^t_i \in D_t \ :\ |D' \cap D_t| < q_t \ \text{or}\ i \le \max_{d^t_j \in D' \cap D_t} j \right\}$$

and $\mathcal{A} = \{D' \subseteq \hat{D} \ :\ |D' \cap D_t| \le q_t \ (\forall t \in [s])\}$. Then, there exists no α-stable algorithm for the $(\mathcal{U}, \Gamma)$-markets if there exists no α-competitive algorithm for the online packing problem with restricted input sequences, i.e.,

$$\alpha \cdot \hat{u}(D') < \max \left\{ \hat{u}(D'') \ :\ D'' \in \hat{\mathcal{I}} | \mathrm{cl}(D') \right\}, \qquad (1)$$

for any $D' \in \hat{\mathcal{I}} \cap \mathcal{A}$.

Proof. Suppose that (1) holds for any $D' \in \mathcal{I} \cap \mathcal{A}$. We set doctors and hospitals as $D = \hat{D}$ and $H = \{h^*\} \cup \bigcup_{t \in [s]} \{h^t_{q_t+1}, \ldots, h^t_{r_t}\}$, respectively. Each doctor's preference over her acceptable hospitals is defined as:

- $\succ_{d^t_i} : h^*, h^t_{q_t+1}, \ldots, h^t_{r_t} \;\; (t \in [s],\ i \in [q_t])$,
- $\succ_{d^t_i} : h^t_i, h^*, h^t_{i+1}, \ldots, h^t_{r_t} \;\; (t \in [s],\ i \in [r_t] \setminus [q_t])$.

Suppose that $u_{h^*}(D') = \hat{u}(D')$ $(\forall D' \subseteq D)$ and each hospital h^t_i has an additive utility function such that $u_{h^t_i}(d^t_j) = (\alpha + 1)^{-j}$ $(t \in [s],\ i \in [r_t] \setminus [q_t],\ j \in [r_t])$. The feasibility constraint of each hospital is defined as: $\mathcal{I}_{h^*} = \hat{\mathcal{I}}$ and $\mathcal{I}_{h^t_i} = \{D' \subseteq D : |D'| \leq 1\}$ $(t \in [s],\ i \in [r_t] \setminus [q_t])$. Note that $(D, H, \succ_D, u_H, \mathcal{I}_H)$ is a $(\mathcal{U}, \Gamma)$-market.

In what follows, we claim that no α-stable matching exists in market $(D, H, \succ_D, u_H, \mathcal{I}_H)$ if (1) holds. Suppose, contrary to our claim, that μ is an α-stable matching.

We show that $h^* \succeq_d \mu(d)$ if $d \in \mathrm{cl}(\mu(h^*))$. Fix $t \in [s]$ and let $D_t \setminus \mu(h^*) = \{d^t_{\sigma(1)}, \ldots, d^t_{\sigma(\ell)}\}$ with $\ell = |D_t \setminus \mu(h^*)|$ and $\sigma(1) < \cdots < \sigma(\ell)$. It is worth mentioning that $\sigma(i) \leq q_t + i$ for all $i \in [\ell]$. By $u_{h^t_{q_t+1}}(d^t_{\sigma(1)}) > \alpha \cdot u_{h^t_{q_t+1}}(d^t_{\sigma(1)+1})$ and $\sigma(1) \leq q_t + 1$, we have $\mu(h^t_{q_t+1}) = d^t_{\sigma(1)}$. Similarly, by induction we conclude that $\mu(h^t_{q_t+i}) = d^t_{\sigma(i)}$ for all $i \in [\ell]$. Here, we have $\mu(d^t_i) \succ_{d^t_i} h^*$ if and only if $i = \sigma(i - q_t)$. Hence, we have $h^* \succeq_{d^t_i} \mu(d^t_i)$ if $\ell > r_t - q_t$ (i.e., $|\mu(h^*) \cap D_t| < q_t$) or $d^t_j \in \mu(h^*)$ for some $j \geq i$ (i.e., $i \leq \max_{d^t_j \in \mu(h^*) \cap D_t} j$).

Therefore, $D^* \in \arg\max\{u_{h^*}(D'') : D'' \in \hat{\mathcal{I}}|\mathrm{cl}(\mu(h^*))\}$ is an α-blocking pair for μ by $D^* \in \mathcal{I}_{h^*}$, $h^* \succeq_d \mu(d)$ $(\forall d \in D^*)$, and $u_{h^*}(D^*) > \alpha \cdot u_{h^*}(\mu(h^*))$, which contradicts the α-stability of μ. □

Theorem 6.1 gives us a general lower bound in (1) and it enables us to derive two novel lower bounds for k-matroid intersection and ρ-dimensional knapsack constraints.

Theorem 6.2. *For any integer $k \geq 2$ and any real $\alpha \in [1, k)$, there exists a $(\mathcal{U}_{\mathrm{add}}, \Gamma^{(k)}_{\mathrm{mat}})$-markets not admitting an α-stable matching.*

Proof. Let $D = D_1 \cup D_2$ with $D_1 = \{d^1_1, \ldots, d^1_k\}$ and $D_2 = \{d^2_1, \ldots, d^2_k\}$. In addition, let $q_1 = q_2 = 1$, $\mathcal{I} = 2^{D_1} \cup 2^{D_2}$ $(\in \Gamma^{(k)}_{\mathrm{mat}})$, and $u(D') = |D'|$ for $D' \subseteq D$.

For set $\mathcal{A}$ in Theorem 6.1, we have $\mathcal{I} \cap \mathcal{A} = \{\emptyset\} \cup \{\{d^1_i\} : i \in [k]\} \cup \{\{d^2_i\} : i \in [k]\}$. Thus, for any $D' \in \mathcal{I} \cap \mathcal{A}$, we have $u(D') = |D'| \leq 1$ and $\max\{u(D'') : D'' \in \mathcal{I}|\mathrm{cl}(D')\} = k$. Hence, by Theorem 6.1, there exists a $(\mathcal{U}_{\mathrm{add}}, \Gamma^{(k)}_{\mathrm{mat}})$-market without α-stable matchings. □

Theorem 6.3. *For any positive integer ρ and any positive real $\epsilon < 1/2$, there exists a $(\mathcal{U}_{\mathrm{add}}, \Gamma^{(\rho,\epsilon)}_{\mathrm{knap}})$-markets not admitting a $\frac{\rho}{2\epsilon}$-stable matching.*

Proof. Let $r := \lceil 1/\epsilon \rceil - 1$. We remark that $\epsilon < 1/r < 2\epsilon$. In addition, let m be an integer such that $(r\rho)^{1-1/m} > \frac{\rho}{2\epsilon}$. Consider a $(\mathcal{U}_{\mathrm{add}}, \Gamma^{(\rho,\epsilon)}_{\mathrm{knap}})$-packing problem with $mr\rho + 1$ doctors $D = D_1 \cup D_2$ where $D_1 = \{d_0\}$ and $D_2 = \{d_1 \ldots, d_{mr\rho}\}$. We define $q_1 = q_2 = 1$.

Let us partition D^2 into $D^{a,b} := \{d_{t+r(a-1)+r\rho(b-1)} : t \in [r]\}$ $(a \in [\rho]$ and $b \in [m])$. Let u be an additive utility function such that $u(d_0) = r\rho$ and $u(d_i) = (r\rho)^{b/m}$ for $d_i \in D^{a,b}$ with $a \in [\rho]$ and $b \in [m]$. The weights for dimension $\ell \in [\rho]$ are defined as:

- $w(d_0, \ell) = 1 - \epsilon$,
- $w(d_i, \ell) = 1/r \;\; (b \in [m];\ d_i \in D^{\ell,b})$, and
- $w(d_i, \ell) = 0 \;\; (a \in [\rho] \setminus \{\ell\};\ b \in [m];\ d_i \in D^{a,b})$.

For set $\mathcal{A}$ in Theorem 6.1, we have $\mathcal{I} \cap \mathcal{A} = \{D' \subseteq D : |D'| \leq 1\}$. We show that (1) holds for any $D' \in \mathcal{I} \cap \mathcal{A}$. If $D' = \emptyset$, we have $\frac{\rho}{2\epsilon} \cdot u(D') = 0 < r\rho = \max\{u(D'') : D'' \in \mathcal{I}|\mathrm{cl}(D')\}$. Hence, we can assume that $D' = \{d_i\}$. If $i = 0$, we have

$$\frac{\max\{u(D'') : D'' \in \mathcal{I}|\mathrm{cl}(D')\}}{u(D')} = \frac{u(D^{1,m} \cup \cdots \cup D^{\rho,m})}{u(\{d_0\})} = \frac{r \cdot \rho \cdot r\rho}{r\rho} = r\rho > \frac{\rho}{2\epsilon}.$$

If $1 \leq i \leq r\rho$ (i.e., $d_i \in D^{a,1}$ for some $a \in [\rho]$), we have

$$\frac{\max\{u(D'') : D'' \in \mathcal{I}|\mathrm{cl}(D')\}}{u(D')} \geq \frac{u(\{d_0\})}{u(\{d_i\})} = \frac{r\rho}{(r\rho)^{\frac{1}{m}}} > \frac{\rho}{2\epsilon}.$$

Finally, if $r\rho < i \leq mr\rho$, let $q^* := \lceil i^*/(r\rho) \rceil$ (i.e., $d_i \in D^{a,q^*}$ for some $a \in [\rho]$), and then we have

$$\frac{\max\{u(D'') : D'' \in \mathcal{I}|\mathrm{cl}(D')\}}{u(D')} \geq \frac{u(D^{1,q^*-1} \cup \cdots \cup D^{\rho,q^*-1})}{u(\{d_i\})} \geq \frac{r \cdot \rho \cdot (r\rho)^{(q^*-1)/m}}{(r\rho)^{q^*/m}} = (r\rho)^{1-1/m} > \frac{\rho}{2\epsilon}.$$

Thus, by Theorem 6.1, there exists a $(\mathcal{U}_{\mathrm{add}}, \Gamma^{(\rho,\epsilon)}_{\mathrm{knap}})$-market without $\frac{\rho}{2\epsilon}$-stable matchings. □

7 CONCLUSION

This paper examined matching with general constraints and analyzed the approximability of stable matchings. We first identified the computational complexity of checking and computing stable matchings according to hospitals' utilities and imposed constraints. Second, we established a useful framework to connect packing problems and its online algorithms with approximately stable matchings in the presence of complicated constraints. Then we successfully built a series of algorithms with good approximation ratios as a variant of GDA. Next, we provided a framework to obtain an α-stable algorithm for the $(\mathcal{U}, \Gamma)$-markets from α-competitive algorithms for the online $(\mathcal{U}, \Gamma)$-packing problem. Conversely, we also show that the nonexistence of the α-competitive algorithm for the online $(\mathcal{U}, \Gamma)$-packing problem (where the input is restricted to certain sequences) implies the nonexistence of α-stable algorithms for the $(\mathcal{U}, \Gamma)$-markets.

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