Asymptotic Bounds on the Optimum Multiuser Efficiency of Randomly Spread CDMA

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***Abstract*—We derive some bounds on the Optimum Asymptotic Multiuser Efficiency (OAME) of randomly spread CDMA as extensions of the result by Tse and Verdú. To this end, random Gaussian and random binary antipodal spreading are considered. Furthermore, the input signal is assumed to be Binary Phase Shift Keying (BPSK). It is shown that in a CDMA system with K-user and N chips when K and $N \to \infty$ and the loading factor, $\frac{K}{N}$, grows logarithmically with K, the OAME converges to 1 almost surely under some condition. It is also shown that a Gaussian randomly spread CDMA system has a performance close to the single user system at high Signal to Noise Ratio (SNR) when the loading factor is kept less than $\frac{\log_3 K}{2}$. Moreover, for random binary antipodal matrices, we show that the loading factor cannot grow faster than $\frac{\log_2 3}{2} \log_3 K$.**

***Index Terms*—Code division multiple access (CDMA), random spreading, multiuser detection, optimum asymptotic multiuser efficiency (OAME), detecting matrices, compressive sensing.**

I. Introduction

Multiuser efficiency [1] is an important performance measure which shows performance loss of CDMA detectors in comparison with the single user system. One interesting asymptotic limit of multiuser efficiency is when the background noise vanishes [1], [2]. This parameter shows the performance of the considered detector when Signal to Noise Ratio (SNR) is very high. For the optimum detector, this parameter is called Optimum Asymptotic Multiuser Efficiency (OAME).

Tse and Verdú in [2] prove that the OAME for a CDMA system with Binary Phase Shift Keying (BPSK) input signal and independent and identically distributed (i.i.d.) random spreading with N chips approaches 1 as the number of users, K, tends to infinity and the loading factor, $\frac{K}{N}$, is kept equal to an arbitrary nonzero constant β. This case, i.e., $N, K \to +\infty$ when $\beta = \frac{K}{N}$, is known as the large system limit. Tanaka in [3] has derived the performance of a CDMA system with finite SNR in the large system limit using the replica method known in the statistical physics. Furthermore, in the related context of compressive sensing [4], the authors of [5] showed that the result of Tse and Verdú is not restricted to binary input signals, but holds for any input alphabet with finite cardinality.

OAME is proportional to the minimum euclidean distance of the N dimensional vectors mapped by the spreading matrix in CDMA systems and has a maximum value 1 [1]. When OAME is 1, the performance of optimum detector is same as the single user system for very high SNR. Therefore, the results in [2] and [5] show that in the large system limit the performance of the optimum detector converges to the performance of single user system when SNR is very high.

Based on the previous works it is not clear whether the finite loading factor is a necessary condition. This question is answered in this paper. We consider randomly spread CDMA with BPSK inputs. Binary antipodal and Gaussian spreading matrices are considered. We show that the finite loading factor condition is not necessary to have an OAME equal to 1. It is shown that the loading factor can grow logarithmically with the number of users. It is also shown that for binary antipodal matrices the loading factor cannot grow faster than $O(\log K)$ to obtain a nonzero OAME.

Although this paper considers CDMA systems with BPSK inputs, the same problem can be investigated for any other types of inputs. Investigating the problem with general discrete inputs is an open problem. One interesting application of such a problem is compressive sensing [4] with discrete inputs.

The rest of this paper is organized as follows. Section II presents the system model. In sections III and IV, the main theorems about OAME for binary antipodal and Gaussian random spreading are presented, respectively. In Section V, an upper bound for the OAME is presented and finally, section VI concludes the paper.

II. System model and preliminaries

Assume a randomly spread CDMA system with a discrete model

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{b} + \boldsymbol{n}, \tag{1}$$

where $\boldsymbol{H}$ is an $N \times K$ spreading matrix whose elements are i.i.d. and have a symmetric probability distribution function (pdf). Note that a random variable x has a symmetric pdf $\rho(x)$ if for every α, $\rho(\alpha) = \rho(-\alpha)$. $\boldsymbol{b}$ is the data vector that $b_i \in \{\pm 1\}$, $\boldsymbol{n} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$ is the additive white Gaussian noise vector and $\boldsymbol{y}$ is the received vector. In the rest of this paper we use the following definition

$$\boldsymbol{u}(\boldsymbol{x}) = [u_1(\boldsymbol{x}), \cdots, u_N(\boldsymbol{x})]^T \triangleq \boldsymbol{H}\boldsymbol{x}. \tag{2}$$

Note that in (1), the number of users is K, the number of chips is assumed to be N and the users are assumed to have unit power. In the considered model, the asymptotic multiuser efficiency is defined as [6]

$$\eta \triangleq 2 \lim_{\sigma \to 0} \sigma^2 \log \left(\frac{1}{\mathrm{P_e}(\sigma)} \right), \tag{3}$$

where $\mathrm{P_e}(\sigma)$ is the bit-error rate of the users. Then, the OAME is calculated as follows [6]

$$\eta = \min_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\boldsymbol{0}\}} \boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x}, \tag{4}$$

where $\boldsymbol{R} \triangleq \boldsymbol{H}^\dagger \boldsymbol{H}$ and $\boldsymbol{x}$ is the error vector. η is in $[0,1]$ for any given K and N. In [2], it is proven that when $K, N \to \infty$ and K/N is kept constant and finite, η converges to 1 almost surely. Therefore, an interesting question is that whether it is necessary to keep K/N finite. In fact, the question is what is the maximum possible K/N to have η converging to 1. This question applies to compressive sensing as well. In compressive sensing, it is desired to find a transfer matrix with minimum number of rows to compress a sparse data vector [4].

Let E_K be the event that $\boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} < \gamma$ for at least one nonzero error vector $\boldsymbol{x} \in \{\pm 1, 0\}^K$ when $\gamma \in [0,1]$. Therefore,

$$\mathrm{P}(E_K) = \mathrm{P}\left(\bigcup_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\boldsymbol{0}\}} \boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} < \gamma \right). \tag{5}$$

In fact instead of calculating the minimum in (4), we prove that $\mathrm{P}(E_K)$ converges to zero for some conditions. By applying the union bound to (5), an upper bound is obtained as

$$\mathrm{P}(E_K) \leq \sum_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\boldsymbol{0}\}} \mathrm{P}(\boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} < \gamma). \tag{6}$$

In the next sections, we consider random binary antipodal and random Gaussian spreading matrices.

III. The OAME for i.i.d. Binary Antipodal Random Spreading

In this section, it is assumed that the entries of the spreading matrix, $H_{i,j}$, are chosen randomly from $\left\{ \pm \frac{1}{\sqrt{N}} \right\}$ with equal probability. The input signal, $\boldsymbol{b}$, is assumed to be BPSK. The summation in (6) is sum of $3^K - 1$ terms. The key point is that the elements of $\boldsymbol{x}$ which are equal to zero, do not effect the term $\boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x}$. Furthermore, since the entries of $\boldsymbol{H}$ are antipodal with equal probability, the sign of nonzero elements of $\boldsymbol{x}$ does not effect the distribution of $\boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x}$. Therefore, we can expand the summation on the weight of $\boldsymbol{x}$. Note that the weight of a vector is defined as the number of nonzero elements of it. By the following lemma, it is shown that to calculate (6), we can only consider the error vectors with even weight.

Lemma 1. *Let $H_{i,j} \in \left\{ \pm \frac{1}{\sqrt{N}} \right\}$. For every error vector $\boldsymbol{x} \in \{\pm 1, 0\}^K$ with odd weight, $\mathrm{P}\left(\boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} < \gamma \right) = 0$ [7].*

In the next lemma, we present an upper bound for the probability $\mathrm{P}(\boldsymbol{x}_j^T \boldsymbol{R} \boldsymbol{x}_j < \gamma)$.

Lemma 2. *Let $\boldsymbol{x}_j$ be an error vector with even weight $2j > 0$ and B_j be the event that the number of nonzero elements of $\boldsymbol{u}(\boldsymbol{x}_j)$ is less than $\frac{\gamma N}{4}$. Then, $\mathrm{P}(\boldsymbol{x}_j^T \boldsymbol{R} \boldsymbol{x}_j < \gamma) \leq \mathrm{P}(B_j)$ [7].*

Note that Lemma 1 and Lemma 2 are only correct for the BPSK input and binary antipodal spreading matrices. As an example, for the case of $\boldsymbol{b} \in \{0, \pm 1\}$ with error vector $\boldsymbol{x} \in \{0, \pm 1, \pm 2\}$, both lemmas are incorrect.

In the following theorem, the main result on the OAME of a CDMA system with binary antipodal random spreading matrices is presented.

Theorem 1. *For the CDMA system* (1) *with $\boldsymbol{b} \in \{\pm 1\}^K$ and $H_{i,j} \in \left\{ \pm \frac{1}{\sqrt{N}} \right\}$, the OAME η is greater or equal than $\min\{1, 4(1 - 2\zeta)\}$ almost surely, when $K, N \to \infty$, and $\zeta \triangleq \frac{K}{N \log_3 K}$.*

Proof: The complete proof is presented in [7] and here we present an intuitive summary of the proof. From the definition of the OAME we know that $\eta \leq 1$. In fact, this can be easily proven as follows

$$\eta = \min_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\boldsymbol{0}\}} \boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} \leq [1, 0, \cdots, 0] \boldsymbol{R} [1, 0, \cdots, 0]^T = 1. \tag{7}$$

Therefore, we only need to prove that $\eta \geq 4(1 - 2\zeta)$. From Lemma 1, it is only required to consider the error vectors with even weight. Furthermore, in [7] it is proven that for all $\boldsymbol{x}$ with the same weight, $\mathrm{P}\left(\boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} < \gamma \right)$ are equal. Therefore,

$$\mathrm{P}(E_K) \leq \sum_{j=1}^{\lfloor \frac{K}{2} \rfloor} \binom{K}{2j} 2^{2j} \mathrm{P}(\boldsymbol{x}_j^T \boldsymbol{R} \boldsymbol{x}_j < \gamma), \tag{8}$$

where $\boldsymbol{x}_j$ is an arbitrary vector with weight $2j$. Using Lemma 2 in (8) results in

$$\begin{aligned} \mathrm{P}(E_K) &\leq \sum_{j=1}^{\lfloor \frac{K}{2} \rfloor} \binom{K}{2j} 2^{2j} \mathrm{P}(B_j) \\ &= \sum_{j=1}^{\lfloor \frac{K}{2} \rfloor} \binom{K}{2j} 2^{2j} \sum_{i=0}^{\frac{N\gamma}{4} - 1} \binom{N}{i} \mathrm{p}(j)^{N-i} (1 - \mathrm{p}(j))^i, \end{aligned} \tag{9}$$

where

$$\mathrm{p}(j) = \mathrm{P}(u_\ell(\boldsymbol{x}_j) = 0) = \binom{2j}{j} 2^{-2j}. \tag{10}$$

The Binomial distribution function $f(i) = \binom{N}{i} \mathrm{p}(j)^{N-i} (1 - \mathrm{p}(j))^i$ is an increasing function for $i < i_\mathrm{m} \triangleq \lfloor N(1 - \mathrm{p}(j)) \rfloor$. Therefore, an upper bound for (9) is derived as

$$\begin{aligned} \mathrm{P}(E_K) &\leq \sum_{j=1}^{\lfloor \frac{K}{2} \rfloor} \binom{K}{2j} 2^{2j} \sum_{i=0}^{\frac{N\gamma}{4} - 1} \binom{N}{i} \mathrm{p}(j)^{N-i} (1 - \mathrm{p}(j))^i \\ &\leq \sum_{j=1}^{\lfloor \frac{K}{2} \rfloor} \binom{K}{2j} 2^{2j} \frac{N\gamma}{4} \binom{N}{\frac{N\gamma}{4}} \mathrm{p}(j)^{N - \frac{N\gamma}{4}} (1 - \mathrm{p}(j))^{\frac{N\gamma}{4}}. \end{aligned} \tag{11}$$

To simplify more, the following inequality is used

$$\binom{m}{r} \leq 2^{m\mathrm{h}\left(\frac{r}{m}\right)}, \quad (12)$$

where

$$\mathrm{h}(t) = -t\log_2 t - (1-t)\log_2(1-t), \quad (13)$$

denotes the binary entropy function. Let $\zeta \triangleq \frac{K}{N\log_3 K}$ be a finite constant. Therefore, (11) can be written as

$$\mathrm{P}(E_K) \leq \sum_{j=1}^{\lfloor \frac{K}{2} \rfloor} \frac{K\gamma}{4\zeta \log_3 K} 2^{K\mathrm{h}\left(\frac{2j}{K}\right)+2j} 2^{\mathrm{h}\left(\frac{\gamma}{4}\right)\frac{K}{\zeta\log_3 K}} \times \mathrm{p}(j)^{\frac{(4-\gamma)K}{4\zeta\log_3 K}} (1-\mathrm{p}(j))^{\frac{\gamma K}{4\zeta \log_3 K}}. \quad (14)$$

It can be seen that the summation in (14) decays exponentially in $\frac{K}{\log_3 K}$ if [7]

$$\zeta < \min\left(\frac{\log_2 3}{2}\left(1-\frac{\gamma}{4}\right), \frac{1-\frac{\gamma}{4}}{2}\right) = \frac{1-\frac{\gamma}{4}}{2}. \quad (15)$$

This can also be obtained intuitively by considering the most populated weight $j = \frac{2K}{3}$. In fact, the dominant weight which has the largest $\mathrm{P}(\boldsymbol{x}_j^T \boldsymbol{R}\boldsymbol{x}_j < 1)$ is the weight corresponds to the maximum number of error vectors, i.e., $j = \frac{2K}{3}$.

By using the Borel-Cantelli lemma [8] it is concluded that η is greater or equal than γ if $\zeta < \frac{1-\frac{\gamma}{4}}{2}$ almost surely. Therefore, it can be shown that $\eta \geq 4(1-2\zeta)$. This together with the fact that $\eta \leq 1$ result in

$$\eta \geq \min\{1, 4(1-2\zeta)\}, \quad (16)$$

which proves the theorem. ■

Note that for $\zeta < \frac{3}{8}$ the result of Theorem 1 is

$$\eta \geq \min\{1, 4(1-2\zeta)\} = 1. \quad (17)$$

Therefore, in this duration η converges to 1 almost surely. This shows that the performance of the optimum receiver is same as the single user performance for very high SNR when the loading factor is less than $\frac{3\log_3 K}{8}$. This means the finite loading factor is not a necessary condition to have $\eta = 1$. For $\frac{3}{8} < \zeta < \frac{1}{2}$, the result in Theorem 1 is a lower bound and for $\zeta \geq \frac{1}{2}$, Theorem 1 is obvious since $\boldsymbol{R}$ is a non-negative definite matrix and $\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x} \geq 0$ for all BPSK inputs.

IV. The OAME for i.i.d. Gaussian spreading

In this section, we investigate the OAME for a randomly spread CDMA when the entries of H are i.i.d. Gaussian distributed.

Theorem 2. *Let $H_{i,j} \sim \mathcal{N}(0, \frac{1}{N})$. The OAME converges to* 1 *almost surely as $K, N \to \infty$, if $\frac{K}{N\log_3 K}$ is kept less than $\frac{1}{2}$.*

Proof: By using (6) and letting $\gamma = 1$ we have

$$\mathrm{P}(E_K) \leq \sum_{j=1}^{K} \binom{K}{j} 2^j \mathrm{P}(\boldsymbol{x}_j^T\boldsymbol{R}\boldsymbol{x}_j < 1), \quad (18)$$

where $\boldsymbol{x}_j$ is an arbitrary vector with weight j. For sake of simplicity, we write (18) as

$$\mathrm{P}(E_K) \leq \underbrace{2K\mathrm{P}(\boldsymbol{x}_1^T\boldsymbol{R}\boldsymbol{x}_1 < 1)}_{\triangleq G_1} + \underbrace{\sum_{j=2}^{K}\binom{K}{j}2^j\mathrm{P}(\boldsymbol{x}_j^T\boldsymbol{R}\boldsymbol{x}_j < 1)}_{\triangleq G_2}, \quad (19)$$

where $\boldsymbol{x}_1$ is an arbitrary vector with weight 1. From [2, eq. (21)], it can be concluded that the term $\mathrm{P}(\boldsymbol{x}_1^T\boldsymbol{R}\boldsymbol{x}_1 < 1)$ decays exponentially in N. Since we assume $\zeta = \frac{K}{N\log_3 K}$ is finite, G_1 can be written as

$$G_1 = \mathrm{O}\left(K\mathrm{e}^{-\alpha\frac{K}{\log_3 K}}\right), \quad (20)$$

where α is a finite positive real number.

Since the channel coefficients are Gaussian we have

$$u_\ell(\boldsymbol{x}_j) \sim \mathcal{N}\left(0, \frac{j}{N}\right). \quad (21)$$

Therefore, $\frac{N}{j}(\boldsymbol{x}_j^T\boldsymbol{R}\boldsymbol{x}_j)$ has a chi-squared distribution with N degrees of freedom. Note that $\boldsymbol{x}_j^T\boldsymbol{R}\boldsymbol{x}_j$ has the average j and variance $\frac{j^2}{N}$. This means that for the error vectors with higher weights, the distribution of $\boldsymbol{x}_j^T\boldsymbol{R}\boldsymbol{x}_j$ shifts to the right but at the same time its variance increases. Therefore, it is not clear which weight results in the largest $\mathrm{P}(\boldsymbol{x}_j^T\boldsymbol{R}\boldsymbol{x}_j < 1)$. It will be shown that in this case also the weight $j = \frac{2K}{3}$ is determinant.

Using the definition of the chi-squared distribution we have

$$\mathrm{P}(\boldsymbol{x}_j^T\boldsymbol{R}\boldsymbol{x}_j < 1) = \int_0^{\frac{N}{j}} \frac{1}{2^{N/2}\Gamma(N/2)} x^{\frac{N}{2}-1}\exp(-x/2)dx. \quad (22)$$

A chi-squared distribution with N degrees of freedom is an increasing function in $[0, N-2]$ for $N > 2$. Therefore, since $j \geq 2$, the term inside of the integration in (22) is an increasing function. Thus,

$$\mathrm{P}(\boldsymbol{x}_j^T\boldsymbol{R}\boldsymbol{x}_j < 1) \leq \frac{N}{j2^{N/2}\Gamma(N/2)}\left(\frac{N}{j}\right)^{\frac{N}{2}-1}\exp\left(-\frac{N}{2j}\right). \quad (23)$$

Without loss of generality we assume that N is an even integer. Based on Stirling's formula a lower bound for $\Gamma(N/2)$ is

$$\Gamma(N/2) = (N/2-1)! = \frac{(N/2)!}{N/2} > 2\sqrt{\pi/N}\left(\frac{N}{2e}\right)^{N/2}, \quad (24)$$

Therefore,

$$G_2 \leq \sum_{j=2}^{K}\binom{K}{j}2^j\frac{1}{2\sqrt{\pi/N}}j^{-\frac{N}{2}}\left(\mathrm{e}^{1-\frac{1}{j}}\right)^{\frac{N}{2}}. \quad (25)$$

To simplify the right hand side of (25), the summation is split to three parts as follows

$$\begin{aligned} G_2 \leq & \sum_{j=2}^{i_0} \binom{K}{j} 2^j \frac{1}{2\sqrt{\pi/N}} j^{-\frac{N}{2}} (\mathrm{e}^{1-\frac{1}{j}})^{\frac{N}{2}} \\ + & \sum_{j=i_0}^{i_1} \binom{K}{j} 2^j \frac{1}{2\sqrt{\pi/N}} j^{-\frac{N}{2}} (\mathrm{e}^{1-\frac{1}{j}})^{\frac{N}{2}} \\ + & \sum_{j=1}^{i_0} \binom{K}{j} 2^j \frac{1}{2\sqrt{\pi/N}} j^{-\frac{N}{2}} (\mathrm{e}^{1-\frac{1}{j}})^{\frac{N}{2}} \end{aligned} \tag{26}$$

where $i_0 = \lfloor \log_2 K \rfloor$ and $i_1 = \left\lfloor \frac{K}{\log_2 K} \right\rfloor$. To simplify (26), first note that the function $\frac{\mathrm{e}^{1-\frac{1}{j}}}{j}$ is a decreasing function for $j \in \{2, \cdots, K\}$. Furthermore, the term $\binom{K}{j}$ can be substituted by the upper-bounded presented in (12). Therefore, it can be written that

$$\begin{aligned} G_2 \leq & \mathrm{O}\left(\sqrt{K \log_2 K} 2^{(\log_2 K)^2 - \frac{1+\frac{1}{2}\log_2 \mathrm{e}}{2\zeta} \frac{K}{\log_3 K}} \right) \\ + & \mathrm{O}\left(\left(\frac{K}{\log_3 K} \right)^{3/2} 2^{\left(1-\frac{\log_2 3}{2\zeta}\right)\frac{K \log_2(\log_2 K)}{\log_2 K} + \frac{K \log_2 \mathrm{e}}{2\zeta \log_3 K}} \right) \\ + & \mathrm{O}\left(\frac{K^{3/2}}{\sqrt{\log_3 K}} \mathrm{e}^{\frac{K}{2\zeta \log_3 K}} 2^{K\left(\mathrm{h}\left(\frac{2}{3}\right)+\frac{2}{3}-\frac{\log_2 3}{2\zeta}\right) + \frac{K \log_2(\log_2 K)}{2\zeta \log_3 K}} \right). \end{aligned} \tag{27}$$

From (27), it is observed that $\lim_{K\to\infty} G_2 = 0$ if

$$\zeta < \min\left(\frac{1}{2}, \frac{\log_2 3}{2} \right) = \frac{1}{2}. \tag{28}$$

Furthermore, from (20), (27) and (28), it can be shown that

$$\sum_{K=1}^{+\infty} \mathrm{P}(E_K) < \infty, \tag{29}$$

which together with the application of the Borel-Cantoli lemma proves that η converges to 1 almost surely if $K \to \infty$ and ζ is kept less than $\frac{1}{2}$. ■

Fig. 1 shows the results of Theorem 1 and Theorem 2.

V. An Upper Bound on the OAME of CDMA Systems with Binary Antipodal Spreading

The theorems in the last sections present some lower bounds on the OAME. Accordingly, in a CDMA system, the loading factor can grow logarithmically with the number of users and still the OAME converges to 1. However, it is not clear whether the loading factor can grow faster than $O(\log K)$. In this section, we answer this question using the concept of detecting matrices in mathematics.

Detecting matrices originate from the coin weighing problem in mathematics [9]- [10]. For a given data set $\mathcal{S}$ such that $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{S}^K$, an $N \times K$ matrix $\boldsymbol{H}$ is called detecting if and only if

$$\boldsymbol{H}\boldsymbol{x}_1 = \boldsymbol{H}\boldsymbol{x}_2 \Rightarrow \boldsymbol{x}_1 = \boldsymbol{x}_2, \tag{30}$$

where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are $K \times 1$ vectors. Another representation form of (30) is

$$\boldsymbol{H}\boldsymbol{x} = \mathbf{0}_{N\times 1} \Rightarrow \boldsymbol{x} = \mathbf{0}_{K\times 1}, \tag{31}$$

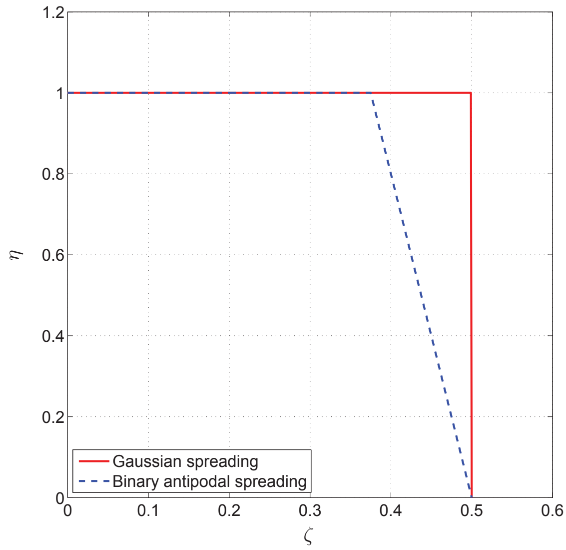

Fig. 1: The optimum asymptotic multiuser efficiency lower bound versus $\zeta = \frac{K}{N \log_3 K}$.

where $\boldsymbol{x} \in \mathcal{S}^K - \mathcal{S}^K$ in which

$$\mathcal{S}^K - \mathcal{S}^K = \left\{ \boldsymbol{x}_1 - \boldsymbol{x}_2 | \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{S}^K \right\}. \tag{32}$$

One can write (31) as

$$\mathrm{Null}(\boldsymbol{H}) \bigcap \mathcal{S}^K - \mathcal{S}^K = \emptyset, \tag{33}$$

where $\mathrm{Null}(\boldsymbol{H})$ is the null space of H and $\emptyset$ is an empty set.

From (4) and (33), it can be observed that there is a connection between η and the concept of detecting matrices. Detecting means that the mapping $\boldsymbol{H}\boldsymbol{b}$ does not map any two data vectors to one output vector. However, the OAME in a CDMA shows the euclidean distance of the mapped vectors. In a CDMA system if the spreading matrix, $\boldsymbol{H}$, is not detecting then there is an error vector $\boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} = 0$. Therefore, if the spreading matrix is not a detecting matrix then the OAME is equal to 0. Note that there might be a matrix which is detecting but its OAME vanishes.

In [11], it is proven that

$$\lim_{K\to\infty} \frac{N_0 \log_2 K}{K} = 2, \tag{34}$$

where N_0 is the minimum possible of N such that an $N \times K$ binary $\{0, 1\}$ or binary antipodal $\{\pm 1\}$ detecting matrix exists for any binary input [12]. Therefore, it is concluded that the OAME is equal to 0 when $K \to \infty$ and $\zeta = \frac{K}{N \log_3 K}$ is kept greater than $\frac{\log_2 3}{2}$. This shows that the loading factor cannot grow faster than $\frac{\log_2 3}{2} \log_3 K$. Note that there is no result for the OAME of a random binary antipodal spread CDMA in $\zeta \in \left(\frac{1}{2}, \frac{\log_2 3}{2} \right)$ so far.

VI. Conclusion and Discussion

We proved that in CDMA systems with BPSK input and Gaussian or binary antipodal spreading matrices, the loading factor can grow logarithmically with the number of users. For Gaussian spreading, the OAME converges to 1 even if the loading factor grows with $\frac{\log_3 K}{2}$. We also proved that for binary antipodal spreading the OAME is greater than $\min\{1, 4(2-\zeta)\}$, where $\zeta = \frac{K}{N \log_3 K}$. It was also proven that for binary antipodal spreading, the loading factor cannot grow faster than $\frac{\log_2 3}{2} \log_3 K$.

Note that in the analysis of OAME, we first take the limit $\sigma^2 \to 0$ and then $K \to +\infty$. Therefore, the analysis does not give any information about the noise power and the SNR. Moreover, the replica analysis presented by Tanaka cannot be used in this case since the loading factor is not finite.

The problem considered in this paper can be generalized to any types of discrete input distribution. However, the presented analysis cannot be applied to other types of input distribution. The limit of the loading factor for a general discrete distribution is an open problem. One motivation for investigating such a problem is compressive sensing with discrete inputs.

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