Strategyproof Mechanisms for Activity Scheduling

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ABSTRACT

Recent years have seen various designs of strategyproof mechanisms in the facility location game and the obnoxious facility game, by considering the facility as a point. In this paper, we extend that point to be an interval and study a novel activity scheduling game to schedule an activity in the time domain $[0, 1]$ based on all agents' time reports. The activity lasts for a time period of d with $0 \leq d \leq 1$, and each agent i wants his private time t_i to be within the activity duration $[y, y + d]$ or at least as close as possible. Thus his cost is the time difference between his time t_i and the activity duration $[y, y + d]$. The social cost is the summation of all agents' costs. Our objective is to choose the activity starting time y so that the mechanisms are strategyproof (truthful) and efficient. We design a mechanism outputting an optimal solution and prove it is group strategyproof. For minimizing the maximum cost, we also design a strategyproof mechanism with approximation ratio 2. In the obnoxious activity scheduling game, each agent prefers his conflict time t_i to be far away from the activity duration $[y, y + d]$. We respectively design deterministic and randomized group strategyproof mechanisms with provable approximation ratios and also show the lower bounds. Besides, for extension, we consider the cost/utility as the characteristic function and find group strategyproof mechanisms for minimizing the social cost and maximizing the social utility.

KEYWORDS

[SCCG] Cooperative games: theory & analysis; [SCCG] Social choice theory

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1 INTRODUCTION

In the literature of activity and resource scheduling (e.g., [10, 16]), the focus is to develop efficient algorithms (e.g., dynamic programming and heuristic search) in a centralized manner to meet task deadlines or improve resource utilization. There is a lack of game theoretic study or strategyproof mechanism design for the scheduler (social planner) to solicit private information from involved human agents in the activity. In this paper, we study a family of novel activity scheduling games with fixed activity duration in the normalized time domain $[0, 1]$. The activity to schedule lasts for a time period of d (e.g., two hours of a day) with $0 \leq d \leq 1$, and calls for the participation of a group of self-interested agents. The activity scheduling scenario is related to the traditional facility location games on a spatial line segment (e.g., [2, 14]), where each agent is self-interested and reports his private location information to influence the social planner's decision on the facility location. In our problem, the social planner collects all agents' private time information and wants to locate the activity in the time domain to minimize the social cost (or maximize the social utility). Our objective is to choose the activity starting time so that the mechanisms are strategyproof and efficient. The technical difference is that the activity has a time window with length d while the facility's location is just a point.

We first study the activity scheduling game whose counterpart is the facility location game. In general, $[y, y + d]$ is an activity's duration and t_i is agent i's own business time. Agent i's own business can be done during the activity in the same location, so agent i wishes t_i to be within $[y, y + d]$. There are many real-life examples for motivating such a game. For example, the activity can be a conference reception session for attendants (agents) to register quickly on the spot and the registration period (activity duration) is $[y, y + d] \subseteq [0, 1]$ by starting at time y and ending at time $y + d$. Other than the conference registration, each agent i has a personal appointment at time t_i in the same location (e.g., meeting some friend there, checking in hotel) and wants t_i to be within the activity duration $[y, y + d]$, for saving the waiting time. Here, we consider the duration of an agent i's personal appointment to be much shorter than the activity duration and thus model it as a time point t_i. If $t_i < y$, he arrives at reception at time t_i for his personal business and then waits for a time period of $y - t_i$ until he can register earliest at time y. Thus, his cost is the extra waiting time $y - t_i$. If $t_i > y + d$, he needs to arrive no latter than time $y + d$ to just catch the session and still needs to wait for at least a time period of $t_i - y - d$ for his personal business at time t_i. Therefore, the cost of an agent is $y - t_i$ if $t_i < y$; $t_i - y - d$ if $t_i > y + d$; and 0 if $y \le t_i \le y + d$. As another example, the activity can be a sales promotion of a brand in a shopping mall for the brand members. The activity organizer asks each member (agent) i to report his available time t_i to be present in the location to determine the activity time $[y, y + d]$. Agent i (if his preferred t_i is outside the activity time window $[y, y + d]$) needs to reschedule his own business to catch the start time or end time of the activity, translating to the inconvenience cost for rescheduling.

In this game, each agent must report his private time t_i to the social planner and he may have a chance to decrease his cost by misreporting t_i. Therefore, we emphasize strategyproofness of a mechanism, which guarantees that an agent cannot acquire any benefit from misreporting. We design a mechanism outputting the optimal solution y to minimize the social cost and prove it is group strategyproof. For another objective of minimizing the maximum cost, we also design a strategyproof mechanism with approximation ratio 2.

We also model and study the obnoxious activity scheduling game whose counterpart is the obnoxious facility location game, we can view t_i as the conflict time for agent i. An agent i wants to do his own business at time t_i (e.g., at another nearby location), and thus prefers t_i to be far from the activity duration $[y, y + d]$ to avoid potential overlap in time. For example, the activity can be a department meeting during $[y, y + d]$, and each attendee (agent) i attends the whole meeting. Each agent i should report his conflict time t_i for doing his own business and prefers t_i to be far away from meeting time. If $y \le t_i \le y + d$, agent i has to give up his own business and thus has zero utility. If $t_i < y$ or $t_i > y + d$, it is still possible that agent i's own business or the meeting may overrun to cause conflict. Therefore, if $t_i < y$, he wants the time gap $y - t_i$ to be as long as possible to reduce the chance of overlap due to possible overrun of his business and thus his utility is $y - t_i$. Similarly, if $t_i > y + d$, he wants long time gap $t_i - y - d$ to reduce the chance of overlap due to possible overrun of the meeting and thus his utility is $t_i - y - d$. An agent may have a chance to increase his utility by misreporting his t_i and thus we aim to design strategyproof mechanisms. We find that the optimal solution to maximize the social utility is no longer strategyproof given $0 \le d < 1$. Therefore, we respectively design deterministic and randomized group strategyproof mechanisms with provable approximation ratios and show some lower bounds. For another objective of maximizing the minimum utility, we find that any strategyproof mechanism achieves an unbounded approximation ratio.

Finally, we extend our model to consider another case that agent i has only binary preference towards the activity schedule $[y, y + d]$ in both normal and obnoxious activity scheduling games. That is, in the normal (or obnoxious) game, each agent i is happy (unhappy) once his t_i is within $[y, y+d]$ and otherwise unhappy (happy). Formally, the cost of agent i in normal game is 0 if $t_i \in [y, y+d]$; 1 if $t_i \in [0, y) \cup (y+d, 1]$. We find group strategyproof mechanisms for minimizing the social cost and maximizing the social utility for the two games, respectively. Besides the above examples, in practice we can imagine many other examples to potentially fit in our basic model.

1.1 Related Work

In the algorithmic view of locating one-facility, [14] first studied strategyproof mechanisms with provable approximation ratios on the line. For the obnoxious facility game, the mechanism design to improve the social utility was first studied by [2]. They presented a deterministic group strategyproof mechanism with approximation ratio 3 and a randomized strategyproof mechanism with approximation ratio 1.5. [21] found the lower bound of any randomized strategyproof mechanisms for maximizing the social utility is 1.077. [11] proved there is no strategyproof mechanism such that the number of candidates is more than two. [23] extended mechanism design for both games with weighted agents on a line and provided lower and upper bounds on the optimal social utility. [8] completely characterized deterministic strategyproof and group strategyproof mechanisms on single-sinked public policy domain. Combining the above two models together, the dual-preference game was studied in [5, 24], where some agents want to be close to the facility while the others want to be far away from the facility. Other variations of single facility location games can be found in [3, 6, 18, 19].

To some extent, our model is related to the two-facility location game, if we fix the gap d between the two facilities. For the two-facility location game, [12] studied the bounds for the scenario of locating two homogeneous facilities and the scenario when one agent possesses multiple locations. [4] considered the requirement of the minimum distance between the two facilities for locating them. [17] proposed a class of percentile mechanisms in the form of generalized median mechanisms. [15] initiated the study on two heterogeneous facility location games in the graph where the cost of an agent is the sum of his distances to both facilities. Other variations on two-facility location games can be found in [1, 7, 22].

Besides, regarding activity scheduling problems, the literature [13, 20] only studied non-strategic agents. To our best knowledge, our paper is the first to study the strategic activity scheduling game, which also generalizes the facility location games from locating points to locating intervals.

As a special case of zero activity duration ($d = 0$), our models will degenerate to the facility location game or obnoxious facility location game, in which agent i's cost or utility is simply $|y - t_i|$. In traditional facility location games, the facility location is just a point, while in our activity scheduling game, the non-trivial duration d of the activity plays an important role in our mechanism design and proofs of bounds. In the obnoxious game, for example, we design mechanisms according to $d \in [0, 1/2]$ or $d \in (1/2, 1)$, and we also prove the lower bounds according to $d \in [0, 1/3]$ or $d \in (1/3, 1)$. This is more challenging than the traditional case where the facility location is considered as a point.

2 SYSTEM MODEL

Let $N = \{1, 2, \cdots, n\}$ be the set of agents, and the time interval is $I = [0, 1]$. We denote $\mathbf{t} = \{t_1, t_2, \cdots, t_n\} \in I^n$ as the n agents' time profile, which is private information and needs to be reported by themselves. Without loss of generality, we assume $t_i \leq t_{i+1}$ for any $1 \leq i \leq n-1$.

In the activity scheduling game, denote the duration of the activity as $d \in [0, 1]$. The activity lasts from the start time y to the end time $y + d$. A deterministic mechanism f outputs the start time y based on a given agents' time profile $\mathbf{t}$, i.e., $y = f(\mathbf{t}, d) : I^n \to I_d = [0, 1-d]$. Any agent i prefers his time t_i to be close to the activity duration. Thus, the cost of agent i is denoted as

$$c_i(f(\mathbf{t}, d)|t_i, d) = \begin{cases} y - t_i, & \text{if } t_i < y; \\ 0, & \text{if } y \leq t_i \leq y + d; \\ t_i - y - d, & \text{if } t_i > y + d. \end{cases} \quad (1)$$

The social cost of a mechanism $f(\mathbf{t}, d)$ on $\mathbf{t}$ is denoted as the sum of costs of n agents, i.e.,

$$SC(f(\mathbf{t}, d)|\mathbf{t}, d) = \sum_{i=1}^{n} c_i(f(\mathbf{t}, d)|t_i, d). \quad (2)$$

Further, a randomized mechanism is a function $f : I^n \to \Delta(I_d)$, where $\Delta(I_d)$ is the set of distributions over I_d. If $f(\mathbf{t}, d) = y \sim P(\mathbf{t}, d)$, where P is a probability distribution, agent i's cost is defined to be the expected cost over such distribution, i.e., $\mathbf{E}_{y\sim P(\mathbf{t},d)}[c_i(y|t_i, d)]$. The social cost of a mechanism $f(\mathbf{t}, d)$ on $\mathbf{t}$ is denoted as the excepted sum of costs of n agents over such distribution, i.e., $\mathbf{E}_{y\sim P(\mathbf{t},d)}[SC(y|\mathbf{t}, d)] = \sum_{i=1}^{n} \mathbf{E}_{y\sim P(\mathbf{t},d)}[c_i(y|t_i, d)]$. The maximum cost of a mechanism $f(\mathbf{t}, d)$ with respect to $\mathbf{t}$ is $MC(f(\mathbf{t}, d)|\mathbf{t}, d) = \max_{i\in N} c_i(f(\mathbf{t}, d)|t_i, d)$.

As agents may misreport their times to change y for their own benefits, strategyproofness is important to ensure. Let $\mathbf{t}_{-i} = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)$ be the time profile without agent i. Let $\mathbf{t}_S$ be the time profile with all agent $i \in S \subseteq N$ and $\mathbf{t}_{-S}$ be the time profile without any agent $i \in S \subseteq N$. Next we formally define the strategyproofness and the group strategyproofness.

Definition 2.1. A mechanism is strategyproof in the activity scheduling game if no agent can benefit from misreporting his time. Formally, given agent i, profile $\mathbf{t} = \{t_i, \mathbf{t}_{-i}\} \in I^n$, and any misreported time $t'_i \in I$, it holds that

$$c_i(f(t_i, \mathbf{t}_{-i}, d)|t_i, d) \leq c_i(f(t'_i, \mathbf{t}_{-i}, d)|t_i, d).$$

Definition 2.2. A mechanism is group strategyproof in the activity scheduling game if for any group of agents, at least one of them cannot benefit if they misreport simultaneously. Formally, given a non-empty set $S \subseteq N$, time profile $\mathbf{t} = \{\mathbf{t}_S, \mathbf{t}_{-S}\} \in I^n$, and the misreported time profile $\mathbf{t}'_S \in I^{|S|}$, there exists $i \in S$, satisfying

$$c_i(f(t_S, \mathbf{t}_{-S}, d)|t_i, d) \leq c_i(f(t'_S, \mathbf{t}_{-S}, d)|t_i, d).$$

For the activity scheduling game, we are interested in designing strategyproof mechanisms that also perform well with respect to minimizing the social cost. For a time profile $\mathbf{t}$, let $OPT(\mathbf{t}, d)$ be the optimal (minimum) social cost. A strategyproof mechanism f has an approximation ratio $\gamma \geq 1$, if for any time profile $\mathbf{t} \in I^n, SC(f(\mathbf{t}, d)|\mathbf{t}, d) \leq \gamma OPT(\mathbf{t}, d)$.

In the obnoxious activity scheduling game, an agent i has his conflict time t_i, when he wants to do his own business and any agent i prefers his time t_i to be far away from the activity duration. We define agent i's utility as $u_i(f(\mathbf{t}, d)|t_i, d)$ (or $\mathbf{E}_{y\sim P(\mathbf{t},d)}[u_i(y|t_i, d)]$), which is the same as $c_i(f(\mathbf{t}, d)|t_i, d)$ in (1) (or $\mathbf{E}_{y\sim P(\mathbf{t},d)}[c_i(y|t_i, d)]$). The objective of this game is to maximize the social utility, which is denoted as $SU(f(\mathbf{t}, d)|\mathbf{t}, d) = \sum_{i=1}^{n} u_i(f(\mathbf{t}, d)|t_i, d)$ (or $\mathbf{E}_{y\sim P(\mathbf{t},d)}[SU(y|\mathbf{t}, d)] = \sum_{i=1}^{n} \mathbf{E}_{y\sim P(\mathbf{t},d)}\ [u_i(y|t_i, d)]$). The minimum utility of a mechanism $f(\mathbf{t}, d)$ with respect to $\mathbf{t}$ is $MU(f(\mathbf{t}, d)|\mathbf{t}, d) = \min_{i\in N} u_i(f(\mathbf{t}, d)|t_i, d)$.

Definition 2.3. A mechanism is strategyproof in the obnoxious activity scheduling game if no agent can benefit from misreporting his time. Formally, given any agent i, profile $\mathbf{t} = \{t_i, \mathbf{t}_{-i}\} \in I^n$, and any misreported time $t'_i \in I$, it holds that

$$u_i(f(t_i, \mathbf{t}_{-i}, d)|t_i, d) \geq u_i(f(t'_i, \mathbf{t}_{-i}, d)|t_i, d).$$

Definition 2.4. A mechanism is group strategyproof in the obnoxious activity scheduling game if for any group of agents, at least one of them cannot benefit if they misreport simultaneously. Formally, given a non-empty set $S \subseteq N$, time profile $\mathbf{t} = \{\mathbf{t}_S, \mathbf{t}_{-S}\} \in I^n$, and the misreported time profile $\mathbf{t}'_S \in I^{|S|}$, there exists $i \in S$, satisfying

$$u_i(f(t_S, \mathbf{t}_{-S}, d)|t_i, d) \geq u_i(f(t'_S, \mathbf{t}_{-S}, d)|t_i, d).$$

For the obnoxious activity scheduling game, we are interested in strategyproof mechanisms that also perform well with respect to maximizing the social utility. For a time profile $\mathbf{t}$, let $OPT(\mathbf{t}, d)$ be the optimal (maximum) social utility. A strategyproof mechanism f has an approximation ratio $\gamma \geq 1$, if for any profile $\mathbf{t} \in I^n, OPT(\mathbf{t}, d) \leq \gamma SU(f(\mathbf{t}, d)|\mathbf{t}, d)$.

3 THE ACTIVITY SCHEDULING GAME

3.1 Minimize the Social Cost

In this section, we study the activity scheduling game. The social cost $SC(y|\mathbf{t}, d)$ in (2) is a continuous function of variable $y \in I_d = [0, 1-d]$ since $c_i(y|t_i, d)$ is a continuous function of variable y. We define $L(y|\mathbf{t}, d) = \{i|t_i \in [0, y), t_i \in \mathbf{t}\}$ and $R(y|\mathbf{t}, d) = \{i|t_i \in (y + d, 1], t_i \in \mathbf{t}\}$. $|L(y|\mathbf{t}, d)|$ and $|R(y|\mathbf{t}, d)|$ are the numbers of agents whose times are in $[0, y)$ and $(y + d, 1]$ respectively. By (1) and (2), the social cost can be rewritten as

$$SC(y|\mathbf{t}, d) = \sum_{i:i\in L(y|\mathbf{t},d)} (y - t_i) + \sum_{i:i\in R(y|\mathbf{t},d)} (t_i - y - d).$$

Denote time interval

$$G_L(\mathbf{t},d)=\{y|\ |L(y|\mathbf{t},d)|<|R(y|\mathbf{t},d)|\},$$
$$G_R(\mathbf{t},d)=\{y|\ |L(y|\mathbf{t},d)|>|R(y|\mathbf{t},d)|\}.$$

As y increases from 0 to $1-d$, $|L(y|\mathbf{t},d)|$ increases from 0 and $|R(y|\mathbf{t},d)|$ decreases to 0, thus $y\in[0,1-d]=G_L(\mathbf{t},d)\cup[\sup G_L(\mathbf{t},d),\inf G_R(\mathbf{t},d)]\cup G_R(\mathbf{t},d)$. By checking the derivative of $dSC(y|\mathbf{t},d)/\,dt=|L(y|\mathbf{t},d)|-|R(y|\mathbf{t},d)|$, we show that for $y\in G_L(\mathbf{t},d)$, as y increases, $SC(y|\mathbf{t},d)$ decreases at rate $|R(y|\mathbf{t},d)|-|L(y|\mathbf{t},d)|>0$; for $y\in[\sup G_L(\mathbf{t},d),\inf G_R(\mathbf{t},d)]$, as y increases, $SC(y|\mathbf{t},d)$ remains unchanged; for $y\in G_R(\mathbf{t},d)$, as y increases, $SC(y|\mathbf{t},d)$ increases at rate $|L(y|\mathbf{t},d)|-|R(y|\mathbf{t},\ d)|>0$. Thus, $SC(y|\mathbf{t},d)$ of y is a continuous piecewise linear function. Denote the optimal time to minimize the social cost as $y^\star$. Therefore, the optimal solution is

$$y^\star\in[\sup G_L(\mathbf{t},d),\inf G_R(\mathbf{t},d)].$$

Mechanism 1. *Given $d\in[0,1)$, return the optimal time $y^\star=\sup G_L(\boldsymbol{t},d)$ or $y^\star=\inf G_R(\boldsymbol{t},d)$.*

Theorem 3.1. *Mechanism 1 is group strategyproof.*

Proof. Without loss of generality, we only need to prove $y^\star=f(\mathbf{t},d)=\sup G_L(\mathbf{t},d)$ is group strategyproof. Denote $S\subseteq N$ to be a coalition and $\mathbf{t}'=\{\mathbf{t}_{-S},\mathbf{t}'_S\}$. Suppose that agent $i\in S$ misreports his time from t_i to t'_i. Denote $y'=f(\mathbf{t}',d)=\sup G_L(\mathbf{t}',d)$. We divide our discussion into four cases.

Case 1: S contains at least one agent $i\in S$ whose time is in $[y^\star,y^\star+d]$. Obviously, $c_i(y^\star|t_i,d)=0\le c_i(y'|t_i,d)$.

Case 2: S contains both agents in $L(y^\star|\mathbf{t},d)$ and in $R(y^\star|\mathbf{t},d)$. Without loss of generality, assume that $y'<y^\star$. Then after misreporting, any agent i in $R(y^\star|\mathbf{t},d)$ increases his cost from $c_i(y^\star|t_i,d)=t_i-y^\star-d$ to $c_i(y'|t_i,d)=t_i-y'-d$.

Case 3: S only contains agents in $L(y^\star|\mathbf{t},d)$. Without loss of generality, assume all agents in $S=L(y^\star|\mathbf{t},d)$ misreport their times one by one. Consider agent i with $i\in S$ as the first to misreport his time t_i and $\mathbf{t}'=\{t_i,\mathbf{t}_{-i}\}$. Let ϵ satisfy $0<\epsilon<y^\star-t_i$. Since $y^\star-\epsilon<y^\star=\sup G_L(\mathbf{t},d)$, we have $y^\star-\epsilon\in G_L(\mathbf{t},d)$, i.e.,

$$|L(y^\star-\epsilon|\mathbf{t},d)|<|R(y^\star-\epsilon|\mathbf{t},d)|. \tag{3}$$

Since $t_i<y^\star-\epsilon$, if $t'_i<y^\star-\epsilon$, we have

$$|L(y^\star-\epsilon|\mathbf{t},d)|=|L(y^\star-\epsilon|\mathbf{t}',d)|; \tag{4}$$

if $t'_i\ge y^\star-\epsilon$,

$$|L(y^\star-\epsilon|\mathbf{t},d)|>|L(y^\star-\epsilon|\mathbf{t}',d)|. \tag{5}$$

Since $t_i<y^\star-\epsilon\le y^\star-\epsilon+d$, if $t'_i\le y^\star-\epsilon+d$,

$$|R(y^\star-\epsilon|\mathbf{t},d)|=|R(y^\star-\epsilon|\mathbf{t}',d)|; \tag{6}$$

if $t'_i>y^\star-\epsilon+d$,

$$|R(y^\star-\epsilon|\mathbf{t},d)|<|R(y^\star-\epsilon|\mathbf{t}',d)|. \tag{7}$$

By combining (3)-(7), we have

$$\begin{aligned}&|L(y^\star-\epsilon|\mathbf{t}',d)|\le|L(y^\star-\epsilon|\mathbf{t},d)|\\ <&|R(y^\star-\epsilon|\mathbf{t},d)|\le|R(y^\star-\epsilon|\mathbf{t}',d)|,\end{aligned}$$

for any $t'_i\in I$, which implies that $y^\star-\epsilon\in G_L(\mathbf{t}',d)$. Thus, $y^\star-\epsilon\le\sup G_L(\mathbf{t}',d)=y'$. Let $\epsilon\to0$, we have $y'\ge y^\star$. Then the second agent's misreporting also makes the activity time greater than or equal to y' and so on. Finally, after all agents misreport, the final activity time $y_{final}\ge\cdots\ge y'\ge y^\star$. Thus, for any agent i in $L(y^\star|\mathbf{t},d)$, his cost $c_i(y'|\mathbf{t},d)=y_{final}-t_i\ge y^\star-t_i=c_i(y^\star|\mathbf{t},d)$ and agent i cannot decrease his cost by misreporting.

Case 4: S only contains agents in $R(y^\star|\mathbf{t},d)$. This case is similar to Case 3.

In conclusion, f is group strategyproof. □

Corollary 3.2. *The complexity of running Mechansim 1 is $O(n)$ in the worst case.*

Proof. Without loss of generality, in Mechanism 1, we use $y^\star=\sup G_L(\mathbf{t},d)$. In fact, given a group of n agents, either $y^\star$ or $y^\star+d$ is at one of the agents' times t_i's. The way to find $y^\star$ is to let y and $y+d$ be $x_1,x_2,\dots x_n$ one by one until some $x_i=\sup G_L(\mathbf{t},d)$ or $x_i=\sup G_L(\mathbf{t},d)+d$. Therefore, we use the sequential search in Mechanism 1, where $2n$ is the length of the list. The complexity of the worst-case performance of running Mechansim 1 is $O(n)$. □

3.2 Minimize the Maximum Cost

For the objective of minimizing the maximum cost, the following lemma shows the optimal solution.

Lemma 3.3. *The optimal maximum cost for $\min_f MC(f(\boldsymbol{t},d)|\boldsymbol{t},d)$ is $(t_n-t_1-d)/2$ if $t_n-t_1\ge d$ and 0 if $t_n-t_1<d$ and the optimal solution is not strategyproof.*

Next, we design a deterministic group strategyproof mechanism.

Mechanism 2. *Given $d\in[0,1)$, return*

$$y=f(\boldsymbol{t},d)=l(\boldsymbol{t},d)=\begin{cases}t_1, & \text{if } t_1+d\le1;\\ 1-d, & \text{if } t_1+d>1,\end{cases}$$

$$\text{or } y=f(\boldsymbol{t},d)=r(\boldsymbol{t},d)=\begin{cases}t_n-d, & \text{if } t_n\ge d;\\ 0, & \text{if } t_n<d.\end{cases}$$

Theorem 3.4. *Mechanism 2 is group strategyproof and has the approximation ratio of 2.*

Similarly, strategyproof Mechanism 1 has the approximation ratio of 2 for minimizing the maximum cost. Inspired by Theorem 2.2 in [14], the next lemma shows the lower bound.

Lemma 3.5. *Given $d\in[0,1)$ for any $n\ge2$ agents, any deterministic strategyproof mechanism f has an approximation ratio of at least 2 for the maximum cost.*

By Theorem 3.4 and Lemma 3.5, the bound 2 is tight.

4 THE OBNOXIOUS ACTIVITY SCHEDULING GAME

In this section, we study the obnoxious activity scheduling game. From the analysis in the last section, it is easy to see that the optimal solution of $\max SU(y|\mathbf{t},d)$ must be 0 or $1-d$. We define

$$SU_l = SU(y=0|\mathbf{t},d) = \sum_{i:t_i \geq d} (t_i - d),$$
$$SU_r = SU(y=1-d|\mathbf{t},d) = \sum_{i:t_i \leq 1-d} (1-d-t_i).$$

Then $OPT(\mathbf{t},d) = \max\{SU_l, SU_r\}$ and the optimal time is: $y=0$ if $SU_l \geq SU_r$; $y=1-d$ if $SU_l < SU_r$. If $d=1$, the optimal solution is $y=0$, and is obviously strategyproof. However, if $d<1$, the optimal solution is not strategyproof, since this game is a generalization of the obnoxious facility game [2]. Next, we design strategyproof mechanisms given $0 \leq d < 1$.

4.1 Deterministic Mechanisms

We design deterministic strategyproof mechanisms for max $SU(y|\mathbf{t},d)$ in this subsection. Given $d \in [0,\frac{1}{2})$, define $Q_1 = \{i|t_i \in [0,\frac{1}{2}]\}$ and $Q_2 = \{i|t_i \in (\frac{1}{2},1]\}$.

Mechanism 3. *Given $d \in [0,\frac{1}{2})$, return $y = f(\mathbf{t},d) = 0$ if $|Q_1| \leq |Q_2|$; $y = f(\mathbf{t},d) = 1-d$ if $|Q_1| > |Q_2|$.*

Theorem 4.1. *Mechanism 3 is group strategyproof and has the approximation ratio of $\frac{3-4d}{1-2d}$.*

Proof. Mechanism 3 is proved group strategyproof by following the similar proof of Theorem 1 in [2].

Given $d < \frac{1}{2}$, without loss of generality, assume $|Q_1| \leq |Q_2|$. Thus, $f=0$ and its social utility is $SU(f(\mathbf{t},d)|\mathbf{t},d) = SU_l$. The optimal solution could still be $y=1-d$. We have

$$\begin{aligned}\gamma &= \frac{OPT(\mathbf{t},d)}{SU(f(\mathbf{t},d)|\mathbf{t},d)} = \frac{\sum_{i:t_i\leq 1-d}(1-d-t_i)}{\sum_{i:t_i\geq d}(t_i-d)} \\ &= \frac{\sum_{i:t_i\in[0,\frac{1}{2}]}(1-d-t_i) + \sum_{i:t_i\in(\frac{1}{2},1-d]}(1-d-t_i)}{\sum_{i:t_i\in[d,\frac{1}{2}]}(t_i-d) + \sum_{i:t_i\in[\frac{1}{2},1]}(t_i-d)} \\ &\leq \frac{(1-d)|Q_1| + (\frac{1}{2}-d)|Q_2|}{0 + (\frac{1}{2}-d)|Q_2|} \leq \frac{(1-d)+(\frac{1}{2}-d)}{\frac{1}{2}-d} \\ &= \frac{3-4d}{1-2d}.\end{aligned}$$

□

Corollary 4.2. *$\gamma = \frac{3-4d}{1-2d}$ is tight for Mechanism 3.*

Proof. Consider a time profile $\mathbf{t} = \{0, \frac{1}{2}+\epsilon\}$, where $0 < \epsilon < \frac{1}{2}$. Obviously, $SU(f(\mathbf{t},d)|\mathbf{t},d) = \frac{1}{2}+\epsilon-d$ with $y=0$ by Mechanism 3 and $OPT(\mathbf{t},d) = \frac{3}{2}-2d-\epsilon$ with $y=1-d$, which implies $\gamma = \frac{3-4d-2\epsilon}{1-2d+2\epsilon} \to \frac{3-4d}{1-2d}$ as $\epsilon \to 0$. □

By following Mechanism 3, we can also design the following strategyproof mechanism given $d \in [\frac{1}{2},1)$. Define $Q_3 = \{i|y_i \in [0,1-d]\}$ and $Q_4 = \{i|y_i \in [d,1]\}$.

Mechanism 4. *Given $d \in [\frac{1}{2},1)$, return $y = f(\mathbf{t},d) = 0$ if $|Q_3| \leq |Q_4|$; $y = f(\mathbf{t},d) = 1-d$ if $|Q_3| > |Q_4|$.*

The following lemma shows the approximation ratio of Mechanism 4 is infinite.

Lemma 4.3. *Given $d \in [\frac{1}{2},1)$ for any $n \geq 2$ agents, any deterministic strategyproof mechanism f which only selects from any two candidate times has an approximation ratio γ of at least $+\infty$.*

Proof. Given $d \in [\frac{1}{2},1)$, let f be a deterministic strategyproof mechanism selecting from two candidate times y_1 and y_2, which satisfy $0 \leq y_1 < y_2 \leq 1-d$. Let ϵ satisfy $0 < \epsilon^2 < 1-d$. Consider the values of y_1 and y_2, we have the three cases.

Case 1: $y_1 < y_2 \leq 1-d-\epsilon^2$. Consider a time profile $\mathbf{t} = \{1-d-\epsilon^2\}$. Then $SU(f(\mathbf{t},d)|\mathbf{t},d) \leq \max\{SU(y_1|\mathbf{t},d), SU(y_2|\mathbf{t},d)\} = 0$ and $OPT(\mathbf{t},d) = SU(1-d|\mathbf{t},d) = \epsilon^2$, implying $\gamma = OPT(\mathbf{t},d)/SU(f(\mathbf{t},d)|\mathbf{t},d) = +\infty$.

Case 2: $\epsilon^2 \leq y_1 < y_2$. This case is similar to Case 1.

Case 3: $y_1 < \epsilon^2 < 1-d-\epsilon^2 < y_2$. Consider a time profile $\mathbf{t} = \{t_1,t_2\} = \{1-d-\epsilon, d+\epsilon\}$. Note that given $d \in [\frac{1}{2},1)$, $0 < t_1 < 1-d \leq d < t_2 < 1$. Without loss of generality, suppose that $f(\mathbf{t},d) = y_1$. Thus, $u_1(y_1|t_1,d) = 0$, due to $y_1 < t_1 < y_1+d$. Consider $t'_1 = 0$ and $\mathbf{t}' = \{t'_1,t_2\}$. Let $y' = f(\mathbf{t}',d)$. As f is strategyproof and agent 1 cannot increase his utility by misreporting from t_1 to t'_1, the utility of agent 1 must satisfy $u_1(y'|t_1,d) \leq u_1(y_1|t_1,d) = 0$, which implies that $y' = y_1$, due to $u_1(y_2|t_1,d) \geq \epsilon - \epsilon^2 > 0$. Hence, for the profile $\mathbf{t}'$, the social utility of $\mathbf{t}'$ under f is $SU(y_1|\mathbf{t}',d) = \epsilon$ and $OPT(\mathbf{t}',d) = SU(1-d|\mathbf{t}',d) = 1-d$. Therefore, the approximation ratio of f is $(1-d)/\epsilon \to +\infty$ as $\epsilon \to 0$.

In conclusion, the approximation ratio γ is $+\infty$. □

By Lemma 4.3, Mechanism 4 achieves the best possible approximation ratio among deterministic strategyproof mechanisms choosing from two candidate times. It also has the possibility of choosing the optimal time 0 or $1-d$ to best serve the agents, thus Mechanism 4 is good. To remedy this, later in the next subsection, we will propose a randomized mechanism with approximation ratio 2.

Discussion: we have two directions to solve the infinite approximation ratio problem. One is to prove the extension of Lemma 4.3, that any deterministic strategyproof mechanism which only selects from $p \geq 3$ candidate times has an infinite approximation ratio. The other one is to prove that there is no p-candidate deterministic strategyproof mechanism for any $p \geq 3$. This means for any strategyproof mechanism with p-candidate, it only selects from two candidate times and the other $(p-2)$ candidate times are never selected. We think this hypothesis is reasonable since it shares some similarity with Theorem 3 in reference [11].

The reason why we have to design two mechanisms divided into $d < 1/2$ and $d \geq 1/2$ in the obnoxious game is as follows: given $d < 1/2$, the two chosen candidate activity duration $[0,d]$ and $[1-d,1]$ have no intersection; but given $d \geq 1/2$, they have intersection. We should design sets Q_1, Q_2 to be different from Q_3, Q_4. The next two lemmas show the lower bounds of deterministic strategyproof mechanisms.

Lemma 4.4. *Given $d \in [0,\frac{1}{3})$ for any $n \geq 2$ agents, any deterministic strategyproof mechanism f has an approximation ratio γ of at least $\frac{4-5d}{2-d} \in (\frac{7}{5},2]$.*

Proof. Assume $N = \{1,2\}$. Let f be a deterministic strategyproof mechanism. Consider the time profile $\mathbf{t} = \{t_1, t_2\} = \{\frac{1+d}{3}, \frac{2-d}{3}\}$ and $f(\mathbf{t}, d) = y$. Note that given $d \in [0, \frac{1}{3})$, $0 \le d < t_1 < t_2 < 1 - d \le 1$. Consider the value of $y \in I_d$, we have the following three cases.

Case 1: $y \in [0, \frac{1-2d}{3}]$. In this case, $u_1(y|t_1, d) = t_1 - y - d$, due to $y + d \le t_1$. Consider $t_1' = 0$ and $\mathbf{t}' = \{t_1', t_2\}$. Let $y' = f(\mathbf{t}', d)$. Note that $SU(0|\mathbf{t}', d) = 0 + (t_2 - d) = \frac{2-4d}{3}$ and $SU(1-d|\mathbf{t}', d) = (1-d)+(1-d-t_2) = \frac{4-5d}{3}$. Thus the optimal social utility of $\mathbf{t}'$ is $OPT(\mathbf{t}', d) = SU(1-d|\mathbf{t}', d) = \frac{4-5d}{3}$. As f is strategyproof and agent 1 cannot increase his utility by misreporting from t_1 to t_1', the utility of agent 1 must satisfy $u_1(y'|t_1, d) \le u_1(y|t_1, d) = t_1 - y - d \le t_1 - d = \frac{1-2d}{3}$, which implies that $y' \in [0, \frac{2-d}{3}]$. Hence, for the profile $\mathbf{t}'$, the social utility of $\mathbf{t}'$ under f is $SU(f(\mathbf{t}', d)|\mathbf{t}', d) \le t_2 = \frac{2-d}{3}$. Therefore, $\gamma \ge OPT(\mathbf{t}', d)/SU(f(\mathbf{t}', d)|\mathbf{t}', d) \ge \frac{4-5d}{2-d}$.

Case 2: $y \in (\frac{1-2d}{3}, \frac{2-d}{3})$. Since $y + d \ge t_1$ and $y \le t_2$, $SU(f(\mathbf{t}, d)|\mathbf{t}, d) \le t_2 - t_1 = \frac{1-2d}{3}$, and $OPT(\mathbf{t}, d) = t_1 + t_2 - 2d = 1-2d$. Thus, we have $\gamma \ge OPT(\mathbf{t}, d)/SU(f(\mathbf{t}, d)|\mathbf{t}, d) \ge 3$.

Case 3: $y \in [\frac{2-d}{3}, 1-d]$. Due to symmetry, this case is similar to Case 1.

Therefore, by combining the above three cases, γ is at least $\min\{\frac{4-5d}{2-d}, 3\} = \frac{4-5d}{2-d} \in (\frac{7}{5}, 2]$. □

Lemma 4.5. *Given $d \in [\frac{1}{3}, 1)$ for any $n \ge 2$ agents, any deterministic strategyproof mechanism f has an approximation ratio γ of at least 2.*

Proof. Assume $N = \{1,2\}$. Let f be a deterministic strategyproof mechanism. Consider the time profile $\mathbf{t} = \{t_1, t_2\} = \{\frac{1-d}{2}, \frac{1+d}{2}\}$ and $f(\mathbf{t}, d) = y$. Note that given $d \in [\frac{1}{3}, 1)$, $0 < t_1 \le d, 1-d \le t_2 < 1$. Consider the value of $y \in I_d$, we have the following two cases.

Case 1: $y \in [0, \frac{1-d}{2}]$. In this case, $u_1(y|t_1, d) = 0$, due to $y \le t_1 < y + d$. Consider $t_1' = 0$ and $\mathbf{t}' = \{t_1', t_2\}$. Let $y' = f(\mathbf{t}', d)$. Note that $SU(0|\mathbf{t}', d) = 0 + (t_2 - d) = \frac{1-d}{2}$ and $SU(1-d|\mathbf{t}', d) = (1-d) + 0 = 1-d$. Thus the optimal social utility of $\mathbf{t}'$ is $OPT(\mathbf{t}', d) = SU(1-d|\mathbf{t}', d) = 1-d$. As f is strategyproof and agent 1 cannot increase his utility by misreporting from t_1 to t_1', the utility of agent 1 must satisfy $u_1(y'|t_1, d) \le u_1(y|t_1, d) = 0$, which implies that $y' \in [0, \frac{1-d}{2}]$. Hence, the social utility of $\mathbf{t}'$ under f is $SU(f(\mathbf{t}', d)|\mathbf{t}', d) = t_2 - d = \frac{1-d}{2}$. Therefore, $\gamma \ge OPT(\mathbf{t}', d)/SU(f(\mathbf{t}', d)|\mathbf{t}', d) = 2$.

Case 2: $y \in (\frac{1-d}{2}, 1-d]$. Due to symmetry, this case is similar to Case 1. □

We find Lemma 4.4 does not work for $d \in [\frac{1}{3}, 1)$, thus we propose Lemma 4.5 with a different time profile. Interestingly, we notice that the lower bound is not continuous when d is close to 1. If $d \in [\frac{1}{3}, 1)$, the lower bound is always 2 but if $d = 1$, the lower bound drops to 1 immediately. The reason is that the utility of agent i is zero if $x_i \in [y, y+d]$. As d approaches 1, the utility of any agent i approaches 0 but with a different speed as that for the optimal solution and thus the limit of the approximation ratio is not 1.

4.2 Randomized Mechanisms

Inspired by Mechanism 3 in [2], we design the following randomized mechanism.

Mechanism 5. *Given $d \in [0, \frac{1}{2})$, return $y = f(\mathbf{t}, d) = 0$ with probability α and $y = f(\mathbf{t}, d) = 1-d$ with probability $(1-\alpha)$, where*

$$\alpha = \frac{2(1-d)|Q_1||Q_2| + (1-2d)|Q_2|^2}{(1-2d)|Q_1|^2 + 4(1-d)|Q_1||Q_2| + (1-2d)|Q_2|^2}.$$

Theorem 4.6. *Mechanism 5 is group strategyproof with approximation ratio $\frac{3-4d}{2-3d} \in [\frac{3}{2}, 2)$ for $d \in [0, \frac{1}{2})$.*

Corollary 4.7. *$\gamma = \frac{3-4d}{2-3d}$ is tight for Mechanism 5.*

Proof. Consider a time profile $\mathbf{t} = \{\underbrace{0.5, \ldots, 0.5}_{n/2}, \underbrace{1, \ldots, 1}_{n/2}\}$. Obviously, $\mathbf{E}_{y\sim P(\mathbf{t},d)}[SU(y|\mathbf{t}, d)] = \frac{2-3d}{4}n$ by Mechanism 5 and $OPT(\mathbf{t}, d) = \frac{3-4d}{4}n$, which implies $\gamma = \frac{3-4d}{2-3d}$. □

In fact, Mechanism 5 only works for $d \in [0, \frac{1}{2})$, but does not work for $d \in [\frac{1}{2}, 1)$, thus we propose the following mechanism.

Mechanism 6. *Given $d \in [\frac{1}{2}, 1)$, return $y = f(\mathbf{t}, d) = 0$ with probability $\frac{1}{2}$ and $y = f(\mathbf{t}, d) = 1-d$ with probability $\frac{1}{2}$.*

Theorem 4.8. *Mechanism 6 is group strategyproof with approximation ratio 2 for $d \in [\frac{1}{2}, 1)$.*

Proof. Mechanism 6 is group strategyproof since the probability distribution of y is unchanged. We obtain

$$\gamma = \frac{OPT(\mathbf{t}, d)}{\mathbf{E}_{y\sim P(\mathbf{t},d)}[SU(y|\mathbf{t}, d)]} = \frac{\max\{SU_l, SU_r\}}{\frac{1}{2}SU_l + \frac{1}{2}SU_r} \le \max\{\frac{2}{1 + \frac{SU_r}{SU_l}}, \frac{2}{\frac{SU_l}{SU_r} + 1}\} \le 2.$$

□

Corollary 4.9. *$\gamma = 2$ is tight for Mechanism 6.*

Proof. Consider a profile $\mathbf{t} = \{\underbrace{1-d, \ldots, 1-d}_{n/2}, \underbrace{1, \ldots, 1}_{n/2}\}$. Obviously, $\mathbf{E}_{y\sim P(\mathbf{t},d)}[SU(y|\mathbf{t}, d)] = \frac{1-d}{4}n$ by Mechanism 6 and $OPT(\mathbf{t}, d) = \frac{1-d}{2}n$, which implies $\gamma = 2$. □

The next two lemmas show the lower bounds of randomized strategyproof mechanisms. Inspired by the idea of Theorem 3 in [21], we have the following lemma.

Lemma 4.10. *Given $d \in [0, \frac{1}{3})$ for any $n \ge 2$ agents, any randomized strategyproof mechanism f has an approximation ratio γ of at least $\frac{14-16d}{13-14d}$.*

Lemma 4.11. *Given $d \in [\frac{1}{3}, 1)$ for any $n \ge 2$ agents, any randomized strategyproof mechanism f has an approximation ratio γ of at least $\frac{4}{3}$.*

Proof. Assume $N = \{1,2\}$. Let f be a randomized strategyproof mechanism. First, assume f follows a continuous distribution. Consider the profile $\mathbf{t} = \{t_1, t_2\} = \{\frac{1-d}{2}, \frac{1+d}{2}\}$ and let $f(\mathbf{t}, d) = y \sim P_1$. Note that given $d \in [\frac{1}{3}, 1)$, $0 < t_1 \le$

d and $1-d \le t_2 < 1$. The utility of agent 1 and the utility of agent 2 are

$$\mathop{\mathbf{E}}_{y\sim P_1}[u_1(y|t_1,d)] = \int_{\frac{1-d}{2}}^{1-d}(y-\frac{1-d}{2})P_1(y)dy,$$

$$\mathop{\mathbf{E}}_{y\sim P_1}[u_2(y|t_2,d)] = \int_{0}^{\frac{1-d}{2}}(\frac{1-d}{2}-y)P_1(y)dy.$$

Without loss of generality, assume $\mathbf{E}_{y\sim P_1}[u_1(y|t_1,d)] \le \mathbf{E}_{y\sim P_1}[u_2(y|t_2,d)]$. In this case, $\mathbf{E}_{y\sim P_1}[SU(y|\mathbf{t},d)] = \mathbf{E}_{y\sim P_1}[u_1(y|t_1,d)] + \mathbf{E}_{y\sim P_1}[u_2(y|t_2,d)] \le OPT(\mathbf{t},d) = 0 + t_2 - d = \frac{1-d}{2}$, which implies $\mathbf{E}_{y\sim P_1}[u_1(y|t_1,d)] \le \frac{1-d}{4}$.

Denote $\mathbf{t}'$ as the time profile after one of the two agents misreports. Consider $t_1' = 0$ and $\mathbf{t}' = \{t_1', t_2\}$. Let $f(\mathbf{t}', d) = y' \sim P_2$. As f is strategyproof and agent 1 cannot increase his utility by misreporting from t_1 to t_1', the utility of agent 1 must satisfy that

$$\mathop{\mathbf{E}}_{y'\sim P_2}[u_1(y'|t_1,d)] = \int_{\frac{1-d}{2}}^{1-d}(y'-\frac{1-d}{2})P_2(y')dy'$$
$$\le \mathop{\mathbf{E}}_{y\sim P_1}[u_1(y|t_1,d)] = \int_{\frac{1-d}{2}}^{1-d}(y-\frac{1-d}{2})P_1(y)dy. \quad (8)$$

For the profile $\mathbf{t}'$, the social utility of $\mathbf{t}'$ under f is

$$\begin{aligned}
&\mathop{\mathbf{E}}_{y'\sim P_2}[SU(y'|\mathbf{t}',d)] \\
&= \mathop{\mathbf{E}}_{y'\sim P_2}[u_1(y'|t_1',d)] + \mathop{\mathbf{E}}_{y'\sim P_2}[u_2(y'|t_2,d)] \\
&= \int_0^{1-d} y'P_2(y')dy' + \int_0^{\frac{1-d}{2}}(\frac{1-d}{2}-y')P_2(y')dy' \\
&= \int_0^{\frac{1-d}{2}} y'P_2(y')dy' + \int_{\frac{1-d}{2}}^{1-d} y'P_2(y')dy' \\
&\quad + \int_0^{\frac{1-d}{2}}(\frac{1-d}{2}-y')P_2(y')dy' \\
&= \int_{\frac{1-d}{2}}^{1-d} y'P_2(y')dy' + \int_0^{\frac{1-d}{2}} \frac{1-d}{2}P_2(y')dy' \\
&= \int_{\frac{1-d}{2}}^{1-d}(y'-\frac{1-d}{2})P_2(y')dy' + \frac{1-d}{2}\int_0^{1-d} P_2(y')dy' \\
&\le \int_{\frac{1-d}{2}}^{1-d}(y-\frac{1-d}{2})P_1(y)dy + \frac{1-d}{2}\times 1 \\
&= \mathop{\mathbf{E}}_{y\sim P_1}[u_1(y|t_1,d)] + \frac{1-d}{2} \le \frac{1-d}{4} + \frac{1-d}{2} = \frac{3}{4}(1-d),
\end{aligned}$$

where the second last inequality is due to (8). The optimal social utility of $\mathbf{t}'$ is $OPT(\mathbf{t}',d) = SU(1-d|\mathbf{t}',d) = 1-d$. Thus, $\gamma \ge OPT(\mathbf{t}',d)/SU(f(\mathbf{t}',d)|\mathbf{t}',d) \ge 4/3$.

On the other hand, if f follows a discrete distribution, we can define probability density functions P_1 and P_2 to be Dirac Delta functions (see Chapter 6. Generalized Functions in [9]) respectively at each one of discrete times. For example, if we select $y = \bar{y}$ with probability $\bar{p}$, then $P_1(\bar{y}) = +\infty$ and $\int_{\bar{y}-\epsilon}^{\bar{y}+\epsilon} g(y)P_1(y)dy = g(\bar{y})\bar{p}$, where $|g(y)| < +\infty$ and $\epsilon > 0$. We can transform each utility function into the integral formula: $E_{y\sim P_j}[u_i(y|t_i,d)] = \sum_{k:y_{j,k}\in A_i}|y_{j,k} - (1-d)/2|p_{j,k} = \int_{y\in A_i}|y-(1-d)/2|P_j(y)dy$, where $i,j = 1,2$, $A_1 = [(1-d)/2, 1-d]$, $A_2 = [0,(1-d)/2]$, $P_j(y)$ is Dirac Delta function and $p_{j,k}$ is the probability of y being selected as $y_{j,k}$. All proofs above follow and we can obtain the same lower bound. □

4.3 Maximize the Minimum Utility

For the objective of $\max_f MU(f(\mathbf{t},d)|\ \mathbf{t},d)$, the optimal solution is not strategyproof. Section 5.4 of [8] proved that if each agent i has a strict preference order over the policy domain which is single-sinked (opposite to single-peaked), for maximizing the minimum utility in the obnoxious facility game, any deterministic strategyproof mechanism has an unbounded approximation ratio. Since the agent's preference order in our work is single-sinked and our utility function (1) is quasi-convex, any deterministic strategyproof mechanism for maximizing the minimum utility has an unbounded approximation ratio.

5 EXTENSION TO THE CHARACTERISTIC FUNCTIONS

For extension, in the activity scheduling game, we further consider the cost of agent i as a characteristic function:

$$c_i(f(\mathbf{t},d)|t_i,d) = \begin{cases} 0, & \text{if } y \le t_i \le y+d; \\ 1, & \text{if } t_i < y \text{ or } t_i > y+d. \end{cases} \quad (9)$$

Our new objective is $\min\sum_{i\in N} c_i(f(\mathbf{t},d)|t_i,d)$. Define $H_1(y|\mathbf{t},d) = \{a|a\in[y,y+d], a\in\mathbf{t}\}$ and $\Omega_1(\mathbf{t},d) = \{y|\ |H_1(y|\mathbf{t},d)| = \max_y|H_1(y|\mathbf{t},d)|, y\in[0,1-d]\}$.

Theorem 5.1. *For minimizing the social cost, the optimal solution can be $y = \sup\Omega_1(\mathbf{t},d)$ or $y = \inf\Omega_1(\mathbf{t},d)$, which is group strategyproof.*

Proof. Without loss of generality, we only need to prove $y = \sup\Omega_1(\mathbf{t},d)$ is the optimal solution and group strategyproof. Denote $S\subseteq N$ to be a coalition and $\mathbf{t}' = \{\mathbf{t}_{-S}, \mathbf{t}'_S\}$. Suppose that agent $i\in S$ misreports his time from t_i to t_i'. Denote $y^\star = \sup\Omega_1(\mathbf{t},d)$ and after misreporting $y' = \sup\Omega_1(\mathbf{t}',d)$.

Obviously, $\Omega_1(\mathbf{t},d) = \arg\min_y\sum_{i\in N} c_i(y|t_i,d)$ and thus is the optimal solution. Since $\Omega_1(\mathbf{t},d)$ is the union of closed intervals, $\sup\Omega_1(\mathbf{t},d)\in\Omega_1(\mathbf{t},d)$ and thus $y = \sup\Omega_1(\mathbf{t},d)$ is the optimal solution.

For group strategyproofness, we have four cases.

Case 1: S contains at least one agent $i\in S$ whose time is in $[y^\star, y^\star+d]$. Obviously, $c_i(y^\star|t_i,d) = 0 \le c_i(y'|t_i,d)$.

Case 2: S contains both agents whose times are in $[0,y^\star)$ and in $(y^\star+d,1]$. If $y' < y^\star$, for any agents whose times are in $(y^\star+d,1]$, after misreporting, by (9), $c_i(y^\star|t_i,d) = 1 = c_i(y'|t_i,d)$; if $y' > y^\star$, for any agents whose times are in $(0,y^\star]$, after misreporting, by (9), $c_i(y^\star|t_i,d) = 1 = c_i(y'|t_i,d)$.

Case 3: S only contains agents whose times are in $[0,y^\star)$. Assume for contradiction f is not group strategyproof. Any agent $i\in S$ must decrease his cost by misreporting. Since $c_i(y^\star|t_i,d) = 1 > c_i(y'|t_i,d) = 0$ from (9), y' must satisfy that $y'\le t_i\le y'+d$ for any $i\in S$ and further, $y'\le t_i < y^\star$.

Since $y^\star = \sup \Omega_1(\mathbf{t}, d)$,

$$|H_1(y^\star|\mathbf{t}, d)| = \max_y |H_1(y|\mathbf{t}, d)| \geq |H_1(y'|\mathbf{t}, d)|. \quad (10)$$

Since $y' = \sup \Omega_1(\mathbf{t}', d)$ and $y' < y^\star$,

$$|H_1(y'|\mathbf{t}', d)| = \max_y |H_1(y|\mathbf{t}', d)| > |H_1(y^\star|\mathbf{t}', d)|. \quad (11)$$

Since $t_i \notin [y^\star, y^\star + d]$ and $t_i \in [y', y' + d]$ for any $i \in S$,

$$|H_1(y^\star|\mathbf{t}', d)| \geq |H_1(y^\star|\mathbf{t}, d)|, \quad (12)$$

$$|H_1(y'|\mathbf{t}', d)| \leq |H_1(y'|\mathbf{t}, d)|. \quad (13)$$

By (10) and (12), we have

$$|H_1(y^\star|\mathbf{t}', d)| \geq |H_1(y'|\mathbf{t}, d)|. \quad (14)$$

By (11) and (13), we have

$$|H_1(y^\star|\mathbf{t}', d)| < |H_1(y'|\mathbf{t}, d)|. \quad (15)$$

However, (14) contradicts (15). Therefore, at least one agent $i \in S$ cannot decrease his cost by misreporting.

Case 4: S only contains agents whose times are in $(y^\star+d, 1]$. This case is similar to Case 3.

In conclusion, f is group strategyproof. □

In the obnoxious activity scheduling game, we further consider the utility of agent i as a characteristic function:

$$u_i(f(\mathbf{t}, d)|t_i, d) = \begin{cases} 0, & \text{if } y < t_i < y + d; \\ 1, & \text{if } t_i \leq y \text{ or } t_i \geq y + d. \end{cases} \quad (16)$$

Our new objective is $\max \sum_{i \in N} u_i(f(\mathbf{t}, d)|t_i, d)$. Define $H_2(y|\mathbf{t}, d) = \{a | a \in (y, y+d), a \in \mathbf{t}\}$ and $\Omega_2(\mathbf{t}, d) = \{y| \; |H_2(y|\mathbf{t}, d)| = \min_y |H_2(y|\mathbf{t}, d)|, y \in [0, 1 - d]\}$.

Theorem 5.2. *For maximizing the social utility, the optimal solution can be $y = \sup \Omega_2(\boldsymbol{t}, d)$ or $y = \inf \Omega_2(\boldsymbol{t}, d)$, which is group strategyproof.*

Proof. Without loss of generality, we only need to prove $y = \sup \Omega_2(\mathbf{t}, d)$ is the optimal solution and group strategyproof. Denote $S \subseteq N$ to be a coalition and $\mathbf{t}' = \{\mathbf{t}_{-S}, \mathbf{t}'_S\}$. Suppose that agent $i \in S$ misreports his time from t_i to t'_i. Denote $y^\star = \sup \Omega_2(\mathbf{t}, d)$ and after misreporting $y' = \sup \Omega_2(\mathbf{t}', d)$.

Obviously, $\Omega_2(\mathbf{t}, d) = \arg\max_y \sum_{i \in N} u_i(y|t_i, d)$ and thus is the optimal solution. Since $\Omega_2(\mathbf{t}, d)$ is the union of closed intervals, $\sup \Omega_2(\mathbf{t}, d) \in \Omega_2(\mathbf{t}, d)$ and thus $y = \sup \Omega_2(\mathbf{t}, d)$ is the optimal solution.

For group strategyproofness, we have two cases.

Case 1: S contains at least one agent $i \in S$ whose time is in $[0, y^\star] \cup [y^\star + d, 1]$. Obviously, from (16), $u_i(y^\star|t_i, d) = 1 \geq u_i(y'|t_i, d)$.

Case 2: S only contains agents whose times are in $(y^\star, y^\star + d)$. Assume for contradiction f is not group strategyproof. Any agent $i \in S$ must increase his utility by misreporting. Since $u_i(y^\star|t_i, d) = 0 < u_i(y'|t_i, d) = 1$ from (16), y' must satisfy that $y' \geq t_i$ for any $i \in S$ or $y' + d \leq t_i$ for any $i \in S$. We have two subcases.

Subcase 1: $y' \geq t_i$ for any $i \in S$. Since $y^\star = \sup \Omega_2(\mathbf{t}, d)$ and $y' \geq t_i > y^\star$,

$$|H_2(y^\star|\mathbf{t}, d)| = \min_y |H_2(y|\mathbf{t}, d)| < |H_2(y'|\mathbf{t}, d)|. \quad (17)$$

Since $y' = \sup \Omega_2(\mathbf{t}', d)$,

$$|H_2(y'|\mathbf{t}', d)| = \min_y |H_1(y|\mathbf{t}', d)| \leq |H_2(y^\star|\mathbf{t}', d)|. \quad (18)$$

Since $t_i \in (y^\star, y^\star + d)$ and $t_i \notin (y', y' + d)$ for any $i \in S$,

$$|H_2(y^\star|\mathbf{t}', d)| \leq |H_2(y^\star|\mathbf{t}, d)|, \quad (19)$$

$$|H_2(y'|\mathbf{t}', d)| \geq |H_2(y'|\mathbf{t}, d)|. \quad (20)$$

By (17) and (19), we have

$$|H_2(y^\star|\mathbf{t}', d)| < |H_2(y'|\mathbf{t}, d)|. \quad (21)$$

By (18) and (20), we have

$$|H_2(y^\star|\mathbf{t}', d)| \geq |H_2(y'|\mathbf{t}, d)|. \quad (22)$$

However, (21) contradicts (22). Therefore, at least one agent $i \in S$ cannot increase his utility by misreporting.

Subcase 2: $y' + d \leq t_i$ for any $i \in S$. This subcase is similar to Subcase 1.

In conclusion, f is group strategyproof. □

We can see that the definitions of (9) and (16) are different. If we define the utility of agent i the same as (9), we have the following remark to find the strategyproof optimal solution. Define $\Omega_3(\mathbf{t}, d) = \{y| \; |H_1(y|\mathbf{t}, d)| = \min_y \; |H_1(y|\mathbf{t}, d)|, \; y \in [0, 1 - d]\}$.

Remark 1. *For maximizing the social utility, the optimal solution can be $y = (\sup \Omega_3(\boldsymbol{t}, d))^-$ or $y = (\inf \Omega_3(\boldsymbol{t}, d))^+$, which is group strategyproof.*

Note that we can use Remark 1 to find the strategyproof optimal solution but Remark 1 is not a mechanism since we can not acquire one-sided limit of a value (i.e., $(\sup \Omega_3(\mathbf{t}, d))^-$, $(\inf \Omega_3(\mathbf{t}, d))^+$).

6 CONCLUSIONS AND FUTURE WORKS

We considered a social planner schedules an activity in the time domain $[0, 1]$. In the activity scheduling game, each agent i wants his time t_i to be close to the activity duration $[y, y+d]$. We designed a group strategyproof mechanism outputting an optimal solution. In the obnoxious activity scheduling game, each agent prefers his time t_i to be far away from the activity duration $[y, y + d]$. We designed deterministic and randomized group strategyproof mechanisms with provable approximation ratios and showed some lower bounds. We also considered the cost/utility as the characteristic function and found group strategyproof mechanisms for minimizing the social cost and maximizing the social utility.

In the future, we will consider agent i's own business domain as an interval $[t_i, t_i + d_i]$, by starting at time t_i and ending at time $t_i + d_i$ ($d_i \geq 0$). The cost/utility is the time to overlap between agent i's interval $[t_i, t_i + d_i]$ and the activity duration $[y, y + d]$. Another insight for the activity scheduling games in the time domain is the potential natural extension to the asymmetric case: before the ideal time point and after the ideal time point by the same time difference might mean differently for an agent. This asymmetric case is hardly justifiable in traditional facility location games.

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