

Constructive algebraic integration theory without choice

Bas Spitters

Radboud University
Nijmegen

Abstract. We present a constructive algebraic integration theory. The theory is constructive in the sense of Bishop, however we avoid the axiom of countable, or dependent, choice. Thus our results can be interpreted in any topos. Since we avoid impredicative methods the results may also be interpreted in Martin-L type theory or in a predicative topos in the sense of Moerdijk and Palmgren.

We outline how to develop most of Bishop's theorems on integration theory that do not mention points explicitly. Coquand's constructive version of the Stone representation theorem is an important tool in this process. It is also used to give a new proof of Bishop's spectral theorem.

Keywords. Algebraic integration theory, spectral theorem, choiceless constructive mathematics, pointfree topology

2000 Mathematics Subject Classification: 46G12 Measures and integration on abstract linear spaces, 03F60 Constructive and recursive analysis, 06D22 Frames, locales.

1 Introduction

The intend of this note is to present the essence of integration theory algebraically, constructively and without using countable choice. Instead of working in a fixed formal system we work informally in Bishop-style mathematics [2], however the axiom of countable choice (CAC) is avoided, as was proposed by Richman [19]. However, a predicative topos in the sense of Moerdijk and Palmgren [16], Aczel's CZF [1] or Martin-L type theory [15] [17] would be a quite suitable foundations for the current article.

Richman [18] states that 'measure theory and the spectral theorem [...] are major challenges for a choiceless development [of constructive mathematics]' and he expects 'a choiceless development of this theory to be accompanied by some surprising insights and a gain of clarity.' We address this challenge by using algebraic and point-free methods. It is somewhat surprising to see that to develop the theory constructively without choice one is invited to state and prove stronger theorems, which makes the general theory more satisfying and the proofs somewhat cleaner.

The main idea is that instead of using functions modulo an equivalence relation, we study the algebraic structure of these spaces. That is, we study them as f-rings. This may be seen as a point-free approach to integration theory. There is an alternative point-free approach where one makes the shift from a measure space to a measure algebra — that is a Boolean algebra with a measure — which in the concrete case is an algebra of sets modulo the null sets; see [8]. These two approaches have counterparts in point-free approaches to topology. The present approach is similar to the C*-algebra approach to topology in the sense that they both study 'functions' algebraically. The Boolean algebra approach is similar to that in formal topology [20] [10] [12], where one uses the algebraic properties of open sets, instead of using the open sets themselves.

Since we only use the algebraic structure our results apply more directly to other areas of functional analysis, for instance to the theory of Abelian algebras of operators, as we shall see in section 8. On the other hand it seems more difficult to relate to more concrete questions in probability theory. It may be that the results on the measurability of Borel sets in [8] are more relevant in this context. However, more research is needed before drawing such conclusions.

This note is a continuation of the work on algebraic integration theory in [23]. The difference with the treatment in [23] is that we do not use the axiom of countable choice and the we start with a general f-algebra instead of the f-algebra $S(A)$ of simple functions on a Boolean algebra A .

This article is organized as follows first we give a short discussion of constructive mathematics without choice. Then we define integration f-algebras and the space L_1 . Then L_0 is defined as its completion with respect to a certain metric. We define L_2 and prove a Radon-Nikodym theorem. Finally, in section 8 we give a new and apparently simpler proof of Bishop's spectral theorem.

2 Completions

In the absence of countable choice the Cauchy and the Dedekind definition of the real numbers are not provably equivalent. In fact, there is a sheaf model where the Cauchy reals are not complete [24] (p. 788). A predicative choice-free definition of the Dedekind reals using Subset collection can be found in [1], section 3.6. Using the Dedekind reals one defines the completion of a metric space using locations as is explained by Richman [19] (sec. 5). A similar construction, the 'flat completion', can be found in [25] using ideas from [13] and [3]. We repeat Richman's definition here. We would like to stress that, given a predicative definition of the Dedekind reals, this definition of the completion is *predicative* in the sense that one does not need to quantify over the all possible subsets. Instead, we use the *set* of uniformly continuous real valued functions on a metric space.

Let S be a metric space. By a *location* in S we mean a real valued function f on S with the properties

1. $f(x) \geq |f(y) - d(x, y)|$ for all $x, y \in S$,
2. $\inf_{x \in S} f(x) = 0$.

Note that (1) is equivalent to $d(x, y) \leq f(x) + f(y)$ and $f(y) \leq f(x) + d(x, y)$, and that it implies that f is nonnegative and uniformly continuous. Moreover if $x \neq y$, that is, if $d(x, y) > 0$, then either $f(x) > 0$ or $f(y) > 0$, so f vanishes on at most one point. Every point z in S gives rise to the location f defined by $f(x) = d(x, z)$. Note that $f(y) = \lim_{f(x) \rightarrow 0} d(x, y)$. This is immediate from the two properties of a location.

We can define a natural metric on the set \hat{S} of locations in S . We define the metric d on \hat{S} by

$$d(f, g) := \sup_{y \in S} |f(y) - g(y)| = \inf_{x \in S} |f(x) + g(x)|.$$

Then \hat{S} is the completion of S . A metric space S is *complete* if the natural map from S to \hat{S} is onto, that is, if every location on S is given by a point of S .

Theorem 1. [Richman] *If $\phi : A \rightarrow B$ is uniformly continuous on bounded subsets, then ϕ extends uniquely to a map from \hat{A} to \hat{B} that is uniformly continuous on bounded subsets. If A is a closed subset of B , and B is complete, then A is complete.*

Richman [19] (p.14) already noted that locations correspond to regular sequences of inhabited subsets. A sequence¹ $\lambda n.S_n$ of subsets is regular when $d(x, y) < 1/n + 1/m$ for all $x \in S_n$ and

¹ We use the notation $\lambda x.y$ for the function which assigns y to x . It is sometimes written as $x \mapsto y$.

$y \in S_m$. Indeed, for any location f the sequence $\lambda n.\{x : f(x) < 1/n\}$ is regular. Conversely, given a regular sequence one can define a location f by letting $f(x)$ be the distance of x to the ‘limit’ of the sequence.

3 Integration f-algebras

We give an algebraic description of some key results Bishop’s integration theory in [2]. We first state some definitions.

Definition 1. A Riesz space is a vector space which has a lattice structure compatible with the vector space structure. That is, $f \leq g$ implies that $f + h \leq g + h$ and $f \geq 0$ implies $\alpha f \geq 0$ whenever α is a non-negative real number. We will only be concerned with Archimedean Riesz spaces L , that is if there exists u in L such that for all n , $|a| \leq (1/n)u$, then $a = 0$. **From now on we drop the adjective Archimedean.**

Definition 2. An f -algebra ² is a Riesz space which has an algebra structure compatible with the vector space structure. That is, $fg \geq 0$, whenever $f, g \geq 0$, and if $f \wedge g = 0$, then $hf \wedge g = 0$ for all h .

One can prove that $a^2 \geq 0$, $|ab| = |a||b|$ and that if $a \wedge b = 0$, then $ab = 0$; see [11] (sec. 352–353). We assume that our f -algebra is a sub- f -algebra of an f -algebra with a unit for the multiplication, which we will denote by 1, and that the f -algebra is closed under the map $\lambda a.a \wedge 1$. This unit is a weak unit with respect to the order, that is if $u \wedge 1 = 0$, then $u = 0$, or alternatively, that 0 is the greatest lower bound of $\{u \wedge (1/n)1\}$; see [11] (353P). Constructively, one may want to distinguish the greatest lower bound from the infimum, the latter having more computational content, however we avoid this problem by Definition 3.

We define $x \wedge q1 := q(\frac{1}{q}x \wedge 1)$, for $q \neq 0$ and write $x \wedge q$ instead of $x \wedge q1$ whenever q is a scalar.

Definition 3. Let L' be a f -algebra with a multiplicative unit 1 and let L be a sub- f -algebra of L' which is closed under the map $\lambda a.a \wedge 1$. An integral I on L is a positive linear functional such that $I(a \wedge n) \rightarrow I(a)$ and $I(a \wedge 1/n) \rightarrow 0$ when n tends to ∞ . An f -algebra with an integral is called an integration f -algebra.

Note that this is not the usual definition of an integral on a Riesz space. The usual definition requires that if $\inf_{a \in A} a = 0$, then $\inf_{a \in A} I(a) = 0$, which does not allow to define non-trivial integrals on the space of test-functions on the unit interval.

Three important examples of integration f -algebras are:

- Bishop’s integration spaces [2];
- A space of test-functions with a positive linear functional: a Daniell integral (which is a special case of the previous one);
- simple functions defined on a measure algebra (for instance [23]).

Given an integration algebra one can make a new integration f -algebra by considering the equality defined by $I(|a - b|) = 0$. We will assume that this has been done.

When α is a positive real number, we write $a \leq \alpha$, or a is bounded above by α , for $a \leq a \wedge \alpha$. We see that $ab \leq \alpha b$, whenever $a \leq \alpha$. An element a of L is said to be bounded if there exists

² For us, an f -algebra is always commutative. In fact, one can prove that an Archimedean f -algebra is commutative; see [6]. In [9] it is indicated how this proof can be made constructive.

$q \in \mathbb{R}$ such that $|a| \wedge q = |a|$, that is $|a| \leq q$. Given any f-algebra one can consider its sub-algebra of bounded elements, which is again an f-algebra. This algebra will be denoted by L_b . If 1 is contained in L_b , then it is a *strong unit* — that is for all a in A , there is a rational number q such that $|a| \leq q1$.

Let (L, I) be an integration f-algebra. As usual the integral I induces a norm $\lambda f. I(|f|)$ on L . The completion of L with respect to this norm is denoted by $L_1(I)$. The space L_1 is a normed Riesz space, but it is not an algebra. An idempotent in L or L_1 is called an integrable set.

We claim that much of Bishop's integration theory can be developed for general integration f-algebras as long as points are not mentioned in the statement of the theorem.

3.1 Segal's integration algebras

Our definition of an integration f-algebra may be compared to a real commutative integration algebra in the sense of Segal [21] [22].

Definition 4. *An integration algebra is a real Abelian algebra A together with a linear functional I such that*

1. $I(a^2) \geq 0$ and $I(a^2) = 0$ if and only if $a = 0$.
2. For all $b \in A$ there exists $\alpha \in \mathbb{R}$ such that $I(ba^2) \leq \alpha I(a^2)$ for all $a \in A$. We say that b is bounded by α and write $b \leq \alpha$.

Any integration f-algebra with a strong unit is an integration algebra. It was stressed by Segal that the algebraic integration theory is the proper way to describe simultaneously observable events in applications.

4 Profiles

We now consider Bishop's profile theorem [2] a key theorem in the Bishop-Cheng integration theory. The profile theorem states in an abstract and positive way the classical fact that an increasing function on the real numbers can have at most countably many discontinuities. This theorem can be translated to the present setting when we use dependent choice, see [23]. However, without choice it seems difficult to prove this theorem. Indeed, it implies that the real numbers³ are uncountable. In the appendix we show that this is not true in the sheaf model over the reals and hence that there is no hope of proving it constructively without using choice.

To see that the profile theorem implies the uncountability of $[0, 1]$ consider a sequence $\lambda n. q_n$ of points in the interval $[0, 1]$ and define the integral $I(f) := \sum_n 2^{-n} f(q_n)$ on the space $C[0, 1]$ of test functions. The function $f(x) := x$ is integrable. A smooth point for this function would be distinct from all the points q_n .

A consequence of the profile theorem is that every integrable function can be approximated arbitrary closely from above and below by integrable simple functions. It is not clear to me whether this theorem can be proved without choice. This approximation theorem is used for instance to prove that for a test function ϕ , $\phi \circ f$ is again integrable. Instead, we will derive this theorem using the Stone representation theorem.

³ As mentioned before, without the axiom of countable choice the Dedekind reals may differ from the Cauchy reals, the argument below does depend on the existence of a limit of a fast converging Cauchy sequence. Such a limit exists in both the Dedekind and the Cauchy reals.

5 Stone representation theorem

When X is a locally compact space, then $C(X)$ denotes the space of test-functions, that is, the space of functions with a compact support.

We present Coquand's constructive version [7] of the Stone representation theorem, which we then specialize to f-algebras. The formal space mentioned in the theorem is a space in the sense of formal topology [20] [10] or locale theory [12]. We recall that the objects of both the categories of frames and locales are complete distributive lattices.

An *ordered ring* is a ring with an ordering which is preserved by $+$ and \cdot . A ring R is *divisible* if for every $n \in \mathbb{N}^*$ and $r \in R$, there exists $s \in R$ such that $ns = r$. To a divisible ordered Archimedean ring R we associate a lattice $\text{Max}(R)$, which intuitively may be thought of as the lattice of opens of the maximal spectrum of the ring. However, instead of using non-constructive set-theoretic tools (e.g. Zorn's Lemma), proof-theoretic tools can be used to study the lattice of opens directly. The lattice $\text{Max}(R)$ will be generated by expressions of the form $D(a)$, a in R , which may be thought of as the opens $\{\phi \in \text{Max}(R) : \hat{a}(\phi) > 0\}$. Here $\hat{\cdot}$ denotes the Gelfand transform $\hat{a}(\phi) := \phi(a)$. The locale $\text{Max}(R)$ is the locale generated by the expressions $D(a)$, a in R , subject to the following relations:

1. $D(a) \wedge D(-a) = 0$;
2. $D(a) = 0$ if $a \leq 0$;
3. $D(a + b) \leq D(a) \vee D(b)$;
4. $D(1) = 1$;
5. $D(ab) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$;
6. $D(a) = \bigvee_r D(a - r)$.

A *continuous function* f on a frame X is defined by two families of elements of X , U_r, L_s , indexed by the rationals satisfying the following properties:

$$\begin{aligned} \bigvee_r U_r &= \bigvee_s L_s = 1 \\ U_r &= \bigvee_{r' > r} U_{r'} \\ L_s &= \bigwedge_{s' < s} L_{s'} \\ U_r \vee L_s &= 1 & \text{if } r < s \\ U_r \wedge L_s &= 0 & \text{if } s \leq r \end{aligned}$$

Intuitively, U_r stands for $f^{-1}(r, \infty)$ and L_s stands for $f^{-1}(-\infty, s)$. It is interesting to note the similarity with the constructive definition of the reals as Dedekind cuts.

We can now formulate a constructive version of the Stone representation theorem which was proved in [7]. We recall that a formal space A is *completely regular* if every a in A is the supremum of all the elements that are really inside it. An open b is *really inside* an open a if there is a continuous function f such that $f = 0$ on b and $f = 1$ outside a . In general, the equivalence of the present definition of 'completely regular' with another common definition of 'completely regular' requires the axiom of choice. However, in the present setting we will be only interested in certain spectra where this property is unproblematic. In view of the following theorem we recall that compact completely regular locales are the pointfree analogues of compact Hausdorff spaces.

Theorem 2. [Stone] *Let R be a divisible ordered Archimedean and commutative ring with strong unit. Let $\text{Max}(R)$ be the formal space of ring homomorphisms $\phi : R \rightarrow \mathbb{R}$ such that $\phi(a) \geq 0$, whenever $a \geq 0$. The space $\text{Max}(R)$ is compact Hausdorff and there is a positive ring homomorphism $\hat{\cdot} : R \rightarrow C(\text{Max}(R))$. Moreover, if $\hat{a} \geq 0$, then $\hat{a} = 0$ if and only if $a \leq \frac{1}{n}$ for all n .*

Finally, the set $\{\hat{a} : a \in R\}$ is dense in $C(\text{Max}(R))$.

Every integration f-algebra L with a strong unit can be densely embedded into a formal $C(X)$, where X is the spectrum of L . This embedding is norm continuous, hence the integral can be extended to an integral on $C(X)$. Also every element can be presented as a point-free function.

6 Measurable functions

Measurable functions can be abstractly treated in at least two ways: either as an order-theoretic limit of the sequence $\lambda n.f \wedge n \vee -n$ of integrable functions, or as the completion of L_1 with respect to a certain metric. The latter approach, the one we will use, has the advantage that $|g| \wedge f$ is integrable for all measurable functions g and all integrable functions f .

Let L be an integration f-algebra. We assume that L has a multiplicative unit, that is the integral is finite. The σ -finite case is treated in [23], however in that article the axiom of dependent choice is used. The restriction to finite integrals is not strictly necessary, but simplifies the presentation.

A *pseudo-metric* ρ on a set X is a binary function such that $\rho(x, y) = \rho(y, x)$ and $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all x, y, z in X . For all $h \in L_1^+$ we define a pseudo-metric by

$$d_h(f, g) := \int |f - g| \wedge h \quad f, g \in L_1.$$

Let $h_1, h_2 \in L_1^+$; then

$$|d_{h_1}(f, g) - d_{h_2}(f, g)| \leq \|h_1 - h_2\|_1$$

for all $f, g \in L_1$.

Lemma 1. *The uniform space $(L_1, \{d_h : h \in L_1^+\})$ is metrically equivalent to the uniform space (L_1, d_1) .*

Proof. We prove that the injection map from $(L_1, \{d_h : h \in L_1^+\})$ to (L_1, d_1) is uniformly continuous. In order to do so fix $\varepsilon > 0$ and $h \in L_1^+$. We pick $c \in \mathbb{R}$ such that $\int (h - c)^+ < \varepsilon/2$. If f, g are elements of L_1 such that $d(f, g) < \varepsilon/2c$, then

$$d_h(f, g) = \int |f - g| \wedge h \leq c \int |f - g| \wedge 1 + \int (h - c)^+ < \varepsilon.$$

We now consider the metric space (L, d_1) . We will write d for d_1 .

Definition 5. *An element in the completion of the metric space (L, d) is called a measurable function. The collection of measurable functions will be denoted by L_0 . Convergence with respect to the metric d is called convergence in measure.*

Because $d(|f|, |g|) \leq d(f, g)$ on L , we can extend the operation $\lambda f. |f|$ from L to L_0 . We then define the operations f^+, f^-, \wedge, \vee and the relation \leq on L_0 , using $|\cdot|$. They extend the already defined operations and relations on L and the usual relations hold. For instance, to see that $|f + g| \leq |f| + |g|$, we have to show that $|f| + |g| - |f + g| \geq 0$, i.e. $\| |f| + |g| - |f + g| \| = |f| + |g| - |f + g|$. But this holds on L and therefore on L_0 . It follows that L_0 is a Riesz space. We will see later that L_0 is actually an f-algebra.

Theorem 3. *Convergence in measure on a set dominated by an integrable function h implies convergence in norm.*

Proof. When $|f|, |g| \leq h$, then $\int |f - g| = \int |f - g| \wedge 2h = d_{2h}(f, g)$. We see that on the set $\{f \mid |f| \leq h\}$ norm convergence is equivalent to d_{2h} -convergence. The latter is a consequence of d -convergence by Lemma 1.

Corollary 1. [Dominated convergence] *Let f be a measurable function, let $\lambda n.F_n$ be a regular sequence of inhabited subsets of L_1 , and let g be an element of L_1 such that for all $n \in \mathbb{N}$ and $f \in F_n$, $|f| \leq g$. Suppose that $F_n \rightarrow f$ in measure. Then $F_n \rightarrow f$ in norm.*

We have not assumed that $f \in L_1$ as Bishop and Bridges did.

Theorem 4. *Let f be a measurable function and let g be an integrable function. If $|f| \leq g$, then $f \in L_1$.*

Proof. Let $h \in L_1^+$. If $f \leq g$, then for all $f' \in L_1$, $d_h(f' \wedge g, f) \leq d_h(f', f)$. Indeed,

$$\begin{aligned} |f' - f| \wedge h &= (f' - f)^+ \wedge h + (f' - f)^- \wedge h \\ &\geq (f' \wedge g - f)^+ \wedge h + (f' \wedge g - f)^- \wedge h \\ &= |f' \wedge g - f| \wedge h. \end{aligned}$$

Let $\lambda n.F_n$ is a regular sequence of inhabited subsets with limit f . Define $F'_n := \{f \wedge g \vee -g \mid f \in F_n\}$. Then $f = \lim_n F_n = \lim_n F'_n$. By Theorem 1 this last limit converges to f in norm. Since L_1 is complete, f is in L_1 .

Theorem 5. *Let $\lambda n.f_n$ be a regular increasing sequence of integrable functions. Then the sequence $\lambda n.f_n$ converges in measure to an integrable function if and only if $\lim_{n \rightarrow \infty} I(f_n)$ exists.*

Proof. Suppose that $l := \lim_{n \rightarrow \infty} I(f_n)$ exists. Then $\int (f_n - f_m) \rightarrow l - l = 0$ when $n, m \rightarrow \infty$. So the sequence $\lambda n.f_n$ is Cauchy in norm and hence converges to some $f \in L_1$.

Conversely, suppose that the sequence $\lambda n.f_n$ converges in measure to an integrable function $f \in L_1$. We may assume that $f_n \geq 0$ for all $n \in \mathbb{N}$. For all $m, n \in \mathbb{N}$, such that $m \geq n$,

$$\int f_m - f_n = \int |f_m - f_n| \wedge f,$$

which converges to 0 when $m, n \rightarrow \infty$, because convergence in measure implies d_f -convergence. So the sequence $\lambda m. \int f_m$ converges.

Fix $m \in \mathbb{R}^+$. Define $L_{\leq m} := \{f \in L_1 : |f| \leq m\}$.

Lemma 2. *Multiplication from $L_{\leq m} \times L_{\leq m}$ to L_0 is uniformly continuous.*

Proof. We prove that the map $\lambda f.f^2$ is uniformly continuous from $L_{\leq m}$ to $L_{\leq m}$. The lemma then follows from the observation that $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$.

Let $f, g \in L_{\leq m}$, $\epsilon > 0$ and suppose that $d(f, g) \leq \epsilon$. Then

$$\begin{aligned} |f^2 - g^2| \wedge 1 &\leq |f - g| |f + g| \wedge 1 \\ &\leq 2m(|f - g| \wedge 1), \end{aligned}$$

consequently $d(f^2, g^2) \leq 2m\epsilon$.

The set $L_{\leq m}$ is an ordered ring with a strong unit. Indeed, if $f, g \geq 0$, then $fg \geq 0$, since this holds in L , multiplication is uniformly continuous and the set of positive elements is closed. By Theorem 2 the ring $L_{\leq m}$ can be embedded into a formal space of continuous functions. Since $L_{\leq m}$ is complete in the uniform topology this embedding is actually an isomorphism. Consequently, $L_{\leq m}$ is an f-algebra. Moreover, we can define $\phi \circ f$ for each continuous function ϕ and $f \in L_{\leq m}$.

Definition 6. Let C be the uniform space of continuous functions on the reals with the sequence $\lambda_n \cdot \rho_n$ of pseudometrics defined by $\rho_n(f) := \sup_{[-n, n]} |f|$ for all $n \in \mathbb{N}$ and $f \in C$.

Let C_0 denote the set of test-functions.

Theorem 6. The map $\circ : C \times L_{0, m} \rightarrow L_0$ defined by $\circ(\phi, f) := \phi \circ f$ is uniformly continuous.

Proof. Let $m \in \mathbb{R}^+$ and fix $\varepsilon > 0$. Suppose that $\phi \in C$ and $|\phi| \leq \varepsilon$ on $[-m, m]$. Then for all $f, g \in L_{0, m}$, $\int |\phi(f) - \phi(g)| \wedge 1 \leq 2\varepsilon \cdot 1$. We see that the map \circ is uniformly continuous on $C \times L_{0, m}$.

For each $f \in L_0$ and for each test-function ψ with a support included in $[-n, n]$ we define $\psi \circ f := \psi \circ (f \wedge n \vee -n)$. This definition does not depend on the choice of n .

Theorem 7. For each $f \in L_0$, the map $\circ_f : C_0 \rightarrow L_0$ defined by $\circ_f(\psi) := \psi \circ f$ is uniformly continuous and can therefore be extended uniquely to a uniformly continuous map from C to L_0 .

Proof. Fix a measurable function f and $\varepsilon > 0$. We assume that $f \geq 0$, the general case is treated by symmetry.

Compute M such that $d(f, f \wedge M) < \varepsilon$. Define $u_n := f \wedge (n+1) - f \wedge n$ and $l_n := 1 - u_n$. Then $0 \leq u_n, l_n \leq 1$ and $\widehat{u}_n = 1$ on $[\widehat{f} \geq n+1]$ and $\widehat{l}_n = 1$ on $[\widehat{f} < n]$. Here $\widehat{\cdot}$ denotes the Gelfand transform defined in Theorem 2 and $[\widehat{f} < n]$ is the formal open where \widehat{f} is smaller than n . Consequently, $u_{n+1} = u_{n+1}u_n$, $u_n \geq u_{n+1}$ and $l_n \leq l_{n+1}$. Observe that if $a + b = 1$ and $a, b \geq 0$, then $|\phi| \wedge 1 = |\phi|(a+b) \wedge 1 \leq |\phi|a \wedge 1 + |\phi|b \wedge 1$, for all ϕ . So

$$\begin{aligned} |\phi f - \phi' f| \wedge 1 &\leq |\phi f - \phi' f|u_n \wedge 1 + |\phi f - \phi' f|l_n \wedge 1 \\ &\leq u_{n-1} + |\phi f - \phi' f|l_n \wedge 1. \end{aligned}$$

Since $f \wedge (n+1) - f \wedge n = (f - f \wedge n) \wedge 1$, we see that $I(u_M) = d(f, f \wedge M) \leq \varepsilon$. This takes care of the first summand.

We now consider the second summand. For all bounded elements f, p and all test functions ϕ , if $fp = f$, then $\phi(fp) = \phi(f)p = \phi f$, since this equality holds for all polynomials ϕ , and the set of polynomials form a dense set. Consequently,

$$\begin{aligned} |\phi f - \phi' f|l_n \wedge 1 &= |(\phi - \phi')f|l_n \wedge 1 \\ &= |(\phi - \phi')(fl_n)|l_n \wedge 1 \\ &\leq \|\phi - \phi'\|_{[0, n]}. \end{aligned}$$

So if we choose test functions ϕ, ϕ' such that $\|\phi - \phi'\|_{[0, M]} \leq \varepsilon$, then $I(|\phi f - \phi' f|l_M \wedge 1) \leq \varepsilon$ and

$$\begin{aligned} I(|\phi f - \phi' f| \wedge 1) &\leq I(u_M) + I(|\phi f - \phi' f|l_M \wedge 1) \\ &\leq 2\varepsilon. \end{aligned}$$

The previous theorem allows us to define the multiplication as in Lemma 2 by $fg := \frac{1}{2}((f+g)^2 - f^2 - g^2)$. Since $\lambda x \cdot x^2$ is the uniform limit of $\lambda x \cdot (x \wedge n \vee -n)^2$ we see that $f^2 = \lim(f \wedge n \vee -n)^2$ and similarly for the multiplication. Consequently, if $f \wedge g = 0$, then $hf \wedge g = 0$ for all h in L_0 , since this holds for all bounded elements. It follows that L_0 is an f-algebra, which, in general, does not have a strong unit.

6.1 L_∞

We saw in the previous section that L_∞ , the bounded elements of L_0 , form an f-algebra with a strong unit. By Theorem 4, the restriction of the multiplication on $L_0 \times L_0$ to $L_1 \times L_\infty$ has codomain L_1 .

7 The Radon-Nikodym theorem

The space L_2 of square integrable elements can be defined as usual by completing L with respect to the norm induced by the inner product $\lambda fg.I(fg)$, L_2 is a Hilbert space and a Riesz space. A linear functional on a Hilbert space H is *normable* when its norm can be computed. The Riesz representation theorem states that for any normable functional f there exists $y \in H$ such that $f(x) = \langle x, y \rangle$. A proof not using the axiom of countable choice can be found in [4] (p.3).

Let I and J be integrals on L , then $I + J$ is also an integral L . The functional J is called *absolutely continuous* with respect to I when J is a continuous linear functional on $L_1(I)$. It is called *normable* when it is normable as a functional on $L_2(I + J)$.

We can prove to following version of the Radon-Nikodym theorem. The idea of the proof is similar to the one in [2].

Theorem 8. *Let I and J be integrals on the same unital f-algebra such that J is absolutely continuous with respect to I and such that J is normable with respect to I . Then there exists a non-negative I -integrable function h such that $J(f) := I(fh)$.*

Proof. The Riesz representation theorem supplies $g \in L_2(I + J)$ such that $J(f) = (I + J)(fg)$. Hence $J(f(1 - g)) = I(fg)$. We will prove that $J(f) = I(fg/(1 - g))$.

Indeed, because $0 \leq J \leq I + J$, we see that $0 \leq g \leq 1$. Moreover, we know that $J(f(1 - g)) = I(fg)$.

Define the sequence $a_n := J(g^n)$ and $b_n := I(g^n)$, then $a_n = a_{n+1} + b_{n+1}$. Since $g \leq 1$, both of these sequences are decreasing. Note that

$$a_0 = a_n + \sum_{k=1}^n b_k. \tag{1}$$

Given $\epsilon > 0$, determine δ such that $J(f) \leq \epsilon$ whenever $I(f) \leq \delta$. By equation (1), for large n , $b_n \leq \delta$, so $a_n \leq \epsilon$. Consequently, a_n converges to 0, so b_n is summable and $g/(1 - g) = \sum_{k=1}^{\infty} g^k$ is I -integrable.

Note that in the point-wise version of [2] quite some effort is spent on the convergence a.e. These problems do not concern us here.

8 Bishop's spectral theorem

In this section we obtain a spectral theorem for bounded integrable functions acting on a commuting algebra of operators on a Hilbert space, which is very similar to Bishop's spectral theorem [2] (Thm. 7.8.22). This theorem defines an embedding of a space of bounded integrable functions into a space of operators. The Stone representation theorem 2 can be used to define such an embedding

for continuous functions. The present proof of the spectral theorem for bounded integrable functions seems to be technically simpler and more natural than the one by Bishop, since we use the spectral theorem for continuous functions to prove the one for bounded measurable functions, as opposed to the converse in Bishop's treatment.

Let H be a Hilbert space with inner product (\cdot, \cdot) . For any finite dimensional subspace F of H , with basis $\{f_1, \dots, f_n\}$, the functional $\text{Tr}_F(A)$ on a bounded linear operator A is defined as $\sum_n (Af_n, f_n)$. The value of $\text{Tr}_F(A)$ is independent of the choice of the basis. If the supremum of $\text{Tr}_F(A)$ over the finite dimensional subspaces F of H exists it is denoted by $\text{Tr } A$, cf. [5].

In this paragraph we use the axiom of dependent choice (DC) to construct a basis, this is needed only to construct the state — a generalized integral — so in case the state can be given in advance, DC is not needed. Alternatively, we could have tried to force a *generic* state, but would have made the connection with Bishop's theorem less clear. Let $\lambda n.e_n$ be a countable basis for a Hilbert space H . Let \mathcal{A} be an algebra of commuting self-adjoint operators. Define the unique operator d such that $d(e_n) = 2^{-n}e_n$ for all $n \in \mathbb{N}$. Define $I(a) := \text{Tr}(da)$. We claim that I is a *faithful state*, i.e. a positive linear functional such that $I(\text{id}) = 1$ and if $\lambda n.a_n$ is a bounded sequence of self-adjoint operators such that $I(a_n^2) \rightarrow 0$, then $a_n x \rightarrow 0$ for all x in H . Indeed,

$$\begin{aligned} I(a_n^2) &= \text{Tr}(da_n^2) \\ &= \sum_k 2^{-k} (a_n^2 e_k, e_k) \\ &= \sum_k 2^{-k} \|a_n e_k\|^2. \end{aligned}$$

It follows that if $I(a_n^2) \rightarrow 0$, then the sequence $\lambda n.a_n e_k$ converges for all k . Since the span of $\{e_k : k \in \mathbb{N}\}$ is dense and the sequence $\lambda n.a_n$ is bounded, the sequence $\lambda n.a_n x$ converges for all x in H .

The pair (\mathcal{A}, I) is an integration algebra. To check the second property, we observe $\text{Tr}(dba^2) \leq \|b\| \text{Tr}(da^2)$. The integral I is positive with respect to the usual order on operators defined by $a \geq_{\text{op}} 0$ if and only if $(ax, x) \geq 0$ for all x in H . Moreover, if $a \geq_{\text{op}} 0$, then $I(ab^2) \geq 0$ for all $b \in \mathcal{A}$, since $(a(be_n), (be_n)) \geq 0$ for all $n \in \mathbb{N}$ and $b \in \mathcal{A}$. We write $a \geq_I 0$ when $I(ab^2) \geq 0$ for all $b \in \mathcal{A}$. Using elementary Hilbert space theory one can prove that $(\mathcal{A}, \geq_{\text{op}})$ is an f-algebra; see for instance [9] for details. By Stone's theorem \mathcal{A} can be densely embedded into an f-algebra $C(X)$ and since $I(\text{id}) = 1$ the integral can also be extended to this f-algebra. We claim that \geq_{op} , the usual order on operators, and the order \geq_I induced by the integral are the same. In fact, this proof works for every integration f-algebra.

Lemma 3. *Let (A, I) be an integration f-algebra. Suppose that $I(ab^2) \geq 0$, for all $b \in A$. Then a is positive in the f-algebra A .*

Proof. By the previous lemma if $\phi \geq 0$ in an f-algebra, then ϕ can be approximated uniformly from below by a sum of squares. Since the algebra has a unit and the integral is positive, it follows that $I(a\phi) \geq 0$, for all positive $\phi \in A$. Let $b = a^+$ and $c = a^-$. Then $I(aa^-) \geq 0$, because $a^- \geq 0$. Since $aa^- = (a^+ - a^-)a^- = -(a^-)^2$, we have $-I((a^-)^2) \geq 0$, so $I((a^-)^2) = 0$. It follows that $a^- = 0$. That is $a = a^+ - 0 \geq 0$.

Consequently, $I(ab^2) \geq 0$ for all $b \in A$ if and only if a is a positive in A . We see that the order relations \geq_{op} and \geq_I agree.

We now obtain a theorem similar to Bishop’s spectral theorem. Recall that using DC one can always construct a faithful state on a separable Hilbert space.

Theorem 9. *Let \mathcal{A} be a unital commuting algebra of self-adjoint operators on a Hilbert space with a faithful state. Then there exists an integral J on the spectrum of \mathcal{A} and bound preserving dense embedding from \mathcal{A} into $L_\infty(J)$. If \mathcal{A} is complete, then the embedding is an isomorphism. Moreover, if $\lambda_n.\widehat{a}_n$ is a bounded sequence which converges in $L_\infty(J)$, then $\lambda_n.a_n$ converges strongly.*

Proof. We consider \mathcal{A} as an integration algebra as above. The Gelfand transform is a bound preserving map from \mathcal{A} into $C(\text{Max}(\mathcal{A}))$. The state can be extended to an integral on $C(\text{Max}(\mathcal{A}))$.

Finally, let \widehat{a}_n be a bounded sequence in $C(\text{Max}(\mathcal{A}))$ converging in $L_\infty(J)$, then \widehat{a}_n converges in $L_2(J)$. Consequently, the sequence $\lambda_n.a_n$ converges strongly.

9 Conclusions

We have shown how to develop a large part of Bishop’s integration theory in an algebraic and pointfree setting. Coquand’s pointfree Stone representation theorem is the key result used both in the development of integration theory and in the proof of the spectral theorem. We have not used the axiom of countable or dependent choice, thus meeting Richman’s challenge to develop a choicefree integration theory.

Since we have avoided choice the results may be interpreted in every topos and in particular in sheaf models. This may be of interest in applications where one uses a ‘continuously’ varying family of integration spaces, as one does, for example, in the theory of stochastic processes.

The proof of the spectral theorem presented above seems to be somewhat easier than the one given by Bishop. The present approach also seems to be more natural in the sense that it starts from the spectral theorem for continuous functions and then proceeds towards the theorem from bounded measurable functions as opposed to Bishop who does the converse.

The present development is mostly directed towards the algebraic aspects of integration theory and their applications in functional analysis. In probability theory sometimes one is more interested in working with Borel sets directly, there the constructive and pointfree approach of Coquand and Palmgren [8] may be more appropriate.

References

1. Peter Aczel and Michael Rathjen. Notes on constructive set theory. Technical Report 40, Mittag-Leffler Institute, 2000/2001.
2. Errett Bishop and Douglas Bridges. *Constructive analysis*, volume 279 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, 1985.
3. M.M. Bonsangue, F. van Breugel, and J.J.M.M. Rutten. Generalized metric spaces: Completion, topology, and powerdomains via the Yoneda embedding. *Theoretical Computer Science*, 193:1–51, 1998.
4. Douglas Bridges, Fred Richman, and Peter Schuster. Adjoints, absolute values and polar decompositions. *Journal of Operator Theory*, 44:243–254, 2000.
5. Douglas Bridges, Fred Richman, and Peter Schuster. Trace-class operators. *Houston Journal of Mathematics*, 28:565–583, 2002.
6. G. Buskes and A. van Rooij. Almost f -algebras: commutativity and the Cauchy-Schwarz inequality. *Positivity*, 4(3):227–231, 2000. Positivity and its applications (Ankara, 1998).
7. Thierry Coquand. About Stone’s notion of spectrum. *J. Pure Appl. Algebra*, 197(1-3):141–158, 2005.

8. Thierry Coquand and Erik Palmgren. Metric boolean algebras and constructive measure theory. *Archive for Mathematical Logic*, 41(7):687–704, 2002.
9. Thierry Coquand and Bas Spitters. Formal topology and constructive mathematics: the Gelfand and Stone-Yosida representation theorems. to appear.
10. M. P. Fourman and R. J. Grayson. Formal spaces. In *The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*, volume 110 of *Stud. Logic Found. Math.*, pages 107–122. North-Holland, Amsterdam, 1982.
11. D.H. Fremlin. *Topological Riesz spaces and Measure theory*. Cambridge University Press, 1974.
12. Peter T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition.
13. Bill Lawvere. Metric spaces, generalised logic, and closed categories. In *Rendiconti del Seminario Matematico e Fisico di Milano*, volume 43. Tipografia Fusi, Pavia, 1973.
14. Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.
15. Per Martin-L. *Intuitionistic type theory*, volume 1 of *Studies in Proof Theory*. Bibliopolis, 1984.
16. Ieke Moerdijk and Erik Palmgren. Type theories, toposes and constructive set theory: predicative aspects of AST. *Annals of Pure and Applied Logic*, 114(1-3):155–201, 2002. Commemorative Symposium Dedicated to Anne S. Troelstra (Noordwijkerhout, 1999).
17. B. Nordström, K. Petersson, and J. M. Smith. *Programming in Martin-Löf’s Type Theory. An Introduction*. Oxford University Press, 1990.
18. Fred Richman. Constructive mathematics without choice. <http://www.math.fau.edu/Richman/docs/nsf.pdf>, 8 March 2005.
19. Fred Richman. The fundamental theorem of algebra: a constructive development without choice. *Pacific Journal of Mathematics*, 196:213–230, 2000.
20. Giovanni Sambin. Intuitionistic formal spaces - a first communication. In D. Skordev, editor, *Mathematical logic and its applications*, pages 187–204. Plenum, 1987.
21. Irvin Segal. Algebraic integration theory. *Bulletin of the American Mathematical Society*, 71:419–489, 1965.
22. Irving E. Segal and Ray A. Kunze. *Integrals and Operators*, volume 228 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin Heidelberg New York, second edition, 1978.
23. Bas Spitters. Constructive algebraic integration theory. In Giovanni Sambin Bernhard Banaschewski, Thierry Coquand, editor, *Proceedings of the Second Workshop on Formal Topology, special issue of Annals of Pure and Applied Logic*, to appear.
24. A.S. Troelstra and D. van Dalen. *Constructivism in mathematics. An introduction. Volume II*. Number 123 in *Studies in Logic and the Foundations of Mathematics*. North-Holland, 1988.
25. Steven Vickers. Localic completion of generalized metric spaces I. submitted for publication, draft available on web at <http://www.cs.bham.ac.uk/~sjv>, 2003.

A Uncountability of the reals

We show that the statement

$$\forall x \in \mathbb{N} \rightarrow \mathbb{R} \exists y \in \mathbb{R} \forall n \in \mathbb{N} \exists m \in \mathbb{N} |x_n - y| \geq 1/m \quad (2)$$

does not hold in the internal logic of the sheaf model over the reals⁴. See for instance [24] or [14] for more information on sheaf models for intuitionistic logic.

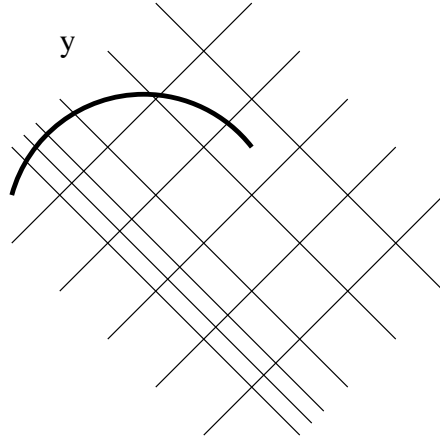
⁴ I would like to thank Pino Rosolini for helping me find this counterexample.

Indeed, suppose that formula 2 would hold in this model. Then given an interval U and any sequence $x \in \mathbb{R}^{\mathbb{N}}(U) \cong U \times \mathbb{N} \rightarrow \mathbb{R}$ of functions there is a cover V_i of U and $y_i : V_i \rightarrow \mathbb{R}$ such that for all n we can choose a cover $V_{i,k}$ of V_i and $m_{i,n,k}$ such that

$$|x(n) - y_i| \geq 1/m_{i,n,k} \text{ on } V_{i,k} \tag{3}$$

for all i and k .

However, if we take $U = \mathbb{R}$ and choose a sequence of linear functions making a dense 45° grid.



Then classically by the intermediate value theorem on any inhabited open V_i there will be a function x_n that intersects y_i . This contradicts formula 3. Using the approximate intermediate value theorem we can make this argument constructive.