Research Article

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On generalized $\alpha-\psi$-Geraghty contractions on b-metric spaces

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Abstract: In this paper, we consider generalized $\alpha-\psi$-Geraghty contractive type mappings and investigate the existence and uniqueness of a fixed point for mappings involving such contractions. In particular, we extend, improve and generalize some earlier results in the literature on this topic. An application concerning the existence of an integral equation is also considered to illustrate the novelty of the main result.

Keywords: Complete b-metric space, fixed point, generalized $\alpha-\psi$-Geraghty contraction

MSC 2010: 74H10, 54H25, 46T99,47H10

1 Introduction

A very interesting extension of the notion of a metric, called b-metric, was proposed by Czerwik [11, 12]. In these pioneer papers, Czerwik observed some fixed point results, including the analog of the Banach contraction principle in the context of complete b-metric spaces. In the sequel, several papers have been reported on the existence (and the uniqueness) of (common) fixed points of various classes of single-valued and multi-valued operators in the setting of b-metric spaces (see, e.g., [2–4, 8–10, 13–15, 17, 18, 24, 27, 30, 31] and the related references therein).

In 2011, Samet, C. Vetro and P. Vetro [29] considered the concept of an admissible mapping to get a very general structure that combines several existing fixed point theorems by introducing $\alpha-\psi$-contractive type mappings in complete metric spaces. Karapınar and Samet [22] improved the results in [29] by defining the notion of generalized $\alpha-\psi$-contractive type mappings. They listed several existing results as consequences of their main results. Following these initial papers, Karapınar [19, 20] introduced $\alpha-\psi$-Geraghty contraction type mappings that generalize the results of Geraghty [16]. For other fixed points via α-admissible mappings, see, e.g., [1, 5–7, 21, 28]. In this paper, we introduce the concept of generalized $\alpha-\psi$-Geraghty contraction type mappings in complete b-metric spaces and investigate the existence and uniqueness of a fixed point for such mappings.

For the sake of completeness, we recall some basic notions, notations and fundamental results. In the sequel, the standard letters $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{N}_0$ and $\mathbb{N}$ will represent the set of all real numbers, the set of all non-negative real numbers, the set of all non-negative integer numbers and the set of all positive integer numbers, respectively.

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Definition 1.1 ([12]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to \mathbb{R}_0^+$ is said to be a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:
(b1) $d(x, y) = 0$ if and only if $x = y$,
(b2) $d(x, y) = d(y, x)$,
(b3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
In this case, the pair (X, d) is called a b-metric space (with constant s).

Remark 1.2. Since a metric space is a b-metric space by taking the constant $s = 1$, the class of b-metric spaces is larger than the class of metric spaces.

The following example shows that there exists a b-metric which is not a metric.

Example 1.3. Let $X = \{0, 1, 2\}$ and let $d\colon X \times X \to [0, \infty)$ be defined by

$$d(0,1) = 1, \quad d(0,2) = \frac{1}{2} \quad \text{and} \quad d(1,2) = 2,$$

with $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. Notice that d is not a metric, since we have $d(1, 2) > d(1, 0) + d(0, 2)$. However, it is easy to see that d is a b-metric with $s \geq \frac{4}{3}$.

Definition 1.4 ([29]). Let $T\colon X \to X$ be a mapping and let $\alpha\colon X \times X \to [0, \infty)$ be a function. The mapping T is said to be α-admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.$$

Definition 1.5 ([21]). A self-mapping $T\colon X \to X$ is called triangular α-admissible if the following hold:
(T1) T is α-admissible,
(T2) $\alpha(x, z) \geq 1, \alpha(z, y) \geq 1 \implies \alpha(x, y) \geq 1, x, y, z \in X$.

Very recently, Popescu [25] has improved the notion of a triangular α-admissible mapping as follows.

Definition 1.6 ([25]). Let $T\colon X \to X$ be a self-mapping and let $\alpha\colon X \times X \to [0, \infty)$ be a function. Then T is said to be α-orbital admissible if the following implication holds:
(T3) $\alpha(x, Tx) \geq 1 \implies \alpha(Tx, T^2x) \geq 1$.

Definition 1.7 ([25]). Let $T\colon X \to X$ be a self-mapping and let $\alpha\colon X \times X \to [0, \infty)$ be a function. Then T is said to be triangular α-orbital admissible if T is α-orbital admissible and the following implication holds:
(T4) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1 \implies \alpha(x, Ty) \geq 1$.

As mentioned in [25], each α-admissible mapping is an α-orbital admissible mapping and each triangular α-admissible mapping is a triangular α-orbital admissible mapping. The converse is false, see, e.g., [25, Example 7].

Definition 1.8 ([25]). Let (X, d) be a b-metric space and let $\alpha\colon X \times X \to X$ be a function. X is said to be α-regular if for every sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ with $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Lemma 1.9 ([25]). *Let $T\colon X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}_0$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$, with $n < m$.*

2 Main results

Now, we are ready to state and prove our main results. Let Ψ be the set of all increasing and continuous functions $\psi\colon [0, \infty) \to [0, \infty)$, with $\psi^{-1}(\{0\}) = \{0\}$. Let $\mathcal{F}$ be the family of all non-decreasing functions $\beta\colon [0, \infty) \to [0, \frac{1}{s})$ which satisfy the condition

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s} \implies \lim_{n\to\infty} t_n = 0 \quad \text{for some } s \geq 1.$$

Definition 2.1. Let (X, d) be a b-metric space and let $T: X \to X$ be a self-map. We say that T is a generalized $\alpha-\psi$-Geraghty contractive mapping whenever there exist $\alpha: X \times X \to [0,\infty)$ and some $L \geq 0$ such that for

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\},$$
$$N(x,y) = \min\{d(x,Tx), d(y,Tx)\},$$

we have

$$\alpha(x,y)\psi(s^3 d(Tx,Ty)) \leq \beta(\psi(M(x,y)))\psi(M(x,y)) + L\phi(N(x,y)) \tag{2.1}$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$.

Remark 2.2. Since the functions belonging to $\mathcal{F}$ are strictly smaller than $\frac{1}{s}$, the expression $\beta(\psi(M(x,y)))$ in (2.1) can be estimated as

$$\beta(\psi(M(x,y))) < \frac{1}{s} \quad \text{for any } x, y \in X, \text{ with } x \neq y.$$

Theorem 2.3. *Let (X, d) be a complete b-metric space and let $T: X \to X$ be a generalized $\alpha-\psi$-Geraghty contractive mapping with the following properties:*
(i) *T is triangular α-orbital admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *T is continuous.*
Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. We construct an iterative sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}_0.$$

If there exists n_0 such that $Tx_{n_0} = x_{n_0}$ for some n_0, then x_{n_0} is a fixed point of T, which completes the proof. Thus, without loss of generality, we assume that

$$x_n \neq x_{n+1} \quad \text{for all } n \in \mathbb{N}_0. \tag{2.2}$$

Since the mapping T is triangular α-orbital admissible, by Lemma 1.9, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}_0. \tag{2.3}$$

By taking $x = x_{n-1}$ and $y = x_n$ in inequality (2.1), using inequality (2.3) and recalling that ψ is an increasing function, we obtain

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n))\\ &\leq \alpha(x_{n-1}, x_n)\psi(s^3 d(Tx_{n-1}, Tx_n))\\ &\leq \beta(\psi(M(x_{n-1}, x_n)))\psi(M(x_{n-1}, x_n)) + L\phi(N(x_{n-1}, x_n))\end{aligned} \tag{2.4}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2s}\right\}\\ &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s}\right\}\\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\right\}\end{aligned}$$

and

$$N(x_{n-1}, x_n) = \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1})\} = \min\{d(x_{n-1}, x_n), d(x_n, x_n)\} = 0. \tag{2.5}$$

Since

$$\frac{d(x_{n-1}, x_{n+1})}{2s} \leq \frac{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{2s} \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

we get

$$M(x_{n-1},x_n) \le \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}. \tag{2.6}$$

Taking (2.6) and (2.5) into account, (2.4) yields

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &\le \psi(s^3 d(x_n,x_{n+1}))\\ &\le \alpha(x_{n-1},x_n)\psi(s^3 d(x_n,x_{n+1}))\\ &\le \beta(\psi(M(x_{n-1},x_n)))\psi(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}).\end{aligned} \tag{2.7}$$

If for some $n \in \mathbb{N}$, we have $\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then, by (2.7) and Remark 2.2, we get

$$\psi(d(x_n,x_{n+1})) \le \beta(\psi(M(x_{n-1},x_n))\psi(d(x_n,x_{n+1}) < \frac{1}{s}\psi(d(x_n,x_{n+1}) < \psi(d(x_n,x_{n+1}),$$

which is a contradiction. Thus, from (2.7) we conclude that

$$\psi(d(x_n,x_{n+1})) \le \beta(\psi(M(x_{n-1},x_n)))\psi(d(x_{n-1},x_n)) < \frac{1}{s}\psi(d(x_{n-1},x_n)) < \psi(d(x_{n-1},x_n)) \tag{2.8}$$

for all $n \in \mathbb{N}$. Hence, $\{\psi(d(x_n,x_{n+1}))\}$ is a non-negative decreasing sequence. Since ψ is increasing, the sequence $\{d(x_n,x_{n+1})\}$ is non-increasing. Consequently, there exists $\delta \ge 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = \delta$. We claim that $\delta = 0$. Suppose, on the contrary, that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = \delta > 0.$$

Since $s \ge 1$, inequality (2.8) can be estimated as

$$\frac{1}{s}\psi(d(x_n,x_{n+1})) \le \psi(d(x_n,x_{n+1})) \le \beta(\psi(M(x_{n-1},x_n)))\psi(d(x_{n-1},x_n)). \tag{2.9}$$

With regard to (2.2), inequality (2.9) implies that

$$\frac{1}{s}\frac{\psi(d(x_n,x_{n+1}))}{\psi(d(x_{n-1},x_n))} \le \beta(\psi(M(x_{n-1},x_n))) < \frac{1}{s}.$$

This yields $\lim_{n\to\infty}\beta(\psi(M(x_{n-1},x_n))) = \frac{1}{s}$. Since $\beta \in \mathcal{F}$, we have $\lim_{n\to\infty}\psi(M(x_{n-1},x_n)) = 0$. We deduce that

$$\lim_{n\to\infty}\psi(d(x_n,x_{n+1})) = 0.$$

Thus, taking into account the fact that $d(x_n,x_{n+1}) \to \delta$ and the continuity of ψ, we derive $\psi(\delta) = 0$. Since $\psi^{-1}(\{0\}) = \{0\}$, we get $\delta = 0$, which is a contradiction. Thus, we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.10}$$

Now, we claim that

$$\lim_{m,n\to\infty} d(x_n,x_m) = 0.$$

Assume, on the contrary, that there exist $\varepsilon > 0$ and subsequences $\{x_{m_i}\}$, $\{x_{n_i}\}$ of $\{x_n\}$, with $n_i > m_i \ge i$, such that

$$d(x_{m_i},x_{n_i}) \ge \varepsilon. \tag{2.11}$$

Additionally, for each m_i, we may choose n_i so that it is the smallest integer satisfying (2.11) and $n_i > m_i \ge i$. Then we have

$$d(x_{m_i},x_{n_i-1}) < \varepsilon. \tag{2.12}$$

From (2.11) and the triangle inequality, we obtain

$$\begin{aligned}\varepsilon \le d(x_{n_i},x_{m_i}) &\le sd(x_{n_i},x_{n_{i+1}}) + sd(x_{n_{i+1}},x_{m_i})\\ &\le sd(x_{n_i},x_{n_{i+1}}) + s^2 d(x_{n_{i+1}},x_{m_{i+1}}) + s^2 d(x_{m_{i+1}},x_{m_i}).\end{aligned} \tag{2.13}$$

Letting $i \to \infty$ and taking (2.10) into account, inequality (2.13) yields

$$\frac{\varepsilon}{s^2} \leq \limsup_{i\to\infty} d(x_{n_{i+1}}, x_{m_{i+1}}). \tag{2.14}$$

By Lemma 1.9, recall that $\alpha(x_{m_i}, x_{n_i}) \geq 1$. Consequently, by (2.1), we have

$$\begin{aligned} \psi(d(x_{n_{i+1}}, x_{m_{i+1}})) &= \psi(d(Tx_{n_i}, Tx_{m_i})) \\ &\leq \psi(s^3\, d(Tx_{n_i}, Tx_{m_i})) \\ &\leq \alpha(x_{m_i}, x_{n_i})\psi(s^3 d(Tx_{n_i}, Tx_{m_i})) \\ &\leq \beta(\psi(M(x_{n_i}, x_{m_i})))\psi(M(x_{n_i}, x_{m_i})) + L\phi(d(x_{m_i}, Tx_{n_i})), \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} M(x_{n_i}, x_{m_i}) &= \max\Big\{d(x_{n_i}, x_{m_i}), d(x_{n_i}, Tx_{n_i}), d(x_{m_i}, Tx_{m_i}), \frac{d(x_{n_i}, Tx_{m_i}) + d(x_{m_i}, Tx_{n_i})}{2s}\Big\} \\ &= \max\Big\{d(x_{n_i}, x_{m_i}), d(x_{n_i}, x_{n_{i+1}}), d(x_{m_i}, x_{m_{i+1}}), \frac{d(x_{n_i}, x_{m_{i+1}}) + d(x_{m_i}, x_{n_{i+1}})}{2s}\Big\} \end{aligned}$$

and

$$N(x_{n_i}, x_{m_i}) = \min\{d(x_{n_i}, Tx_{n_i}), d(x_{m_i}, Tx_{n_i})\} = \min\{d(x_{n_i}, x_{n_i+1}), d(x_{m_i}, x_{n_i+1})\}.$$

Notice that

$$\frac{d(x_{n_i}, x_{m_{i+1}}) + d(x_{m_i}, x_{n_{i+1}})}{2s} \leq \frac{s[d(x_{n_i}, x_{m_i}) + d(x_{m_i}, x_{m_{i+1}})] + s[d(x_{m_i}, x_{n_i}) + d(x_{n_i}, x_{n_{i+1}})]}{2s} \tag{2.16}$$

and

$$d(x_{n_i}, x_{m_i}) \leq s[d(x_{n_i}, x_{n_i-1}) + d(x_{n_i-1}, x_{m_i})] < sd(x_{n_i}, x_{n_i-1}) + s\varepsilon. \tag{2.17}$$

Taking (2.12), (2.16) and (2.17) into account, we find that

$$\limsup_{i\to\infty} M(x_{n_i}, x_{m_i}) \leq s\varepsilon, \tag{2.18}$$

$$\lim_{i\to\infty} N(x_{n_i}, x_{m_i}) = 0. \tag{2.19}$$

By taking the upper limit as $i \to \infty$ and using condition (T4) together with expressions (2.14), (2.18) and (2.19), inequality (2.15) becomes

$$\begin{aligned} \frac{1}{s}\psi(s\varepsilon) \leq \psi(s\varepsilon) &\leq \limsup_{i\to\infty} \psi(s^3\, d(x_{n_{i+1}}, x_{m_{i+1}})) \\ &\leq \limsup_{i\to\infty} \alpha(x_{m_i}, x_{n_i})\psi(s^3 d(x_{n_{i+1}}, x_{m_{i+1}})) \\ &= \limsup_{i\to\infty} \alpha(x_{m_i}, x_{n_i})\psi(s^3 d(Tx_{n_i}, Tx_{m_i})) \\ &\leq \limsup_{i\to\infty} [\beta(\psi(M(x_{n_i}, x_{m_i})))\psi(M(x_{n_i}, x_{m_i})) + L\phi(N(d(x_{n_i}, x_{m_i})))] \\ &\leq \psi(s\varepsilon) \limsup_{i\to\infty} \beta(\psi(M(x_{n_i}, x_{m_i}))) \\ &\leq \frac{1}{s}\psi(s\varepsilon). \end{aligned}$$

Then $\limsup_{i\to\infty} \beta(\psi(M(x_{n_i}, x_{m_i}))) = \frac{1}{s}$. Due to the fact that $\beta \in \mathcal{F}$, we have

$$\limsup_{i\to\infty} \psi(M(x_{n_i}, x_{m_i})) = 0.$$

Thus, we conclude that

$$\lim_{i\to\infty} \psi(d(x_{n_i}, x_{m_i})) = 0.$$

Therefore, by the continuity of ψ and the fact that $\psi^{-1}(\{0\}) = \{0\}$, we have

$$\lim_{i\to\infty} d(x_{n_i}, x_{m_i}) = 0,$$

which contradicts (2.11). We deduce that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete b-metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. The mapping T is continuous and it is obvious that $Tx^* = x^*$. □

We replace the continuity of the mapping T in the above theorem by a suitable condition on X.

Theorem 2.4. *Let (X, d) be a complete b-metric space and let $T: X \to X$ be a generalized $\alpha-\psi$-Geraghty contractive mapping with the following properties:*
(i) *T is triangular α-orbital admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *X is α-regular.*
Then T has a fixed point.

Proof. Following the lines in the proof of Theorem 2.3, we conclude that $\lim_{n\to\infty} x_n = x^*$. If X is α-regular, then, since $\alpha(x_n, x_{n+1}) \geq 1$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n_k}, x^*) \geq 1 \tag{2.20}$$

for all k. By the triangle inequality, we have

$$d(x^*, Tx^*) \leq sd(x^*, x_{n_k+1}) + sd(x_{n_k+1}, Tx^*) = sd(x^*, x_{n_k+1}) + sd(Tx_{n_k}, Tx^*).$$

Letting k tend to infinity yields

$$d(x^*, Tx^*) \leq \liminf_{k\to\infty} sd(Tx_{n_k}, Tx^*). \tag{2.21}$$

Using the fact that $\psi \in \Psi$, (2.20) and (2.21), we get

$$\begin{aligned} \psi(s^2 d(x^*, Tx^*)) &\leq \lim_{k\to\infty} \psi(s^3 d(Tx_{n_k}, Tx^*)) \\ &\leq \lim_{k\to\infty} \alpha(x_{n_{k+1}}, x^*)\psi(s^3 d(Tx_{n_k}, Tx^*)) \\ &\leq \lim_{k\to\infty} [\beta(\psi(M(x_{n_k}, x^*)))\psi(M(x_{n_k}, x^*)) + L\phi(N(x_{n_k}, x^*))]. \end{aligned} \tag{2.22}$$

We have

$$\begin{aligned} M(x_{n_k}, x^*) &= \max\Big\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2s}\Big\} \\ &= \max\Big\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_{k+1}}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_{k+1}})}{2s}\Big\} \end{aligned}$$

and

$$N(x_{n_k}, x^*) = \min\{d(x_{n_k}, Tx_{n_k}), d(x^*, Tx_{n_k})\} = \min\{d(x_{n_k}, x_{n_{k+1}}), d(x^*, x_{n_{k+1}})\}.$$

Recall that

$$\frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_{k+1}})}{2s} \leq \frac{sd(x_{n_k}, x^*) + sd(x^*, Tx^*) + d(x^*, x_{n_{k+1}})}{2s}.$$

Then, by (2.10), we get

$$\limsup_{k\to\infty} \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_{k+1}})}{2s} \leq \frac{d(x^*, Tx^*)}{2}.$$

When k tends to infinity, we deduce

$$\lim_{k\to\infty} M(x_{n_k}, x^*) = d(x^*, Tx^*)$$

and

$$\lim_{k\to\infty} N(x_{n_k}, x^*) = 0.$$

Since $\beta(\psi(M(x_{n_k}, x^*))) \le \frac{1}{s}$ for all $k \in \mathbb{N}$, from (2.22), we obtain

$$\psi(s^2 d(x^*, Tx^*)) \le \frac{1}{s}\psi(d(x^*, Tx^*)) \le \psi(d(x^*, Tx^*)).$$

Since $\psi \in \Psi$, the above holds unless $d(x^*, Tx^*) = 0$, that is, $Tx^* = x^*$ and x^* is a fixed point of T. □

For the uniqueness of a fixed point of a generalized $\alpha - \psi$ contractive mapping, we will consider the following hypothesis:
(H) For all $x, y \in \mathrm{Fix}(T)$, either $\alpha(x, y) \ge 1$ or $\alpha(y, x) \ge 1$.
Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 2.5. *Adding condition* (H) *to the hypotheses of Theorem 2.3 (respectively, Theorem 2.4), we obtain the uniqueness of the fixed point of T.*

Proof. Suppose that x^* and y^* are two fixed points of T. It is obvious that $M(x^*, y^*) = d(x^*, y^*)$ and $N(x^*, y^*) = 0$. Hence,

$$\begin{aligned}\psi(d(x^*, y^*)) &\le \psi(s^3 d(Tx^*, Ty^*))\\ &\le \alpha(x^*, y^*)\psi(s^3 d(Tx^*, Ty^*))\\ &\le \beta(\psi(M(x^*, y^*)))\psi(M(x^*, y^*)) + L\phi(N(x^*, y^*))\\ &< \frac{1}{s}\psi(d(x^*, y^*))\\ &\le \psi(d(x^*, y^*)),\end{aligned}$$

which is contradiction. □

Definition 2.6. Let (X, d) be a b-metric space and let $T\colon X \to X$ be a self-mapping. We say that T is a generalized $\alpha - \psi$-Geraghty contractive mapping of type (B) whenever there exists $\alpha\colon X \times X \to [0, \infty)$ such that for

$$M(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\Big\},$$

we have

$$\alpha(x, y)\psi(s^3 d(Tx, Ty)) \le \beta(\psi(M(x, y)))\psi(M(x, y))$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi \in \Psi$.

From the proofs of Theorems 2.3, 2.4 and 2.5, we get the following results.

Theorem 2.7. *Let (X, d) be a complete b-metric space and let $T\colon X \to X$ be a generalized $\alpha - \psi$-Geraghty contractive mapping of type* (B) *with the following properties:*
(i) *T is triangular α-orbital admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$,*
(iii) *either T is continuous or X is α-regular.*
Then T has a fixed point.

Theorem 2.8. *Adding condition* (H) *to the hypotheses of Theorem 2.7, we obtain the uniqueness of the fixed point of T.*

Example 2.9. Let X be a set of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |x(t)|\, dt < 1.$$

Define $d\colon X \times X \to [0, \infty)$ by

$$d(x, y) = \Big(\int_0^1 |x(t) - y(t)|\, dt\Big)^2.$$

Then d is a b-metric on X with $s = 2$.

The operator $T\colon X\to X$ is defined by

$$Tx(t)=\frac{1}{4}\ln(1+|x(t)|).$$

Consider the mappings $\alpha\colon X\times X\to[0,\infty)$, $\beta\colon[0,\infty)\to[0,\frac{1}{2})$ and $\psi\colon[0,\infty)\to[0,\infty)$ defined by

$$\alpha(x,y)=\begin{cases}1 & \text{if } x(t)\ge y(t) \text{ for all } t\in[0,1],\\ 0 & \text{otherwise.}\end{cases}\qquad \beta(t)=\frac{(\ln(1+\sqrt{t}))^2}{2t}\quad\text{and}\quad \psi(t)=t.$$

Evidently, $\psi\in\Psi$ and $\beta\in\mathcal{F}$. Moreover, T is a triangular α-orbital admissible mapping and $\alpha(1,T1)\ge 1$.

Now, we shall prove that T is a generalized $\alpha-\psi$-Geraghty contractive mapping. Indeed, for all $t\in[0,1]$, we have

$$\begin{aligned}\sqrt{\alpha(x(t),y(t))\psi(s^3d(Tx(t),Ty(t)))}&\le\sqrt{2^3\Big(\int_0^1|Tx(t)-Ty(t)|\,dt\Big)^2}\\&\le 2\sqrt{2}\int_0^1\Big|\frac{1}{4}\ln(1+|x(t)|)-\frac{1}{4}\ln(1+|y(t)|)\Big|\,dt\\&=\frac{1}{\sqrt{2}}\int_0^1\Big|\ln\Big(\frac{1+|x(t)|}{1+|y(t)|}\Big)\Big|\,dt\\&=\frac{1}{\sqrt{2}}\int_0^1\Big|\ln\Big(1+\frac{|x(t)|-|y(t)|}{1+|y(t)|}\Big)\Big|\,dt\\&\le\frac{1}{\sqrt{2}}\int_0^1\big|\ln(1+|x(t)|-|y(t)|)\big|\,dt.\end{aligned}$$

By Lemma A.1 (given in Appendix A), we get

$$\int_0^1\big|\ln(1+|x(t)|-|y(t)|)\big|\,dt\le\ln\Big(\int_0^1(1+|x(t)-y(t)|)\,dt\Big)=\ln\Big(1+\int_0^1|x(t)-y(t)|\,dt\Big).$$

Therefore,

$$\sqrt{\alpha(x(t),y(t))\psi(s^3d(Tx(t),Ty(t)))}\le\frac{1}{\sqrt{2}}\ln\Big(1+\int_0^1|x(t)-y(t)|\,dt\Big)\le\frac{1}{\sqrt{2}}\ln(1+\sqrt{d(x,y)}).$$

So, we obtain

$$\begin{aligned}\alpha(x(t),y(t))\psi(s^3d(Tx(t),Ty(t)))&\le\frac{1}{2}(\ln(1+\sqrt{d(x,y)}))^2\\&\le\frac{1}{2}(\ln(1+\sqrt{M(x,y)}))^2\\&=\frac{(\ln(1+\sqrt{M(x,y)}))^2}{2M(x,y)}M(x,y)\\&=\beta(\psi(M(x,y)))\psi(M(x,y)).\end{aligned}$$

Thus, by Theorem 2.7, we see that T has a fixed point.

3 Consequences

In this section, we demonstrate that several existing results in the literature can be easily concluded from Theorem 2.5.

3.1 Standard fixed point theorems in a *b*-metric

By taking $\alpha(x, y) = 1$ in Theorem 2.5, for all $x, y \in X$, we immediately obtain the following corollary.

Corollary 3.1. *Let (X, d) be a complete b-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping on X. If there exists $L \geq 0$ such that for all $x, y \in X$,*

$$\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) + L\phi(N(x, y)),$$

where $\beta \in \mathcal{F}$, $\psi, \phi \in \Psi$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\},$$
$$N(x, y) = \min\{d(x, Tx), d(y, Tx)\},$$

then T has a unique fixed point.

By taking $\alpha(x, y) = 1$ in Theorem 2.8, for all $x, y \in X$, we immediately obtain the following fixed point result.

Corollary 3.2. *Let (X, d) be a complete b-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping on X such that for all $x, y \in X$,*

$$\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where $\beta \in \mathcal{F}$, $\psi \in \Psi$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Then T has a unique fixed point.

If we put $\alpha(x, y) = 1$ for all $x, y \in X$, $L = 0$ and $\psi(t) = t$ in Theorem 2.5, we may state the following result.

Corollary 3.3. *Let (X, d) be a complete b-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping on X such that for all $x, y \in X$,*

$$s^3 d(Tx, Ty) \leq \beta(M(x, y))M(x, y),$$

where $\beta \in \mathcal{F}$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Then T has a unique fixed point.

If we take $s = 1$ and $\beta(t) = \frac{1}{t+1}$ for $t > 0$ in Corollary 3.3, we deduce the following result.

Corollary 3.4. *Let (X, d) be a complete metric space and let $T: X \to X$ be a mapping on X such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \frac{M(x, y)}{1 + M(x, y)}.$$

Then T has a unique fixed point.

3.2 Fixed point theorems on *b*-metric spaces endowed with a partial order

In the last decade, several exciting developments have been reported in the field of existence of a fixed point on metric spaces endowed with partial orders, see, e.g., [23, 26, 32]. In this section, from Theorem 2.5 (and also from Theorem 2.8), we shall easily conclude some fixed point results on a *b*-metric space endowed with a partial order. First of all, we recall some basic concepts.

Definition 3.5. Let $(X, \preceq)$ be a partially ordered set and let $T: X \to X$ be a given mapping. We say that T is non-decreasing with respect to $\preceq$ if

$$x, y \in X, \ x \preceq y \implies Tx \preceq Ty.$$

Definition 3.6. Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be non-decreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 3.7. Let $(X, \preceq)$ be a partially ordered set and let d be a b-metric on X. We say that $(X, \preceq, d)$ is regular if for every non-decreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

We have the following result.

Corollary 3.8. *Let $(X, \preceq)$ be a partially ordered set and let d be a b-metric on X such that (X, d) is complete. Let $T\colon X \to X$ be a non-decreasing mapping with respect to $\preceq$. Suppose that there exist functions $\beta \in \mathcal{F}$ and $\psi \in \Psi$ such that*

$$\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y))$$

and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:
(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(ii) *T is continuous or $(X, \preceq, d)$ is regular.*
Then T has a fixed point. Moreover, if for all $x, y \in$ Fix(T) either $x \preceq y$ or $y \preceq x$, then the fixed point is unique.

Proof. Define the mapping $\alpha\colon X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \text{ or } x \succeq y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, T is a generalized $\alpha - \psi$ contractive mapping, that is,

$$\alpha(x, y)\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y))$$

for all $x, y \in X$. From condition (i), we have $\alpha(x_0, Tx_0) \geq 1$. On the other hand, for all $x, y \in X$, from the monotone property of T, we have

$$\alpha(x, y) \geq 1 \implies x \succeq y \text{ or } x \preceq y \implies Tx \succeq Ty \text{ or } Tx \preceq Ty \implies \alpha(Tx, Ty) \geq 1.$$

So T is α-admissible. If T is continuous, the existence of a fixed point is concluded from Theorem 2.7. Now, assume that $(X, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. From the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. From the definition of α, we have that $\alpha(x_{n(k)}, x) \geq 1$ for all k. In this case, the existence of a fixed point follows from Theorem 2.7. To prove the uniqueness, let $x, y \in X$. Due to the hypothesis, we have $\alpha(x, y) \geq 1$ and $\alpha(y, x) \geq 1$. Hence, by Theorem 2.8, we conclude the uniqueness of the fixed point. □

The following results are immediate consequences of Corollary 3.8.

Corollary 3.9. *Let $(X, \preceq)$ be a partially ordered set and let d be a b-metric on X such that (X, d) is complete. Let $T\colon X \to X$ be a non-decreasing mapping with respect to $\preceq$. Suppose that there exist functions $\beta \in \mathcal{F}$ and $\psi \in \Psi$ such that*

$$\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

for all $x, y \in X$, with $x \succeq y$. Suppose also that the following conditions hold:
(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(ii) *T is continuous or $(X, \preceq, d)$ is regular.*
Then T has a fixed point. Moreover, if for all $x, y \in$ Fix(T) either $x \preceq y$ or $y \preceq x$, then the fixed point is unique.

Remark 3.10. In fact, in all the results above, one can take $s = 1$ to conclude the existing results in the literature.

4 Application

As an application, we consider the following integral equation:

$$x(t) = h(t) + \int_0^1 k(t,\xi) f(\xi, x(\xi))\, d\xi \quad \text{for all } t \in [0,1]. \tag{4.1}$$

Let Ω denote the class of non-decreasing functions $\omega\colon [0,\infty) \to [0,\infty)$ satisfying

$$(\omega(t))^r \le t^r \omega(t^r) \quad \text{for all } r \ge 1 \text{ and all } t \ge 0.$$

We will analyze equation (4.1) under the following assumptions:

(a1) $h\colon [0,1] \to \mathbb{R}$ is a continuous function.

(a2) $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t,x) \ge 0$ and there exists $\omega \in \Omega$ such that for all $x, y \in \mathbb{R}$,

$$|f(t,x) - f(t,y)| \le \omega(|x-y|),$$

with $w(t_n) \to \frac{1}{2^{r-1}}$ as $n \to \infty$ implying $\lim_{n\to\infty} t_n = 0$.

(a3) $k\colon [0,1] \times [0,1] \to \mathbb{R}$ is continuous in $t \in [0,1]$ for every $\xi \in [0,1]$ and is measurable in $\xi \in [0,1]$ for all $t \in [0,1]$ such that $k(t,x) \ge 0$ and

$$\int_0^1 k(t,\xi)\, d\xi \le \frac{1}{2^{3-\frac{3}{r}}}.$$

Consider the space of continuous functions $X = C([0,1])$, with the standard metric given by

$$\rho(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)| \quad \text{for all } x, y \in C([0,1]).$$

Now, for $r \ge 1$, we define

$$d(x,y) = (\rho(x,y))^r = \Big(\sup_{t\in[0,1]} |x(t)-y(t)|\Big)^r = \sup_{t\in[0,1]} |x(t)-y(t)|^r \quad \text{for all } x,y \in C([0,1]).$$

Note that (X,d) is a complete b-metric space with $s = 2^{r-1}$.

Theorem 4.1. *Under assumptions* (a1)–(a3), *equation* (4.1) *has a unique solution in* $C([0,1])$.

Proof. We consider the operator $T\colon X \to X$ defined by

$$T(x)(t) = h(t) + \int_0^1 k(t,\xi) f(\xi, x(\xi)) d\xi, \quad t \in [0,1].$$

By virtue of our assumptions, T is well defined (this means that if $x \in X$, then $Tx \in X$). Also, for $x, y \in X$, we have

$$\begin{aligned} |T(x)(t) - T(y)(t)| &= \Big| h(t) + \int_0^1 k(t,\xi) f(\xi,x(\xi))\, d\xi - h(t) - \int_0^1 k(t,\xi) f(\xi,x(\xi))\, d\xi \Big| \\ &\le \int_0^1 k(t,\xi) |f(\xi,x(\xi)) - f(\xi,y(\xi))|\, d\xi \\ &\le \int_0^1 k(t,\xi) \omega(|x(\xi)-y(\xi)|)\, d\xi. \end{aligned}$$

Since the function ω is non-decreasing, we get

$$\omega(|x(\xi)-y(\xi)|) \le \omega\Big(\sup_{t\in[0,1]} |x(\xi)-y(\xi)|\Big) = \omega(\rho(x,y)).$$

Therefore,

$$|T(x)(t)-T(y)(t)| \le \frac{1}{2^{3-\frac{3}{r}}}\omega(\rho(x,y)).$$

Now, we have

$$\begin{aligned} d(Tx,Ty) &= \sup_{t\in[0,1]} |T(x)(t)-T(y)(t)|^r \le \Big[\frac{1}{2^{3-\frac{3}{r}}}\omega(\rho(x,y))\Big]^r \\ &\le \frac{1}{2^{3r-3}} d(x,y)\omega(d(x,y)) \le \frac{1}{2^{3r-3}}\omega(M(x,y))M(x,y), \end{aligned}$$

that is,

$$s^3 d(Tx,Ty) \le \beta(M(x,y))M(x,y),$$

where $s=2^{r-1}$ and $\beta(t)=\omega(t)$. Notice that if $\omega\in\mathcal{F}$, then $\beta\in\mathcal{F}$. By Corollary 3.3, equation (4.1) has a unique solution in $C[0,1]$ and the proof is completed. □

A Appendix

Lemma A.1. *Let (X,μ) be a measure space such that $\mu(X)=1$. Let $f\in L^1(X,\mu)$, with $f(x)>0$ for all $x\in X$. Then $\ln(f)\in L^1(X,\mu)$ and*

$$\int \ln(f)\,d\mu \le \ln\Big(\int f\,d\mu\Big).$$

Proof. Put $g(t):=t-1-\ln(t)$ and $h(t):=1-\frac{1}{t}-\ln(t)$ for $t>0$. Then $g'(t)=1-\frac{1}{t}$ and $h'(t)=\frac{1}{t^2}-\frac{1}{t}$. Clearly, we have

$$g(t)\ge g(1)=0 \quad \text{and} \quad h(t)\le h(1)=0 \quad \text{for all } t>0.$$

We deduce

$$t-1\ge \ln(t)\ge 1-\frac{1}{t} \quad \text{for all } t>0. \tag{A.1}$$

Since f is measurable and ln is continuous, $\ln(f)$ is measurable. Now, for all $x\in X$, let $t=\frac{f(x)}{\|f\|_1}$ in (A.1). So, we have

$$1-\frac{\|f\|_1}{f(x)} \le \ln(f(x))-\ln(\|f\|_1) \le \frac{f(x)}{\|f\|_1}-1.$$

Since the right-hand and the left-hand expression in the above estimations are both integrable, we have that $\ln(f(x))-\ln(\|f\|_1)$ is integrable as well. We also have

$$\int (\ln(f(x))-\ln(\|f\|_1))\,d\mu \le \int\Big(\frac{f(x)}{\|f\|_1}-1\Big)\,d\mu = 0.$$

Therefore,

$$\int \ln(f)\,d\mu \le \ln\Big(\int f\,d\mu\Big).$$ □

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