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State reconstruction of piecewise linear maps using a clustering machine

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Abstract. State reconstruction of piecewise linear systems is addressed. The description of such a family of systems involves, for each region of the partitioned state space, an affine description and a switching rule which orchestrates the way the dynamics changes from a linear form to another. It results on two distinct states : the continuous state the discrete state. An observer of piecewise linear systems must recover both of them. It is shown that the discrete state can be recovered by a clustering technique. The continuous state reconstruction is formulated as set of Linear Matrix Inequalities to be solved. They are derived from the notion of poly-quadratic stability and ensure global convergence of the observer.

1 Introduction

Piecewise linear systems have received a growing attention in control theory [1][2]. For those systems, the state space is partitioned into distinct regions. Their description not only involves an affine form related to a partition with a one-to-one correspondence, but also a switching rule which orchestrates the way the dynamics changes from a linear form to another. It results on two distinct states : the continuous state describing the local dynamics in each partition and the discrete state characterized by an indicator vector associated to the visited region of the continuous state vector. Observer of piecewise linear system must recover the complete state vector. Two main problems arise. Firstly, apart from a restricted class of systems, the discrete state is not known a priori and must be recovered from available information, that are, the inputs and the outputs of the system. Secondly, assuming that the discrete state is recovered, the computation of the observer gain matrix must ensure global stability of a time-varying reconstruction error equation involving switches among a finite set of constant dynamical matrices. For the time being, most of rigorous proofs of convergence relies on Lyapunov approach often involving quadratic Lyapunov functions.
In this paper, an attempt to provide a solution to the open problem, detailed above, of state reconstruction of piecewise linear systems is suggested. On one hand, it is shown that the problem of discrete state reconstruction is equivalent

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to clustering and a neural classifier achieves such a task. On the other hand, as conditions of convergence based upon quadratic Lyapunov function are conservative, a recent and novel notion named poly-quadratic stability, consisting in checking for a parameter Lyapunov function provided in [3][4] is recalled. The interest of the combination of those both points relies on the fact that it enables to enlarge the class of discrete time piecewise linear systems for which an observer can be designed.
The layout of this paper is as the following. In section 2, problem of observer design for piecewise linear systems is stated. In section 3, a solution to discrete and continuous state reconstruction is presented. Finally, 4 is devoted to an illustrative example through a chaos synchronization problem.

2 Problem formulation

In the remainder of the paper, '^{T}' stands for transposition. For some symmetric matrices X, $X > 0$ indicates that X is positive definite and the symbol $(\bullet)^T$ denotes each of its symmetric block.
Consider the discrete time system :

$$\begin{array}{lllllll} x_{k+1} & = & A_\alpha x_k & + & E_\alpha & + & B_\alpha u_k \\ y_k & = & C_\alpha x_k & & & & \end{array} \tag{1}$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^m$, $u_k \in \mathbb{R}^q$. α is a piecewise constant function of x_k from $\mathbb{R}^n$ to a finite index set $I = \{1, \ldots, P\}$. The state space $\mathbb{R}^n$ is partitioned into P regions denoted R_p with $\bigcup_{p=1}^{p=P} \subseteq \mathbb{R}^n$. α is thus a switching rule expressing that a unique constant matrix A_p is assigned to the region R_p visited by x_k at the discrete time k.
Consider the observer described by :

$$\begin{array}{lllllllll} \hat{x}_{k+1} & = & A_\beta \hat{x}_k & + & E_\beta & + & B_\beta u_k & + & L_\beta(y_k - \hat{y}_k) \\ \hat{y}_k & = & C_\beta \hat{x}_k & & & & & & \end{array} \tag{2}$$

β is a switching rule which will be discussed in the forthcoming section.
Introducing a switching rule to describe a piecewise linear dynamics, instead of involving the running index of the matrices at each discrete time k is more convenient for defining a so-called "indicator vector". This quantity enables to conveniently tackle the problem of state reconstruction.

Definition 1 *An indicator vector associated to both a switching rule γ and a vector v_k is the quantity $\xi_k^{\gamma,v} = (\xi_k^1, \ldots, \xi_k^P)^T$ whose components are given by :*

$$\xi_k^p = \begin{cases} 1 \text{ if } \gamma(v_k) = p \\ 0 \text{ otherwise} \end{cases}$$

For (1), $\gamma = \alpha$ and $v_k = x_k$ while for (2), $\gamma = \beta$ and v_k will be defined later. Exploiting indicator vectors related to the respective switching rules α and β, all matrices Y_γ of appropriate dimensions involved in (1) and (2) can be rewritten as follows :

$$Y_\gamma = \sum_{p=1}^{P} \xi_k^p Y_p \tag{3}$$

System (1) is piecewise linear and is completely defined at time k by both x_k referred to as the "continuous state" and the indicator vector $\xi_k^{\alpha,x}$ referred to as the "discrete state".
Problem is to design an observer such that $\hat{x}_k$ tends asymptotically towards x_k for any initial state $\hat{x}_0$, that is :

$$\lim_{k\to\infty} \|x_k - \hat{x}_k\| = 0 \quad \forall \hat{x}_0 \tag{4}$$

The observer is completely determined by the function β and the gain $L_\beta = \sum_{p=1}^{P} \xi_k^p L_p$ according to (3).

3 Piecewise linear observer

The proposed design involves two successive steps : the recovering of the discrete state $\xi_k^{\alpha,x}$, then the reconstruction of the continuous state x_k.

3.1 Discrete state reconstruction

Reconstruction must be achieved in such a way that both discrete states $\xi_k^{\alpha,x}$ of (1) and $\xi_k^{\beta,v}$ of (2) coincide. Due to the switching rules, an explicit recursion of the form $\xi_{k+1}^{\alpha,x} = f(x_k, \xi_k^{\alpha,x})$ cannot be obtained. Furthermore, the argument v_k of β cannot be the same as the one of α since x_k is not available. Discrete state reconstruction aims at building a switching function β and finding an argument v_k depending on available information which guarantees $\xi_k^{\alpha,x} = \xi_k^{\beta,v}$.
Reconstruction can be viewed as a clustering process. Indeed, let assume that there exist integers M, N and a nonlinear clustering function g from $\mathbb{R}^{M+N}$ to $\{0,1\}^P$ such that $\xi_k^{\alpha,x} = g(y_k, , \ldots, y_{k-N}, u_k, , \ldots, u_{k-M})$. Next, we can define β and $v_k = [y_k, , \ldots, y_{k-N}, u_k, , \ldots, u_{k-M}]^T$ as respectively the switching rule and its corresponding argument such that the indicator $\xi_k^{\beta,v}$ coincides with $g(v_k)$.
As in general, g cannot be found explicitly, it must be estimated from a finite set of l pairs $(\xi_k^{\alpha,x}, v_k)$, generated from the simulation of (1). v_k constitutes the regressor vector while $\xi_k^{\alpha,x}$ is the target. We must resort to classifiers to accomplish such a task. Support Vector Machines or Artificial Neural Networks can be candidates as clustering machines.
Now, assume that the clustering function g achieves the discrete state reconstruction. As a consequence, the equation governing the convergence error $\epsilon_k = x_k - \hat{x}_k$ can be obtained first by subtracting (1) and (2). In addition, taking into account (3) and the fact that the indicator vectors $\xi_k^{\alpha,x}$ and $\xi_k^{\beta,v}$ coincide yields :

$$\epsilon_{k+1} = \sum_{p=1}^{P} \xi_k^p (A_p - L_p C_p)\epsilon_k \tag{5}$$

Global convergence of (5) amounts to an achievement of the continuous state reconstruction. This later requires the computation of suitable gains L_p. This is done through the poly-quadratic stability notion recalled in the following subsection.

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3.2 Continuous state reconstruction

The reconstruction of the continuous state x_k relies on the recent and novel results of poly-quadratic stability of [4][3] from which a definition and a theorem are recalled. $\xi_k^{\alpha,x}$ and $\xi_k^{\beta,v}$ coinciding, as a shorthand, they will be conveniently denoted ξ_k.

Definition 2 *System (5) is said to be Poly-Quadratically stable if there exists a Positive Definite Parameter Dependent Quadratic Lyapunov Function (PDLF) $V(\epsilon_k, \xi_k) = \epsilon_k^T \mathcal{P}(\xi_k)\epsilon_k$ with $\mathcal{P}(\xi_k) = \sum_{p=1}^{P} \xi_k^p P_p$ whose difference along the solution of (5) satisfies*

$$\mathcal{L} = V(\epsilon_{k+1}, \xi_{k+1}) - V(\epsilon_k, \xi_k) \leq -\alpha_0(\|\epsilon_k\|) \quad (6)$$

with α_0 a κ_∞ function[1].

The various P_p, $p = 1, ..., P$, are symmetric positive definite (SPD) matrices of appropriate order.
The following theorem gives a necessary and sufficient condition for the piecewise linear system (5) to be Poly-Quadratically stable and then ensures global convergence.

Theorem 1 *Global convergence is achieved, if $\forall (i,j) \in \{1, \ldots, P\}$, there exist symmetric matrices S_i, matrices F_i and G_i which are solutions of:*

$$\begin{bmatrix} G_i + G_i^T - S_i & (\bullet)^T \\ A_i^T G_i - C_i^T F_i & S_j \end{bmatrix} > 0 \quad (7)$$

The resulting gains L_i are given by $L_i = (G_i^{-1})^T F_i^T$. In this case, the time varying PDLF corresponds to $\mathcal{P}(\xi_k) = \sum_{p=1}^{P} \xi_k^p S_p^{-1}$.

Remark 1 *Poly-Quadratic stability is sufficient for asymptotic stability. It is less conservative than Quadratic stability, in spite of the fact that it involves more LMIs. Indeed, Quadratic stability corresponds to $G_i = S_i = S$, a constant matrix.*

4 Illustrative example

For the purpose of illustration, we address the problem of chaos synchronization of piecewise linear systems. It is a particular case of the more general problem described in the paper since autonomous dynamics are considered.
The considered map is characterized by :

- $x_k = [x_k^1 \; x_k^2]^T$ and $\hat{x}_k = [\hat{x}_k^1 \; \hat{x}_k^2]^T$
- $A_i = \begin{bmatrix} 0 & 1 \\ h_i & 1 \end{bmatrix}$, $h_1 = -1.05$, $h_2 = 2$, $E_1 = [0 \; 0]^T$, $E_2 = [0 \; -18.3]^T$

[1] A function $\alpha : [0, \infty) \to [0, \infty)$ is a κ_∞ function if it is continuous, strictly increasing, zero at zero and unbounded ($\alpha(s) \to \infty$ as $s \to \infty$).

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Two regions R_1 and R_2 are respectively assigned to A_1 and A_2. R_1 is the set $\{x_k|x_k^1 < 6\}$ and R_2 is the set $\{x_k|x_k^2 \geq 6\}$. The output signal y_k corresponds to a constant matrix $C = [2 \;\; 2]$.
For this example, 700 pairs $(\xi_k^{\alpha,\bar{x}}, v_k)$ have been collected to train a conventional Multilayer Perceptron (MLP) with one hidden layer of 25 sigmodal units. Here, $M = 0$ as autonomous case is treated and we set $N = 3$. In chaos synchronization context, N is related to the so-called "embedding dimension". The ANN implements the clustering function g ensuring discrete state reconstruction. Continuous state reconstruction is achieved by suitable observer gains computed by solving the set a Linear Matrix Inequalities (7). Solution of (7) gives the gains $L_1 = [0.1326 \;\; -0.0256]^T$ and $L_2 = [0.1328 \;\; 0.4336]^T$.
Simulations of synchronization have been performed over 3000 points split into 700 training points and 2300 test points.
Results are depicted on Fig 1. It can be noted that some test errors on the discrete state reconstruction are likely to occur. On average, occurrence is few (0.6%) and the errors are well cancelled (Fig 1D).

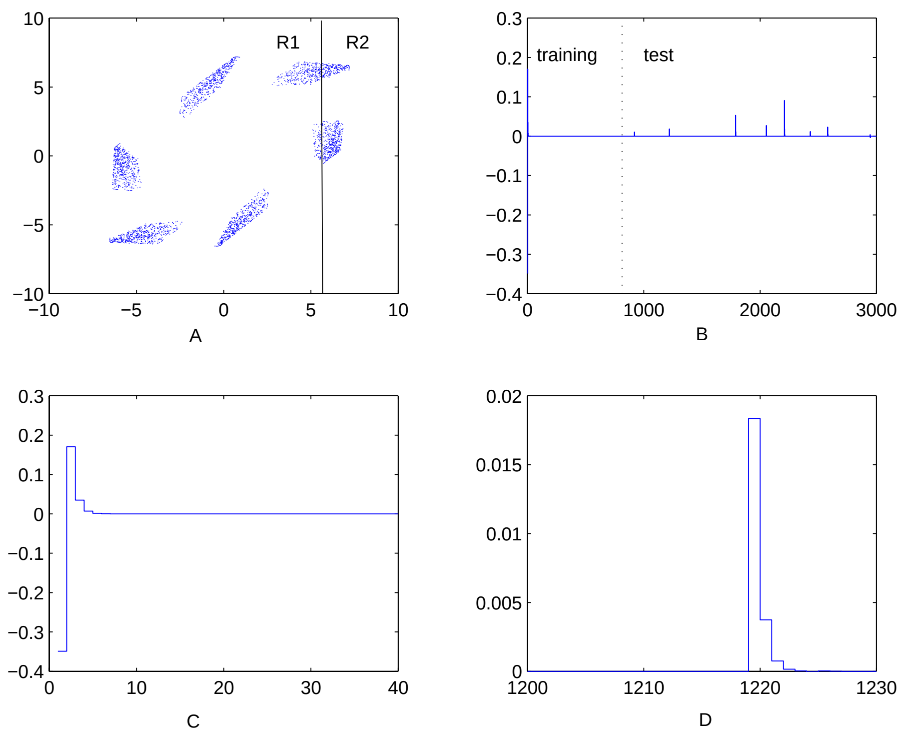

Figure 1: A : chaotic attractor and regions R_1, R_2. B : error of reconstruction $\|\epsilon_k\|$. C,D : transients towards zero for learning sequence (C) and for an error occurring in the test sequence (D).

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5 Discussion and concluding remarks

Throughout this paper, a observer design procedure for the reconstruction of the state of piecewise linear systems has been proposed. Those systems are characterized by both a continuous state vector for which local linear dynamics can be described and a discrete state vector for which no explicit dynamics is available. It has been shown that discrete state reconstruction amounts to build a clustering machine which must be trained such that indicators vectors of the system and the observer coincide. This allows to express the dynamics of the reconstruction error as a piecewise linear dynamics. Then, continuous state reconstruction can be tackled by a Lyapunov theory. The adopted theory is less conservative than the usual quadratic one : it is called poly-quadratic and involves a Parameter Dependent Lyapunov Function.
The assessment of the proposed reconstruction approach essentially depend on the machine learning performance. Indeed, problem is to guarantee a powerful generalization ability in order to avoid as many misclassified patterns as possible. To cope with this problem, the classical trade-off between complexity the machine and accuracy of the training error must be satisfied. Nevertheless, as opposed to "smooth" nonlinear systems, as a small perturbation can cause a misleading switch, it seems that further works would require a thorough effort directed to derive confidence bounds on the prediction and to refine strategies when detecting an error.

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