Towards third generation HOTT

Part 2: Symmetries and semicartesian cubes

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Up today

Plan for the three talks:

1. Basic syntax of H.O.T.T.
2. Symmetries and semicartesian cubes
3. Semantics of univalent universes

Outline

Review

- Last week I described the "Book" version of H.O.T.T., starting with simple ideas, and introducing complexity only as necessary.
- By way of review, let's reformulate the resulting theory more concisely and cleanly.

In particular, we eventually ended up with n-variable ap (and Id) that bind a finite list of variables:

$$\frac{\Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash t : B \qquad \cdots}{\Gamma \vdash \mathsf{ap}_{x_1,\ldots,x_n.t}(p_1, \ldots, p_n) : \mathsf{Id}_B(\cdots)}$$

Such a "context suffix" is also called a telescope.
We now reify these into a "telescope calculus".

Telescopes

Telescopes are defined inductively as finite lists of types:

$$\frac{}{\Gamma \vdash \epsilon \ \mathsf{tel}} \qquad \frac{\Gamma \vdash \Delta \ \mathsf{tel} \qquad \Gamma, \Delta \vdash A : \mathsf{U}}{\Gamma \vdash (\Delta, x : A) \ \mathsf{tel}}$$

The "elements" of a telescope are substitutions:

$$\frac{}{() : \epsilon} \qquad \frac{\delta : \Delta \qquad \Delta \vdash A : \mathsf{U} \qquad a : A[\delta]}{(\delta, a) : (\Delta, x : A)}$$

These are defined mutually with their action on terms (and types):

$$\frac{\Delta \vdash a : A \qquad \delta : \Delta}{a[\delta] : A[\delta]}$$

Dependent Id and ap with telescopes

Now we can define identity telescopes from identity types:

$$\frac{\Delta\ \mathsf{tel} \qquad \delta : \Delta \qquad \delta' : \Delta}{\mathsf{Id}_{\Delta}(\delta, \delta')\ \mathsf{tel}}$$

$$\mathsf{Id}_{\epsilon}((), ()) \equiv \epsilon$$

$$\mathsf{Id}_{(\Delta, x:A)}((\delta, a), (\delta', a')) \equiv \big(\varrho : \mathsf{Id}_{\Delta}(\delta, \delta'),\ \alpha : \mathsf{Id}^{\varrho}_{\Delta.A}(a, a')\big)$$

These are defined mutually with *n*-ary Id, which depends on them:

$$\frac{\varrho : \mathsf{Id}_{\Delta}(\delta, \delta') \qquad \Delta \vdash A : \mathsf{U} \qquad a : A[\delta] \qquad a' : A[\delta']}{\mathsf{Id}^{\varrho}_{\Delta.A}(a, a') : \mathsf{U}}$$

We write $\mathsf{Id}_A(a, a') \equiv \mathsf{Id}^{()}_{\epsilon.A}(a, a')$ in the non-dependent case.

(Last time I defined dependent Id in terms of ap; here we postulate it separately and then make them coincide later.)

Computing Id

As we saw last time, Id computes on all type formers:

$$\mathsf{Id}^{\varrho}_{\Delta.A\times B}(s,t) \equiv \mathsf{Id}^{\varrho}_{\Delta.A}(\pi_1 s,\pi_1 t)\times \mathsf{Id}^{\varrho}_{\Delta.B}(\pi_2 s,\pi_2 t)$$

$$\mathsf{Id}^{\varrho}_{\Delta.\sum_{(x:A)}B}(s,t) \equiv \sum\nolimits_{(q:\mathsf{Id}^{\varrho}_{\Delta.A}(\pi_1 s,\pi_1 t))}\mathsf{Id}^{\varrho,q}_{(\Delta,x:A).B}(\pi_2 s,\pi_2 t)$$

$$\mathsf{Id}^{\varrho}_{A\to B}(f,g) \equiv \prod\nolimits_{(u:A)}\prod\nolimits_{(v:A)}\prod\nolimits_{(q:\mathsf{Id}^{\varrho}_{\Delta.A}(u,v))}\mathsf{Id}^{\varrho}_{\Delta.B}(fu,gv)$$

$$\mathsf{Id}^{\varrho}_{\prod_{(x:A)}B}(f,g) \equiv \prod\nolimits_{(u:A)}\prod\nolimits_{(v:A)}\prod\nolimits_{(q:\mathsf{Id}^{\varrho}_{\Delta.A}(u,v))}\mathsf{Id}^{\varrho,q}_{(\Delta,x:A).B}(fu,gv)$$

Id is a 1-1 correspondence

All identity types are 1-1 correspondences:

$$\frac{\varrho : \mathsf{Id}_\Delta(\delta, \delta') \qquad \Delta \vdash A : \mathsf{U} \qquad a : A[\delta]}{\overrightarrow{\mathsf{corr}}^{\varrho}_{\Delta.A}(a) : \mathsf{isContr}\left(\sum_{(a' : A[\delta'])} \mathsf{Id}^{\varrho}_{\Delta.A}(a, a')\right)}$$

$$\frac{\varrho : \mathsf{Id}_\Delta(\delta, \delta') \qquad \Delta \vdash A : \mathsf{U} \qquad a' : A[\delta']}{\overleftarrow{\mathsf{corr}}^{\varrho}_{\Delta.A}(a') : \mathsf{isContr}\left(\sum_{(a : A[\delta])} \mathsf{Id}^{\varrho}_{\Delta.A}(a, a')\right)}$$

The centers of contraction constitute transport:

$$\frac{\varrho : \mathsf{Id}_\Delta(\delta, \delta') \qquad \Delta \vdash A : \mathsf{U} \qquad a : A[\delta]}{\overrightarrow{\mathsf{tr}}^{\varrho}_{\Delta.A}(a) : A[\delta'] \qquad \overrightarrow{\mathsf{lift}}^{\varrho}_{\Delta.A}(a) : \mathsf{Id}^{\varrho}_{\Delta.A}(a, \overrightarrow{\mathsf{tr}}^{\varrho}_{\Delta.A}(a))}$$

These witnesses compute on type formers: $\overrightarrow{\mathsf{corr}}^{\varrho}_{\Delta.A \times B}(a) \equiv \cdots$,

hence also $\overrightarrow{\mathsf{tr}}^{\varrho}_{\Delta.A \times B}(a) \equiv \cdots$, etc.

Computing ap

A term can be applied to Id of any telescope it depends on:

$$\frac{\varrho : \mathsf{Id}_{\Delta}(\delta, \delta') \qquad \Delta \vdash t : B}{\mathsf{ap}_{\Delta.t}(\varrho) : \mathsf{Id}^{\varrho}_{\Delta.B}(t[\delta], t[\delta'])}$$

This higher-dimensional explicit substitution computes on all* terms:

$$\mathsf{ap}_{\Delta.(s,t)}(\varrho) \equiv (\mathsf{ap}_{\Delta.s}(\varrho), \mathsf{ap}_{\Delta.t}(\varrho))$$

$$\mathsf{ap}_{\Delta.\pi_1 s}(\varrho) \equiv \pi_1\, \mathsf{ap}_{\Delta.s}(\varrho) \qquad \mathsf{ap}_{\Delta.\pi_2 s}(\varrho) \equiv \pi_2\, \mathsf{ap}_{\Delta.s}(\varrho)$$

$$\mathsf{ap}_{\Delta.f\,b}(\varrho) \equiv \mathsf{ap}_{\Delta.f}(p)\big(b[a/x],\ b[a'/x],\ \mathsf{ap}_{\Delta.b}(\varrho)\big).$$

$$\mathsf{ap}_{\Delta.(\lambda y.t)}(\varrho) \equiv \lambda u.\lambda v.\lambda q.\mathsf{ap}_{\Delta.y.t}(\varrho, q).$$

We define reflexivity as the 0-ary ap: $\mathsf{refl}_a \equiv \mathsf{ap}_{\epsilon.a}()$.

Univalence

$\mathsf{Id}_{\mathsf{U}}(A, B)$ contains as a retract the type of 1-1 correspondences:

$$\text{1-1-Corr}(A, B) :\equiv \sum_{(R:A\to B\to \mathsf{U})} \Big(\prod_{(a:A)} \mathsf{isContr}\big(\sum_{(b:B)} R(a,b)\big)\Big) \times \Big(\prod_{(b:B)} \mathsf{isContr}\big(\sum_{(a:A)} R(a,b)\big)\Big).$$

$$\text{1-1-Corr}(A, B) \xrightarrow{\uparrow} \mathsf{Id}_U(A, B) \xrightarrow{\downarrow} \text{1-1-Corr}(A, B) \qquad p{\uparrow}{\downarrow} \equiv p$$

We identify dependent Id with ap into the universe:

$$\begin{aligned}
\mathsf{Id}^{\varrho}_{\Delta.B}(b, b') &\equiv \pi_1(\mathsf{ap}_{\Delta.B}(\varrho)\!\downarrow)(b, b') \\
\overrightarrow{\mathsf{corr}}^{\varrho}_{\Delta.B}(b, b') &\equiv \pi_1\pi_2(\mathsf{ap}_{\Delta.B}(\varrho)\!\downarrow)(b, b') \\
\overleftarrow{\mathsf{corr}}^{\varrho}_{\Delta.B}(b, b') &\equiv \pi_2\pi_2(\mathsf{ap}_{\Delta.B}(\varrho)\!\downarrow)(b, b')
\end{aligned}$$

(Last time, we defined the LHS as the RHS. Separating them is more natural for Tarski universes, and permits types not lying in any universe.)

That asterisk: Neutral reflexivities

I claimed that ap is never a normal form, but there's one exception:

When y is a variable, refl_y is neutral (hence normal).

Since refl is nullary ap, the rule that would apply is

$$\mathsf{ap}_{x_1.\cdots x_n.y}(p_1, \ldots, p_n) \equiv \mathsf{refl}_y \text{ (if } y \text{ is a variable } \notin \{x_1, \ldots, x_n\})$$

where $n = 0$, but this just reduces $\mathsf{refl}_y \equiv \mathsf{ap}_{().y}()$ to itself!

This includes other terms that obviously must also be neutral:

- $\mathsf{ap}_{x.f(x)}(p) \equiv \mathsf{refl}_f(a_0, a_1, p)$ for a variable $f : A \to B$.
- $\mathsf{Id}_A(a_0, a_1) \equiv (\pi_1\, \mathsf{refl}_A)(a_0, a_1)$ for a variable $A : \mathsf{U}$.

Similarly, $\mathsf{refl}_{\mathsf{refl}_x}$, $\mathsf{refl}_{\mathsf{refl}_{\mathsf{refl}_x}}$, etc., are also neutral.

Outline

Squares and cubes

H.O.T.T. is not a “cubical type theory”: there are no explicit cubes in the syntax. But like any other type theory with dependent identity types (including Book HoTT!), it has an emergent notion of cube:

$$a_{02} : \mathsf{Id}_A(a_{00}, a_{01}) \qquad a_{12} : \mathsf{Id}_A(a_{10}, a_{11}) \qquad a_{20} : \mathsf{Id}_A(a_{00}, a_{10})$$

$$a_{21} : \mathsf{Id}_A(a_{01}, a_{11}) \qquad a_{22} : \mathsf{Id}^{a_{02}, a_{12}}_{x.y.\mathsf{Id}_A(x,y)}(a_{20}, a_{21})$$

$$\begin{array}{ccc} a_{10} & \xrightarrow{a_{12}} & a_{11} \\ {\scriptstyle a_{20}}\uparrow & a_{22} & \uparrow{\scriptstyle a_{21}} \\ a_{00} & \xrightarrow[a_{02}]{} & a_{01} \end{array}$$

Similarly, $\mathsf{Id}_{\mathsf{Id}_{\mathsf{Id}_A}}$ is a type of 3-dimensional cubes, etc.

Very important point

The roles of a_{02}, a_{12} and a_{20}, a_{21} are asymmetrical!

Cubical horn-fillers

Given a_{02}, a_{12}, a_{20}, we have fillers of left-to-right cubical horns:

$$\overrightarrow{\mathsf{tr}}^{a_{02},a_{12}}_{x.y.\mathsf{Id}_A(x,y)}(a_{20}) : \mathsf{Id}_A(a_{01}, a_{11})$$

$$\overrightarrow{\mathsf{lift}}^{a_{02},a_{12}}_{x.y.\mathsf{Id}_A(x,y)}(a_{20}) : \mathsf{Id}^{a_{02},a_{12}}_{x.y.\mathsf{Id}_A(x,y)}(a_{20}, \overrightarrow{\mathsf{tr}}^{a_{02},a_{12}}_{x.y.\mathsf{Id}_A(x,y)}(a_{20}))$$

$$\begin{array}{ccc}
a_{10} & \xrightarrow{\quad a_{12} \quad} & a_{11} \\
{\scriptstyle a_{20}}\uparrow & \overrightarrow{\mathsf{lift}}^{a_{02},a_{12}}_{x.y.\mathsf{Id}_A(x,y)}(a_{20}) & \uparrow{\scriptstyle \overrightarrow{\mathsf{tr}}^{a_{02},a_{12}}_{x.y.\mathsf{Id}_A(x,y)}(a_{20})} \\
a_{00} & \xrightarrow[\quad a_{02} \quad]{} & a_{01}
\end{array}$$

Similarly, $\overleftarrow{\mathsf{tr}}$ and $\overleftarrow{\mathsf{lift}}$ fill right-to-left cubical horns.
And $\overrightarrow{\mathsf{tr}}_{\mathsf{Id}_{\mathsf{Id}_A}}$, etc. fill higher-dimensional left-right horns.

Problem #1

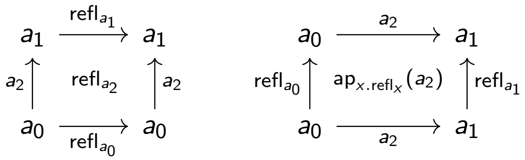
We don't seem to have top-to-bottom or bottom-to-top fillers.

Degenerate cubes

Given $a_2 : \mathrm{Id}_A(a_0, a_1)$, there are two degenerate squares:

$$\mathrm{refl}_{a_2} : \mathrm{Id}_{\mathrm{Id}_A(a_0,a_1)}(a_2, a_2) \equiv \mathrm{Id}^{\mathrm{refl}_{a_0},\mathrm{refl}_{a_1}}_{x.y.\mathrm{Id}_A(x,y)}(a_2, a_2)$$

$$\mathrm{ap}_{x.\mathrm{refl}_x}(a_2) : \mathrm{Id}^{a_2}_{x.\mathrm{Id}_A(x,x)}(\mathrm{refl}_{a_0}, \mathrm{refl}_{a_1}) \equiv \mathrm{Id}^{a_2,a_2}_{x.y.\mathrm{Id}_A(x,y)}(\mathrm{refl}_{a_0}, \mathrm{refl}_{a_1})$$



Degenerate cubes

Given $a_2 : \mathsf{Id}_A(a_0, a_1)$, there are two degenerate squares:

$$
\begin{array}{ccc}
a_1 & \xrightarrow{\mathsf{refl}_{a_1}} & a_1 \\
{\scriptstyle a_2}\uparrow & \mathsf{refl}_{a_2} & \uparrow{\scriptstyle a_2} \\
a_0 & \xrightarrow[\mathsf{refl}_{a_0}]{} & a_0
\end{array}
\qquad\qquad
\begin{array}{ccc}
a_0 & \xrightarrow{\quad a_2 \quad} & a_1 \\
{\scriptstyle \mathsf{refl}_{a_0}}\uparrow & \mathsf{ap}_{x.\mathsf{refl}_x}(a_2) & \uparrow{\scriptstyle \mathsf{refl}_{a_1}} \\
a_0 & \xrightarrow[\quad a_2 \quad]{} & a_1
\end{array}
$$

Problem #2

For $a : A$, the two doubly-degenerate squares

$$
\begin{array}{ccc}
a & \xrightarrow{\mathsf{refl}_a} & a \\
{\scriptstyle \mathsf{refl}_a}\uparrow & \mathsf{refl}_{\mathsf{refl}_a} & \uparrow{\scriptstyle \mathsf{refl}_a} \\
a & \xrightarrow[\mathsf{refl}_a]{} & a
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \xrightarrow{\quad \mathsf{refl}_a \quad} & a \\
{\scriptstyle \mathsf{refl}_a}\uparrow & \mathsf{ap}_{x.\mathsf{refl}_x}(\mathsf{refl}_a) & \uparrow{\scriptstyle \mathsf{refl}_a} \\
a & \xrightarrow[\quad \mathsf{refl}_a \quad]{} & a
\end{array}
$$

seem to be definitionally unrelated.

Stuck degeneracies break canonicity

Problem #3

Our rules so far compute refl_{a_2} based on the structure of a_2, but $\mathsf{ap}_{x.\mathsf{refl}_x}(a_2)$ is stuck, even if a_2 is very concrete.

- refl_x doesn't reduce when x is a variable.
- ap doesn't inspect its identification argument.

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- refl_x doesn't reduce when x is a variable.
- ap doesn't inspect its identification argument.

A bit nonobviously, this also breaks canonicity for $\mathbb{N}$.

Intuitive homotopy-theoretic reason

For a type $A : \mathsf{U}$, the square $\mathsf{ap}_{x.\mathsf{refl}_x}(\mathsf{refl}_A)$ in U is essentially a self-homotopy of the identity equivalence of A, i.e. $\prod_{(a:A)} \mathsf{Id}_A(a, a)$. Taking $A = S^1$ we get a stuck loop in $\mathsf{Id}_{S^1}(\mathsf{base}, \mathsf{base})$, hence in $\mathbb{Z}$.

(There's also an explicit argument using two universes instead of S^1.)

Outline

Symmetry

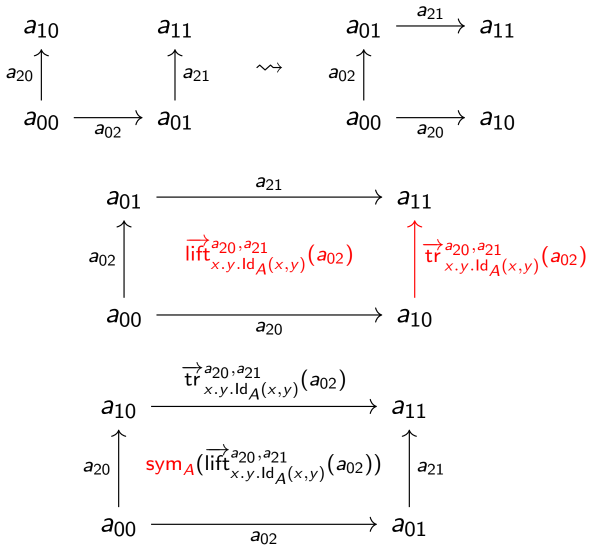
To solve these problems, we introduce a symmetry operation that transposes squares:

$$\begin{array}{ccc} a_{10} & \xrightarrow{a_{12}} & a_{11} \\ {\scriptstyle a_{20}}\uparrow & a_{22} & \uparrow{\scriptstyle a_{21}} \\ a_{00} & \xrightarrow[a_{02}]{} & a_{01} \end{array} \quad\mapsto\quad \begin{array}{ccc} a_{01} & \xrightarrow{a_{21}} & a_{11} \\ {\scriptstyle a_{02}}\uparrow & \mathsf{sym}_A(a_{22}) & \uparrow{\scriptstyle a_{12}} \\ a_{00} & \xrightarrow[a_{20}]{} & a_{10} \end{array}$$

$$\frac{a_{22} : \mathsf{Id}^{a_{02},a_{12}}_{x.y.\mathsf{Id}_A(x,y)}(a_{20}, a_{21})}{\mathsf{sym}_A(a_{22}) : \mathsf{Id}^{a_{20},a_{21}}_{x.y.\mathsf{Id}_A(x,y)}(a_{02}, a_{12})}$$

The other Kan operations

Now we can fill other cubical horns, solving problem #1:



Computing symmetry

To solve problem #3, we define

$$\mathsf{ap}_{x.\mathsf{refl}_x}(a_2) \equiv \mathsf{sym}_A(\mathsf{refl}_{a_2}).$$

This computes based on a_2… if sym also computes!

Computing symmetry

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This computes based on a_2... if sym also computes!

For the most part, computing symmetry is straightforward, e.g.:

$$\begin{aligned}
&\mathsf{Id}^{s_{02},s_{12}}_{u.v.\mathsf{Id}_{A\times B}(u,v)}(s_{20},s_{21}) \\
&\qquad \equiv \mathsf{Id}^{s_{02},s_{12}}_{u.v.\mathsf{Id}_A(\pi_1 u,\pi_1 v)\times \mathsf{Id}_B(\pi_2 u,\pi_2 v)}(s_{20},s_{21}) \\
&\qquad \equiv \mathsf{Id}^{s_{02},s_{12}}_{u.v.\mathsf{Id}_A(\pi_1 u,\pi_1 v)}(\pi_1 s_{20},\pi_1 s_{21}) \times \mathsf{Id}^{s_{02},s_{12}}_{u.v.\mathsf{Id}_B(\pi_2 u,\pi_2 v)}(\pi_2 s_{20},\pi_2 s_{21}) \\
&\qquad \equiv \mathsf{Id}^{\pi_1 s_{02},\pi_1 s_{12}}_{x.w.\mathsf{Id}_A(x,w)}(\pi_1 s_{20},\pi_1 s_{21}) \times \mathsf{Id}^{\pi_2 s_{02},\pi_2 s_{12}}_{y.z.\mathsf{Id}_B(y,z)}(\pi_2 s_{20},\pi_2 s_{21}).
\end{aligned}$$

So we can define

$$\mathsf{sym}_{A\times B}((p,q)) \equiv (\mathsf{sym}_A(p),\mathsf{sym}_B(q))$$

Dependent symmetry

To generalize this to Σ-types, we need dependent symmetry over a square in a telescope (don't worry too much about the syntax):

$$\frac{\delta_{22} : \mathsf{Id}^{\delta_{02},\delta_{12}}_{\delta.\delta'.\mathsf{Id}_{\Delta}(\delta,\delta')}(\delta_{20},\delta_{21}) \qquad a_{22} : \mathsf{Id}^{\delta_{02},\delta_{12},\delta_{22},a_{02},a_{12}}_{\delta.\delta'.\varrho.u.v.\mathsf{Id}^{\varrho}_{\Delta.A}(u,v)}(a_{20},a_{21})}{\mathsf{sym}^{\delta_{22}}_{\Delta.A}(a_{22}) : \mathsf{Id}^{\delta_{20},\delta_{21},\mathsf{sym}(\delta_{22}),a_{20},a_{21}}_{\delta.\delta'.\varrho.u.v.\mathsf{Id}^{\varrho}_{\Delta.A}(u,v)}(a_{02},a_{12})}$$

Then we can define

$$\mathsf{sym}^{\delta_{22}}_{\Delta.\sum_{(x:A)}B}((p,q)) \equiv (\mathsf{sym}^{\delta_{22}}_{\Delta.A}(p), \mathsf{sym}^{\delta_{22},p}_{(\Delta,x:A).B}(q))$$

Symmetry for functions

$$\mathsf{Id}^{f_{02},f_{12}}_{f.g.\mathsf{Id}_{A\to B}(f,g)}(f_{20},f_{21}) \equiv \mathsf{Id}^{f_{02},f_{12}}_{f.g.\prod_{(x_0:A)}\prod_{(x_1:A)}\prod_{(x_2:\mathsf{Id}_A(x_0,x_1))}\mathsf{Id}_B(fx_0,gx_1)}(f_{20},f_{21})$$

$$\equiv \prod_{(x_{00}:A)}\prod_{(x_{01}:A)}\prod_{(x_{02}:\mathsf{Id}_A(x_{00},x_{01}))}$$

$$\prod_{(x_{10}:A)}\prod_{(x_{11}:A)}\prod_{(x_{12}:\mathsf{Id}_A(x_{10},x_{11}))}$$

$$\prod_{(x_{20}:\mathsf{Id}_A(x_{00},x_{10}))}\prod_{(x_{21}:\mathsf{Id}_A(x_{01},x_{11}))}\prod_{(x_{22}:\mathsf{Id}^{x_{02},x_{12}}_{x.y.\mathsf{Id}_A(x,y)}(x_{20},x_{21}))}$$

$$\mathsf{Id}^{f_{02}x_{02},f_{12}x_{12}}_{u.v.\mathsf{Id}_B(u,v)}(f_{20}x_{20},f_{21}x_{21})$$

So $f_{22} : \mathsf{Id}^{f_{02},f_{12}}_{f.g.\mathsf{Id}_{A\to B}(f,g)}(f_{20},f_{21})$ is a function from squares in A, with arbitrary boundary, to squares in B with specified boundary.
Thus we define $\mathsf{sym}_{A\to B}$ by transposing both input and output:

$$\mathsf{sym}_{A\to B}(f_{22})(x_{00},x_{10},x_{20},x_{01},x_{11},x_{21},x_{02},x_{12},x_{22})$$
$$\equiv \mathsf{sym}(f_{22}(x_{00},x_{01},x_{02},x_{10},x_{11},x_{12},x_{20},x_{21},\mathsf{sym}(x_{22})))$$

Symmetry for Π-types is similar, using dependent symmetry.

Rules for symmetry

Some obvious rules for symmetry are that it should be an involution:

$$\mathsf{sym}_A(\mathsf{sym}_A(a_{22})) \equiv a_{22}$$

and it should commute with iterated ap on squares:

$$\mathsf{sym}_B(\mathsf{ap}_{\mathsf{ap}_f}(a_{22})) \equiv \mathsf{ap}_{\mathsf{ap}_f}(\mathsf{sym}_A(a_{22}))$$

The nullary case of the latter is $\mathsf{sym}(\mathsf{refl}_{\mathsf{refl}_a}) \equiv \mathsf{refl}_{\mathsf{refl}_a}$.
This solves problem #2:

$$\mathsf{ap}_{x.\mathsf{refl}_x}(\mathsf{refl}_a) \equiv \mathsf{sym}(\mathsf{refl}_{\mathsf{refl}_a}) \equiv \mathsf{refl}_{\mathsf{refl}_a}$$

Higher-dimensional symmetry

For n-dimensional cubes (i.e. n-fold iterated Id-types):

- We would expect symmetries to permute all n dimensions. The symmetric group S_n should act on n-cubes.
- We have transpositions of adjacent dimensions, from our sym. (E.g. $\mathsf{sym}_{\mathsf{Id}_A} : \mathsf{Id}_{\mathsf{Id}_{\mathsf{Id}_A}} \to \mathsf{Id}_{\mathsf{Id}_{\mathsf{Id}_A}}$ and $\mathsf{ap}_{\mathsf{sym}_A} : \mathsf{Id}_{\mathsf{Id}_{\mathsf{Id}_A}} \to \mathsf{Id}_{\mathsf{Id}_{\mathsf{Id}_A}}$.)

Fortunately, S_n is generated by adjacent transpositions!

$$S_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_k \sigma_k = 1 \\ \sigma_j \sigma_k = \sigma_k \sigma_j \quad (j+1 < k) \\ \sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1} \end{array} \right\rangle$$

The first two relations follow from the equations on the last slide. To obtain the third, we assert

$$\mathsf{sym}_{\mathsf{Id}_A}(\mathsf{ap}_{\mathsf{sym}_A}(\mathsf{sym}_{\mathsf{Id}_A}(a_{222}))) \equiv \mathsf{ap}_{\mathsf{sym}_A}(\mathsf{sym}_{\mathsf{Id}_A}(\mathsf{ap}_{\mathsf{sym}_A}(a_{222}))).$$

Outline

Towards computation by gluing

Symmetry computes the previously stuck term $\mathsf{ap}_{x.\mathsf{refl}_x}(a_2)$.
But how do we know there aren't other stuck terms?

Obviously, by proving canonicity/normalization.

We haven't done this yet, but the first step (from a modern perspective) is constructing a set-based semantic model to be the codomain for Artin gluing.

Identity contexts

Question

What categorical structure corresponds to our identity types?

- The objects of a category $\mathcal{C}$ correspond to syntactic contexts.
- The fundamental operation on contexts takes Δ to

$$\mathsf{ID}_\Delta :\equiv \big(\delta : \Delta, \delta' : \Delta, \varrho : \mathsf{Id}_\Delta(\delta, \delta,')\big).$$

  which factors the diagonal (i.e. is a path object):

$$\Delta \xrightarrow{\mathsf{refl}} \mathsf{ID}_\Delta \to (\delta : \Delta, \delta' : \Delta) \cong \Delta \times \Delta.$$

- This operation is functorial (via ap).
- We have natural symmetries $\mathsf{ID}_{\mathsf{ID}_\Delta} \cong \mathsf{ID}_{\mathsf{ID}_\Delta}$, yielding an S_n-action on n-fold identity contexts..

Cubical actions

Thus, an ID-structure on $\mathcal{C}$ is the same as

- A functor $\mathrm{ID} : \mathcal{C} \to \mathcal{C}$
- Nat. trans. $r : 1_{\mathcal{C}} \to \mathrm{ID}$ and $s, t : \mathrm{ID} \rightrightarrows 1_{\mathcal{C}}$ with $sr = tr = 1_{1_{\mathcal{C}}}$
- Natural symmetries $\mathrm{ID} \circ \mathrm{ID} \cong \mathrm{ID} \circ \mathrm{ID}$ satisfying S_n relations.

Definition

Let $\square^{\mathrm{op}}$ be the monoidal category freely generated by an object $\mathbb{I}$, morphisms $r : \mathbb{1} \to \mathbb{I}$ and $s, t : \mathbb{I} \to \mathbb{1}$ with $sr = tr = 1_{\mathbb{1}}$, where $\mathbb{1}$ is the unit, and symmetries $\mathbb{I} \otimes \mathbb{I} \cong \mathbb{I} \otimes \mathbb{I}$ satisfying S_n relations.

Then an ID-structure on $\mathcal{C}$ is also equivalently

- A monoidal functor $\square^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]$

and therefore also equivalently

- A coherent action $\square^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.

The semicartesian cube category

- $\square$ is a semicartesian monoidal category: symmetric monoidal and its unit $\mathbb{1}$ is terminal. Projections, but no diagonals.
- It is also the semicartesian monoidal category freely generated by an object $\mathbb{I}$ and morphisms $s, t : \mathbb{1} \to \mathbb{I}$.

We call $\square$ the semicartesian cube category.

This is the category used by:

- Bernardy–Coquand–Moulin, for internal parametricity (actually they used a unary version, this would be the binary one)
- Bezem–Coquand–Huber, for the original cubical model
- Cavallo–Harper, for the parametricity direction of parametric cubical type theory

Enrichment

The presheaf category $\widehat{\square} = \mathrm{Set}^{\square^{\mathrm{op}}}$ inherits a Day convolution monoidal structure (also semicartesian):

$$(X \otimes Y)_n = \int^{k,\ell} X_k \times Y_\ell \times \square(n, k \oplus \ell).$$

We write $\square^n$ for the representable $\square(-, \mathbb{I}^{\otimes n})$. Note $\square^0$ is terminal.

Theorem

An action $\triangleright : \square^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ is the same as an enrichment of $\mathcal{C}$ over $\widehat{\square}$ that has powers by representables (write $\square^n \pitchfork X \equiv \mathbb{I}^{\otimes n} \triangleright X$).

$$\mathrm{Map}(A, B)_n := \mathcal{C}(A, \square^n \pitchfork B)$$

$$\widehat{\square}(X, \mathrm{Map}(A, \square^n \pitchfork B)) \cong \widehat{\square}(X \otimes \square^n, \mathrm{Map}(A, B))$$

$\widehat{\square}$-enriched categories are the natural home for H.O.T.T. semantics.

Cubical objects

Of course, $\widehat{\square}$ is enriched over itself.

Similarly, any category $\mathcal{E}^{\square^{\mathrm{op}}}$ of cubical objects is $\widehat{\square}$-enriched, with powers and copowers if $\mathcal{E}$ is complete and cocomplete:

$$
\begin{aligned}
(A \odot X)_n &= \int^{k,\ell} (A_k \times \square(n, k \oplus \ell)) \cdot X_\ell \\
(A \pitchfork X)_n &= \int_{k,\ell} (X_k)^{A_\ell \times \square(k, n \oplus \ell)} \\
(\square^m \pitchfork X)_n &= X_{n \oplus m} \\
\mathrm{Map}(X, Y)_n &= \mathcal{E}^{\square^{\mathrm{op}}}(X, \square^n \pitchfork Y)
\end{aligned}
$$

More about the cube category

Up to equivalence:

- The objects of $\square$ are finite sets.
- A morphism $\phi \in \square(m, n)$ is a function $\phi : n \to m \sqcup \{-, +\}$ that is injective on the preimage of m.
- The monoidal structure $m \oplus n$ is disjoint union.

Sometimes use a skeletal version with objects $\underline{n} = \{0, 1, \ldots, n-1\}$, but often the non-skeletal version with all finite sets is better.

- The coface $\delta_{k,\pm} \in \square(n \setminus \{k\}, n)$ is the identity on $n \setminus \{k\}$ and sends k to $\pm$.
- The codegeneracy $\sigma_k \in \square(n, n \setminus \{k\})$ is the inclusion.
- The endomorphism monoid $\square(n, n)$ is the symmetric group S_n.

The magic of semicartesian cubes

The monoidal structure of $\square$ is "almost" cartesian; only the injectivity requirement spoils it. If it were cartesian we would have

$$¿ \quad \square(n, k \oplus \ell) \cong \square(n, k) \times \square(n, \ell). \quad ?$$

Instead, we have

$$\square(n, k \oplus \ell) \cong \sum_{\phi : \square(n,k)} \square(n \setminus \phi(k), \ell).$$

Removing $\phi(k)$ from the second domain ensures the copaired function $k \sqcup \ell \to n \sqcup \{-, +\}$ is still injective on the preimage of n.

But in some ways this is even better!

Copowers by representables

For $A \in \widehat{\square}$ and $X \in \mathcal{E}^{\square^{\mathrm{op}}}$, we have

$$(A \odot X)_n = \int^{k,\ell} (A_k \times \square(n, k \oplus \ell)) \cdot X_\ell$$

$$\begin{aligned}
(\square^m \odot X)_n &= \int^{k,\ell} (\square(k, m) \times \square(n, k \oplus \ell)) \cdot X_\ell \\
&= \int^{\ell} \square(n, m \oplus \ell) \cdot X_\ell \\
&= \int^{\ell} \Big(\textstyle\sum_{\phi \in \square(n,m)} \square(n \setminus \phi(m), \ell) \Big) \cdot X_\ell \\
&= \sum_{\phi \in \square(n,m)} \int^{\ell} \square(n \setminus \phi(m), \ell) \cdot X_\ell \\
&= \sum_{\phi \in \square(n,m)} X_{n \setminus \phi(m)}.
\end{aligned}$$

Semicartesian cylinders

Taking $m = 1$, we get

$$(\square^1 \odot X)_n = \sum_{\phi \in \square(n,1)} X_{n \smallsetminus \phi(1)}.$$

A morphism $\phi \in \square(n, 1)$ is a function $1 \to n \sqcup \{-, +\}$, so either:

- some $k \in n$, in which case $n \smallsetminus \phi(1) = n \smallsetminus \{k\}$, or
- $+$ or $-$, in which case $n \smallsetminus \phi(1) = n$. Thus:

$$(\square^1 \odot X)_n = X_n + X_n + \sum_{k \in n} X_{n \smallsetminus \{k\}}.$$

An n-cube in $\square^1 \odot X$ is either an n-cube in the left-hand copy of X, an n-cube in the right-hand copy of X, or an $(n-1)$-cube in X stretched out in some dimension along the cylinder.

There is almost no other cube category for which this holds.

Outline

Semantic identity types

In a $\widehat{\mathbb{I}}$-enriched category with representable powers, we also need:

1. Coherence theorems. $\leftarrow$ next time
2. Transport and lifting ("fibrancy"). $\leftarrow$ next time
3. Categorical computation rules for Id, up to isomorphism.

Semantic identity types

In a $\widehat{\square}$-enriched category with representable powers, we also need:

1. Coherence theorems. $\leftarrow$ next time
2. Transport and lifting ("fibrancy"). $\leftarrow$ next time
3. Categorical computation rules for Id, up to isomorphism.

It's tempting to think that, at least in $\widehat{\square}$, we can just define $\mathsf{Id}_{A\times B}$, $\mathsf{Id}_{A\to B}$, etc., to be whatever we want. But we can't: Id_X must be defined as $\square^1 \pitchfork X$. What we can define is the individual sets of n-cubes in a particular $X \in \widehat{\square}$. But:

- It can be non-obvious how these lead to a categorical characterization of the entire cubical set Id_X.
- For type formers like $A \times B$, $A \to B$, we don't even have this much choice: they are determined by their universal properties.

The computation rules for Id are non-trivial theorems about $\mathcal{E}^{\square^{\mathrm{op}}}$.

Identity types of products

Note $x : A, y : A \vdash \mathsf{Id}_A(x, y) : \mathsf{U}$ is represented semantically by the projection from the representable power $\square^1 \pitchfork A \to A \times A$.

Since $(\square^1 \pitchfork -)$ is a right adjoint, it preserves products:

$$\begin{array}{ccc} \square^1 \pitchfork (A \times B) & \xrightarrow{\cong} & (\square^1 \pitchfork A) \times (\square^1 \pitchfork B) \\ \downarrow & & \downarrow \\ (A \times B) \times (A \times B) & \xrightarrow[\cong]{} & (A \times A) \times (B \times B) \end{array}$$

Syntactically, this gives

$$\mathsf{Id}_{A \times B}(u, v) \cong \mathsf{Id}_A(\pi_1 u, \pi_1 v) \times \mathsf{Id}_B(\pi_2 u, \pi_2 v).$$

Same idea works for Σ-types. A coherence theorem will improve $\cong$ to $=$.

Up next

Plan for the three talks:

1. Basic syntax of H.O.T.T.
2. Symmetries and semicartesian cubes
3. Univalent universes