Graded concept lattices in fuzzy rough set theory

Alexander Šostak[1,2,*], Māris Krastiņš[2] and Ingrīda Uļjane[1,2]

[1]Institute of Mathematics and CS University of Latvia, Riga, LV-1459, Latvia
[2]Department of Mathematics, University of Latvia, Riga, LV-1004, Latvia

Abstract
Noting certain limitations of fuzzy concept lattices in rough set theory in terms of their possible applications, we present here a more flexible notion of a graded fuzzy concept lattice in rough set theory. We establish initial facts about the object-oriented version of graded fuzzy concept lattice and illustrate them with some examples, of both theoretical and practical nature.

Keywords
Fuzzy relations, fuzzy rough sets, measure of inclusion, object-oriented fuzzy concepts, graded fuzzy concept lattices in rough set theory

1. Introduction

Formal concept analysis was initiated by R. Wille and B. Ganter [28], [8] in 80-ties of the previous century. The principal subject of the study and at the same time the main tool of research in formal concept analysis are formal concepts and concept lattices. At present concept analysis and corresponding theory of concept lattices is a well developed field of mathematics with many practical applications. The basics of the fuzzy concept analysis and the corresponding theory of fuzzy concept lattices were mainly developed by R. Bělohlávek [1], see also [2], etc.

I. Düntch and G. Gediga [7] who worked in the field of model logic introduced approximation-type operators by means of a binary relation. In addition, they used these operators for an alternative notion of a concept lattice, now known as a property-oriented concept lattice, see, e.g. [29], [16] et. al. Afterwards, Y.Y. Yao [29] defined an object-oriented concept lattice, in a certain sense dual to the property-oriented concept lattice. Besides, Y.Y. Yao has highlighted the important feature of these lattices, namely, that they can be realized as the analogue of "classic" Wille-Ganter-Bělohlávek concept lattices in rough set theory. The fuzzy version of object and property-oriented concept lattices was considered in [6]. In addition, this paper contains deep analysis of relations between the three kinds of (fuzzy) concept lattices, i.e. between Wille-Ganter-Bělohlávek (or WGB for short) (fuzzy) concept lattices, and two (fuzzy) concept lattices in rough set theory, namely the property-oriented and the object-oriented (fuzzy) concept lattices.

Published in Pablo Cordero, Ondrej Kridlo (Eds.): The 16th International Conference on Concept Lattices and Their Applications, CLA 2022, Tallinn, Estonia, June 20–22, 2022, Proceedings, pp. 19–33.
*Corresponding author.
aleksandrs.sostaks@lu.lv (A. Šostak); mk18032@lu.lv (M. Krastiņš); ingrida.uljane@lu.lv (I. Uļjane)
0000-0002-0252-8306 (I. Uļjane)

Concept lattices, both "classic" and concept lattices in rough set theory, have found many important applications outside pure mathematics. Specifically, the technique based on concept lattices is used in medicine, biology, healthcare, sociology, etc. On the other hand, we know examples of only fragmentary applications of fuzzy concept lattices (of both types) in practice. The reason for this is that the precise matching between extents and intents in fuzzy environment is nearly impossible, see some comments in this concern in Subsection 4.3. To overcome this problem we suggest to replace the notion of a fuzzy concept by a more general notion of a graded fuzzy concept and to develop a more flexible theory of graded fuzzy concept lattices. To realize this approach we start with the notion of a preconcept, viewing it as a certain "potential concept", and then estimate "conceptuality" of a fuzzy preconcept by means of fuzzy logic tools. In the result we obtain a graded fuzzy concept lattice. We study basic properties of graded fuzzy concept lattices and illustrate them by some specific examples.

In the paper [26] we used similar ideas to develop a graded approach to WGB-fuzzy concept lattices. However, as different from graded WGB-fuzzy concept lattices, here we base on forward and backward powerset operators induced by fuzzy relations. As the result the structure of graded fuzzy concept lattices in rough set theory relays on isotone Galois connection (or adjunction) as different from WGB-fuzzy concept lattices constructed in accordance with antitone Galois connection.

The paper is structured as follows. In the second, preliminary, section, we recall and refine definitions used in the paper and clarify the framework in which our research is carried out. In the third section forward and backward operators induced by fuzzy relations are defined and used for studying derivation operators, which in turn form the basis of fuzzy concept lattices in fuzzy rough set theory. In section 4, which is central to the article, we develop a gradation approach to concept lattices in fuzzy rough set theory. The fifth section is devoted to examples illustrating our approach in particular cases determined by specific choice of a t-norm involved in the definition of a fuzzy relation and the choice of fuzzy sets of potential objects and potential properties. In the sixth section we consider some examples of practical nature where graded fuzzy concepts could be useful. In the last, conclusion section some prospects are sketched for the further studies on the basis of this paper.

2. Preliminaries

Lattices, quantales and residuated lattices. We recall here some well known concepts from the theory of lattices see, e.g. [3], [13], [17], in order to clarify the terms used in the paper.

In our paper $L = (L, \leq, \wedge, \vee)$ denotes a complete lattice, that is a lattice in which joins $\bigvee M$ and meets $\bigwedge M$ of all subsets $M \subseteq L$ exist. In particular $0 \in L$ and $1 \in L$ are the bottom and the top elements of L respectively. A complete lattice L is called join-distributive if $a \wedge \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \wedge b_i)$ for every $a \in L$ and every $\{b_i \mid i \in I\} \subseteq L$. Dually, a complete lattice L is called meet-distributive if $a \vee \left(\bigwedge_{i\in I} b_i\right) = \bigwedge_{i\in I}(a \vee b_i)$. A complete lattice is called bi-distributive if it is join- and meet-distributive.

Let L be a complete lattice and $* : L \times L \to L$ be a binary associative monotone operation. The tuple $(L, \leq, \wedge, \vee, *)$ is called *a quantale* [21] if $*$ distributes over arbitrary joins:
$a * \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a * b_i), \left(\bigvee_{i\in I} b_i\right) * a = \bigvee_{i\in I}(b_i * a) \;\; \forall a \in L,\; \{b_i | i \in I\} \subseteq L.$

A quantale is integral if the top element acts as the unit, i.e. $1 * a = a$. A quantale is commutative, if $a * b = b * a$ for all $a, b \in L$. In what follows by *a quantale* we mean *a commutative integral quantale*. A typical example of a quantale is the unit interval endowed with a lower semi-continuous t-norm, see, e.g. [15].

In a quantale a further binary operation $\mapsto : L \times L \to L$, the residuum, can be introduced as associated with operation $*$ of the quantale $(L, \leq, \wedge, \vee, *)$ via the Galois connection, that is $a * b \leq c \iff a \leq b \mapsto c$ for all $a, b, c \in L$.
A quantale $(L, \leq, \wedge, \vee, *)$ provided with the derived operation $\mapsto$, that is the tuple $(L, \leq, \wedge, \vee, *, \mapsto)$, is known also as a (complete) residuated lattice [17].

In the next proposition we list basic properties of the residuum that can be found in the works of different authors, see, e.g. [12].

Proposition 1. Basic properties of the residium:

(1) $\left(\bigvee_i a_i\right) \mapsto b = \bigwedge_i (a_i \mapsto b)$ *for all* $\{a_i \mid i \in I\} \subseteq L$, *for all* $b \in L$;
(2) $a \mapsto (\bigwedge_i b_i) = \bigwedge_i (a \mapsto b_i)$ *for all* $a \in L$, *for all* $\{b_i \mid i \in I\} \subseteq L$;
(3) $1_L \mapsto a = a$ *for all* $a \in L$;
(4) $a \mapsto b = 1_L$ *whenever* $a \leq b$;
(5) $a * (a \mapsto b) \leq b$ *for all* $a, b \in L$;
(6) $(a \mapsto b) * (b \mapsto c) \leq a \mapsto c$ *for all* $a, b, c \in L$;
(7) $a \mapsto b \leq (a * c \mapsto b * c)$ *for all* $a, b, c \in L$;
(8) $a * b \leq a \wedge b$ *for any* $a, b \in L$;
(9) $(a * b) \mapsto c = a \mapsto (b \mapsto c)$ *for any* $a, b, c \in L$.

Fuzzy sets and fuzzy relations. The concept of a fuzzy set was introduced by L.A. Zadeh [31] and then extended to a more general concept of an L-fuzzy set by J.A. Goguen [11] where L is a complete lattice, in particular a quantale. Given a set X its L-fuzzy subset is a mapping $A : X \to L$. The lattice and the quantale structure of L is extended point-wise to the L-exponent of X, that is to the set L^X of all L-fuzzy subsets of X. An L-fuzzy relation between two sets X and Y is an L-fuzzy subset of the product $X \times Y$, that is a mapping $R : X \times Y \to L$, see, e.g. [27], [32]. An L-fuzzy relation R is called left connected if $\bigwedge_{y \in Y} \bigvee_{x \in X} R(x, y) = 1$. If for every $y \in Y$ there exists $x \in X$ such that $R(x, y) = 1$, then R is called *strongly left connected*. An L-fuzzy relation R is called right connected if $\bigwedge_{x \in X} \bigvee_{y \in Y} R(x, y) = 1$. If for every $x \in X$ there exists $y \in Y$ such that $R(x, y) = 1$, then R is called *strongly right connected*. An L-fuzzy relation $R : X \times Y \to L$ is called connected (strongly connected) if it is both left and right connected (respectively strongly left and strongly right connected). Since in the paper L is an arbitrary but a fixed lattice, we shall omit the prefix L and speak just of fuzzy sets and fuzzy relations.

Measure of inclusion of L-fuzzy sets. The gradation of a preconcept lattice presented below is based on the fuzzy inclusion between fuzzy sets. We give here a brief introduction into this field.

In order to fuzzify the inclusion relation $A \subseteq B$ "a fuzzy set A is a subset of a fuzzy set B", we have to interpret it as a certain fuzzy inclusion $\hookrightarrow$ based on "if - then" rule, that is on some implication $\Rightarrow$ on the lattice L. In the result we come to the formula $A \hookrightarrow B = \inf_{x \in X}(A(x) \Rightarrow B(x))$. As far as we know, this approach was applied for the first time in [23], see also [24], where it was based on the Kleene-Dienes implication $\Rightarrow$. Later the fuzzified relation of inclusion between fuzzy sets was studied and used by many authors, see, e.g. [14], [22], [4], [5], et al. In most of the papers the implication $\Rightarrow$ was defined by means of residuum $\mapsto$ of the underlying quantale $(L, \leq, \wedge, \vee, *)$.

Definition 1. *By setting $A \hookrightarrow B = \bigwedge_{x \in X}(A(x) \mapsto B(x))$ for all $A, B \in L^X$, we obtain a mapping $\hookrightarrow\colon L^X \times L^X \to L$. We call $A \hookrightarrow B$ by the measure of inclusion of a fuzzy set A into the fuzzy set B. Let $A \hookleftarrow B =_{def} B \hookrightarrow A$. We denote $A \cong B =_{def} (A \hookrightarrow B) \wedge (B \hookrightarrow A)$ and view it as the degree of equality of fuzzy sets A and B.*

Proposition 2. [10]. Properties of the mapping $\hookrightarrow\colon L^X \times L^X \to L$:

(1) $\left(\bigvee_i A_i\right) \hookrightarrow B = \bigwedge_i (A_i \hookrightarrow B)$ *for all* $\{A_i \mid i \in I\} \subseteq L^X$ *and for all* $B \in L^X$;
(2) $A \hookrightarrow (\bigwedge_i B_i) = \bigwedge_i (A \hookrightarrow B_i)$ *for all* $A \in L^X$, *and for all* $\{B_i \mid i \in I\} \subseteq L^X$;
(3) $A \hookrightarrow B = 1$ *whenever* $A \leq B$;
(4) $1_X \hookrightarrow A = \bigwedge_x A(x)$ *for all* $A \in L^X$ *where* $1_X : X \to L$ *is a constant function with the value* $1 \in L$;
(5) $(A \hookrightarrow B) \leq (A * C \hookrightarrow B * C)$ *for all* $A, B, C \in L^X$;
(6) $(A \hookrightarrow B) * (B \hookrightarrow C) \leq (A \hookrightarrow C)$ *for all* $A, B, C \in L^X$;
(7) $\left(\bigwedge_i A_i\right) \hookrightarrow \left(\bigwedge_i B_i\right) \geq \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i : i \in I\}, \{B_i : i \in I\} \subseteq L^X$;
(8) $\left(\bigvee_i A_i\right) \hookrightarrow \left(\bigvee_i B_i\right) \geq \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i : i \in I\}, \{B_i : i \in I\} \subseteq L^X$.

3. Forward and backward operators induced by fuzzy relations

Let $R : X \times Y \to L$ be a fuzzy relation. We refer to the set X as the domain and to the set Y as the codomain of the fuzzy relation R. The forward and backward lower and upper operators induced by fuzzy relations can be found in the works of different authors and under various names. Since our goal is to apply them in the framework of concept analysis we interpret them here also as derivation operators in the definition of a fuzzy concept.

Definition 2.
($^\square$) - $R^\square$ *denotes the lower forward operator* $R^\Rightarrow : L^X \to L^Y$. *That is, given* $A \in L^X$ *and* $y \in Y$, *let* $R^\square(A)(y) =_{def} A^\square(y) = \bigwedge_{x \in X}(R(x, y) \mapsto A(x))$.
($^\lozenge$) - $R^\lozenge$ *denotes the upper forward operator* $R^\rightarrow : L^X \to L^Y$. *That is, given* $A \in L^X$ *and* $y \in Y$, *let* $R^\lozenge(A)(y) =_{def} A^\lozenge(y) = \bigvee_{x \in X}(R(x, y) * A(x))$.
($^\blacksquare$) - $R^\blacksquare$ *denotes the lower backward operator* $R^\Leftarrow : L^Y \to L^X$. *That is, given* $B \in L^Y$ *and* $x \in X$, *let* $R^\blacksquare(B)(x) =_{def} B^\blacksquare(x) = \bigwedge_{y \in Y}(R(x, y) \mapsto B(y))$.
($^\blacklozenge$) - $R^\blacklozenge$ *denotes the upper backward operator* $R^\leftarrow : L^Y \to L^X$. *That is, given* $B \in L^Y$ *and* $x \in X$, *let* $R^\blacklozenge(B)(x) =_{def} B^\blacklozenge(x) = \bigvee_{y \in Y}(R(x, y) * B(y))$.

The proof of the following three propositions is easy and can be found in [25]:

Proposition 3. *Let X, Y be sets and $R : X \times Y \to L$ a fuzzy relation. Then*
(1) $A_1, A_2 \in L^X, A_1 \leq A_2 \implies A_1^\square \leq A_2^\square; A_1^\lozenge \leq A_2^\lozenge$;
(2) $B_1, B_2 \in L^Y, B_1 \leq B_2 \implies B_1^\blacksquare \leq B_2^\blacksquare; B_1^\blacklozenge \leq B_2^\blacklozenge$.

Proposition 4. *If a fuzzy relation is strongly left connected, then* $R^{\square} \leq R^{\Diamond}$. *If* $R : X \times Y \to L$ *is strongly right connected, then* $R^{\blacksquare} \leq R^{\blacklozenge}$.

Proposition 5.
(1) $R^{\square}(1_X) = 1_Y$ *and* $R^{\Diamond}(0_X) = 0_Y$. *If R is left connected, then* $R^{\square}(a_X) = R^{\Diamond}(a_X) = a_Y$ *for every* $a \in L$.
(2) $R^{\blacksquare}(1_Y) = 1_X$ *and* $R^{\blacklozenge}(0_Y) = 0_X$. *If R is right connected, then* $R^{\blacksquare}(b_Y) = R^{\blacklozenge}(b_Y) = b_X$ *for every* $b \in L$.

Proposition 6. *Let* $\{A_i \mid i \in I\} \subseteq L^X$ *and* $\{B_i \mid i \in I\} \subseteq L^Y$. *Then:*

(1) $\left(\bigwedge_{i\in I} A_i\right)^{\square} = \bigwedge_{i\in I} A_i^{\square}$;
(2) $\left(\bigvee_{i\in I} A_i\right)^{\Diamond} = \bigvee_{i\in I} A_i^{\Diamond}$;
(3) $\left(\bigwedge_{i\in I} B_i\right)^{\blacksquare} = \bigwedge_{i\in I} B_i^{\blacksquare}$;
(4) $\left(\bigvee_{i\in I} B_i\right)^{\blacklozenge} = \bigvee_{i\in I} B_i^{\blacklozenge}$.

Proof. Let $\{A_i \mid i \in I\} \subseteq L^X$ and $y \in Y$. Then $\left(\bigwedge_{i\in I} A_i^{\square}\right)(y) = \bigwedge_{i\in I} A_i^{\square}(y) = \bigwedge_{i\in I}\left(\bigwedge_{x\in X}(R(x,y) \mapsto A_i(x))\right) = \bigwedge_{x\in X}\left(\bigwedge_{i\in I}(R(x,y) \mapsto A_i(x))\right) =$
$\bigwedge_{x\in X}\left(R(x,y) \mapsto \bigwedge_{i\in I} A_i(x)\right) = \left(\bigwedge_{i\in I} A_i\right)^{\square}(y)$.
$\left(\bigvee_{i\in I} A_i^{\Diamond}\right)(y) = \bigvee_{i\in I} A_i^{\Diamond}(y) = \bigvee_{i\in I}\left(\bigvee_{x\in X}(R(x,y) * A_i(x))\right) =$
$\bigvee_{x\in X}\left(\bigvee_{i\in I}(R(x,y) * A_i(x))\right) = \bigvee_{x\in X}\left((R(x,y) * \bigvee_{i\in I} A_i(x)\right) = \left(\bigvee_{i\in I} A_i\right)^{\Diamond}(y)$. □

Our next goal is to consider the successive composition of these operators. We denote compositions $R^{\blacklozenge} \circ R^{\square} = R^{\square\blacklozenge}$; $R^{\Diamond} \circ R^{\blacksquare} = R^{\blacksquare\Diamond}$, $R^{\square} \circ R^{\blacklozenge} = R^{\blacklozenge\square}$ $R^{\blacksquare} \circ R^{\Diamond} = R^{\Diamond\blacksquare}$. Respectively, given fuzzy sets $A \in L^X$ and $B \in L^Y$, denote $R^{\square\blacklozenge}(A) = A^{\square\blacklozenge}$, $R^{\Diamond\blacksquare}(A) = A^{\Diamond\blacksquare}$; $R^{\blacksquare\Diamond}(B) = B^{\blacksquare\Diamond}$ and $R^{\blacklozenge\square}(B) = B^{\blacklozenge\square}$.

Theorem 1. *For every* $A \in L^X$ *and every* $B \in L^Y$ *it holds:*
(1) $A^{\square\blacklozenge} \leq A \leq A^{\Diamond\blacksquare}$;
(2) $B^{\blacksquare\Diamond} \leq B \leq B^{\blacklozenge\square}$.

Proof. Let $y \in Y$. Then $A^{\square\blacklozenge}(x) = \bigvee_{y\in Y}\left(R(x,y) * A^{\square}(y)\right) =$
$\bigvee_{y\in Y}\left(R(x,y) * (\bigwedge_{x'\in X}(R(x',y) \mapsto A(x'))\right) \leq$
$\bigvee_{y\in Y}(R(x,y) * (R(x,y) \mapsto A(x))) \leq A(x)$;
on the other hand $A^{\Diamond\blacksquare}(x) = \bigwedge_{y\in Y}\left(R(x,y) \mapsto A^{\blacklozenge}(x)\right) =$
$\bigwedge_{y\in Y}\left(R(x,y) \mapsto \left(\bigvee_{x'\in X} R(x',y) * A(x')\right)\right) \geq$
$\bigwedge_{y\in Y}(R(x,y) \mapsto (R(x,y) * A(x))) \geq A(x)$.
In a similar way the second inequality can be proved. □

Theorem 2. *For every* $A \in L^X$ *and every* $B \in L^Y$

(1)$A^{\Diamond\blacksquare\Diamond} = A^{\Diamond}$;
(2) $A^{\square\blacklozenge\square} = A^{\square}$;
(3) $B^{\blacklozenge\square\blacklozenge} = B^{\blacklozenge}$;
(4) $B^{\blacksquare\Diamond\blacksquare} = B^{\blacksquare}$.

Proof. Applying Theorem 1 we have $A^{\Diamond\blacksquare\Diamond} = (A^{\Diamond\blacksquare})^{\Diamond} \geq A^{\Diamond}$; $A^{\Diamond\blacksquare\Diamond} = (A^{\Diamond})^{\blacksquare\Diamond} \leq A^{\Diamond}$ and hence $A^{\Diamond\blacksquare\Diamond} = A^{\Diamond}$. On the other hand, again by Theorem 1, $A^{\square\blacklozenge\square} = (A^{\square\blacklozenge})^{\square} \leq A^{\square}$; $A^{\square\blacklozenge\square} = (A^{\square})^{\blacklozenge\square} \geq A^{\square}$ and hence $A^{\square\blacklozenge\square} = A^{\square}$. The last two equalities can be proved in a similar way. □

From Theorem 2 we get

Corollary 1. *For every $A \in L^X$ and every $B \in L^Y$*

(1) $A^{\Diamond\blacksquare\Diamond\blacksquare} = A^{\Diamond\blacksquare}$; (3) $B^{\blacklozenge\Box\blacklozenge\Box} = B^{\blacklozenge\Box}$;
(2) $A^{\Box\blacklozenge\Box\blacklozenge} = A^{\Box\blacklozenge}$; (4) $B^{\blacksquare\Diamond\blacksquare\Diamond} = B^{\blacksquare\Diamond}$,

and hence the operators $R^{\Box\blacklozenge}, R^{\Diamond\blacksquare}, R^{\blacksquare\Diamond}, R^{\blacklozenge\Box}$ are idempotent.

Theorem 3. *Operators $R^{\Diamond} : L^X \to L^Y$ and $R^{\blacksquare} : L^Y \to L^X$ form an adjoint pair $(R^{\Diamond}, R^{\blacksquare})$, that is $A^{\Diamond} \le B \Longleftrightarrow A \le B^{\blacksquare}$ for any $A \in L^X, B \in L^Y$.*
Operators $R^{\blacklozenge} : L^Y \to L^Y$ and $R^{\Box} : L^X \to L^Y$ form an adjoint pair $(R^{\blacklozenge}, R^{\Box})$, that is $B^{\blacklozenge} \le A \Longleftrightarrow B \le A^{\Box}$ for any $A \in L^X, B \in L^Y$.

Proof. To show that $A^{\Diamond} \le B \Longleftrightarrow A \le B^{\blacksquare}$ assume first that $A^{\Diamond} \le B$. Then by Proposition 3(2) $A^{\Diamond\blacksquare} \le B^{\blacksquare}$ and by Theorem 1(1) $A \le A^{\Diamond\blacksquare} \le B^{\blacksquare}$. Conversely, assume that $A \le B^{\blacksquare}$. Then $A^{\Diamond} \le B^{\blacksquare\Diamond}$ by Proposition 3(1) and $A^{\Diamond} \le B^{\blacksquare\Diamond} \le B$ by Theorem 1(2).

To show that $B^{\blacklozenge} \le A \Longleftrightarrow B \le A^{\Box}$ assume first that $B^{\blacklozenge} \le A$. Then by Proposition 3(2) $B^{\blacklozenge\Box} \le A^{\Box}$ and by Theorem 1(1) $B \le B^{\blacklozenge\Box} \le A^{\Box}$. Conversely, assume that $B \le A^{\Box}$. Then $B^{\blacklozenge} \le A^{\Box\blacklozenge}$ by Proposition 3(1) and $B^{\blacklozenge} \le A^{\Box\blacklozenge} \le A$ by Theorem 1(2). □

4. Degree of conceptuality of fuzzy preconcepts in fuzzy rough sets theory

Following the main lines of research in formal concept analysis, we start with a pair of sets X and Y and a fuzzy relation $R : X \times Y \to L$. Now the set X is interpreted as the set of some objects and the set Y as the set of some properties, the value $R(x, y) \ge \alpha \in L$ means that the object x has the property y at least to the degree α. The quadruple (X, Y, R, L) is called a formal fuzzy context.

4.1. Preconcepts and preconcept lattices

Viewing $A \in L^X$ as a fuzzy set of potential objects and $B \in L^X$ as a fuzzy set of properties we interpret the pair (A, B) as a potential fuzzy concept. These leads to the following definition:

Definition 3. [26][1] *Given a formal fuzzy context (X, Y, R, L), a pair $P = (A, B) \in L^X \times L^Y$ is called a fuzzy preconcept.*

To view the set $L^X \times L^Y$ of all fuzzy preconcepts as a lattice we must first introduce an order on it. To do this reasonably, we must take into account that an increase in the set of objects A naturally leads to a decrease in the set B of properties that these objects satisfy. Therefore we introduce a partial order $\preceq$ as follows. Given $P_1 = (A_1, B_1)$ and $P_2 = (A_2, B_2)$, we set $P_1 \preceq P_2$ if and only if $A_1 \le A_2$ and $B_1 \ge B_2$. Let $(\mathbb{P}, \preceq)$ be the set $L^X \times L^Y$ endowed with this partial order. Further, given a family of fuzzy preconcepts $\{P_i = (A_i, B_i) : i \in I\} \subseteq L^X \times L^Y$, we define its join (supremum) by $\underline{\vee}_{i\in I} P_i = (\bigvee_{i\in I} A_i, \bigwedge_{i\in I} B_i)$ and its meet (infimum) as $\overline{\wedge}_{i\in I} P_i = (\bigwedge_{i\in I} A_i, \bigvee_{i\in I} B_i)$.

[1]This notion of a fuzzy preconcept is not related to the notion of a preconcept as it is defined in [8]

Theorem 4. [26]. $(\mathbb{P}, \preceq, \veebar, \barwedge)$ *is a complete lattice. Besides, if* L *is an infinitely bi-distributive lattice, then* $(\mathbb{P}, \preceq, \veebar, \barwedge)$ *is also an infinitely bi-distributive lattice.*

4.2. Concept lattices in rough set theory.

Patterned after [29], and using our notations, we define an object-oriented fuzzy concept as the pair $(A, B) \in \mathbb{P}$ such that $A = B^{\blacklozenge}$ and $A^{\Box} = B$. Similarly, a property-oriented fuzzy concept is defined as a pair $(A, B) \in L^X \times L^Y$ such that $A = B^{\blacksquare}$ and $A^{\Diamond} = B$. In what follows, we restrict to considering object-oriented fuzzy concepts; the case of property-oriented fuzzy concepts can be easily obtained just by replacing fuzzy relation $R : X \times Y \to L$ by its inverse $R^{-1} : Y \times X \to L$. Let $\mathbb{O}(X, Y) \subseteq L^X \times L^Y$ be the family of object-oriented fuzzy concepts.

In order to explain the meaning of an object-oriented fuzzy concept, we consider the simplest non-trivial case. Let L be a lattice whose all elements are isolated from below, let $R : X \times Y \to \{0, 1\} \subseteq L$ and let $A \in L^X, B \in L^Y$. We denote $A_\alpha = \{x \in X \mid A(x) \geq \alpha\}; B_\alpha = \{y \in Y \mid B(y) \geq \alpha\}$. Thus A_α is the set of objects $x \in X$ belonging to A at least to degree α, respectively B_α is the set of properties $y \in Y$ belonging to B at least to degree α. Now $A_\alpha^{\Box} = \{y \mid A^{\Box}(y) \geq \alpha\}$ can be described as the set of properties $y \in Y$ such that if y is related to some object x, then $x \in A_\alpha$. In turn, $B_\alpha^{\blacklozenge} = \{x \in X \mid B^{\blacklozenge}(x) \geq \alpha\}$ is the set of objects $x \in X$ which are related to some $y \in B_\alpha$. Now $(A, B) \in \mathbb{O}(X, Y)$ means that $A_\alpha^{\Box} = B_\alpha$ and $A_\alpha = B_\alpha^{\blacklozenge}$ for all $\alpha \in L$.

Following the principles of concept analysis, the set $(\mathbb{O}(X, Y), \preceq)$ should be realized as a complete lattice, where $\preceq$ is the relation induced from the partially ordered set $(\mathbb{P}, \preceq, \barwedge, \veebar)$, that is $(A, B) \preceq (A', B')$ iff $A \leq A'$ and $B \geq B'$. Since the coordinate-wise infimum and supremum of (even two) fuzzy concepts need not be a fuzzy concept, we must define infimum $\curlywedge$ and supremum $\curlyvee$ operations on $(\mathbb{O}(X, Y), \leq)$ in a different way. We do it patterned after definitions of such operations in case of "classical" (that is Wille-Ganter-Bělohlávek) concepts and concepts in rough set theory, see [29]. Namely, given a family $\mathfrak{F} = \{(A_i, B_i) \mid i \in I\} \subseteq \mathbb{O}(X, Y)$, we define the infimum and the supremum of this family respectively by
$\curlywedge \mathfrak{F} = \left((\bigwedge_{i \in I} A_i)^{\Box \blacklozenge}, \bigwedge_{i \in I} B_i \right)$ and $\curlyvee \mathfrak{F} = \left(\bigvee_{i \in I} A_i, (\bigvee_{i \in I} B_i)^{\Diamond \blacksquare} \right)$.

Theorem 5. $(\mathbb{O}(X, Y), \leq, \curlywedge, \curlyvee)$ *is a complete lattice.*

Proof. We have to show that if $\mathfrak{F} = \{(A_i, B_i) \mid i \in I\} \subseteq \mathbb{O}(X, Y)$, then $\left((\bigwedge_{i \in I} A_i)^{\Box \blacklozenge}, \bigwedge_{i \in I} B_i \right)$ and $\left(\bigvee_{i \in I} A_i, (\bigvee_{i \in I} B_i)^{\Diamond \blacksquare} \right)$ are object-oriented fuzzy concepts. However, we get it immediately by applying statements (2) and (4) of Theorem 2 and recalling that $A_i = B_i^{\blacklozenge}$ and $A_i^{\Box} = B_i$ for every $i \in I$. □

4.3. Problems of practical applications of concept lattices.

Unfortunately the scope of practical applications of object-oriented (as well as property-oriented) fuzzy concepts is rather restricted. For example, if $A \subseteq X, B \subseteq Y$ are crisp sets and $R : X \times Y \to L$, then (A, B) can be an object-oriented fuzzy concept only in case $B^{\blacklozenge} = A$ and $A^{\Box} = B$. This actually means that also R must be a crisp relation and hence there is no even any "flavour" of fuzziness in this case. As a reasonable improvement of this situation we suggest to replace the notion of an (object-oriented, property-oriented) fuzzy concept by a graded (object-oriented, property oriented) fuzzy preconcept. We do it in the next subsection.

4.4. Graded fuzzy concept lattices in rough set theory

As above, let X and Y be sets of objects and properties respectively and $R : X \times Y \to L$ be a fuzzy relation. We aim to determine *for every pair* $(A, B) \in L^X \times L^Y$ the extent, or the degree to which this pair (A, B) is a fuzzy object-oriented concept (resp. a fuzzy property-oriented concept). This can be done along the lines of our article in [26], which proposes a graded approach in "classical" fuzzy concept analysis.

Definition 4. *The degree of (object-oriented) conceptuality of a pair* $(A, B) \in L^X \times L^Y$ *is defined by* $\mathrm{D_{ob}}(A, B) = (A^{\Box} \cong B) \wedge (A \cong B^{\blacklozenge})$.
The degree of (property-oriented) conceptuality of a pair $(A, B) \in L^X \times L^Y$ *is defined by* $\mathrm{D_{pr}}(A, B) = (A^{\Diamond} \cong B) \wedge (A \cong B^{\blacksquare})$.

By changing pairs (A, B) over $L^X \times L^Y$ we obtain operators $\mathrm{D_{ob}} : L^X \times L^Y \to L$ and $\mathrm{D_{pr}} : L^X \times L^Y \to L$.

Definition 5. *A graded fuzzy object-oriented concept lattice is the lattice* $\mathbb{P}$ *endowed with operator* $\mathrm{D_{ob}} : L^X \times L^Y \to L$*, that is the tuple* $(\mathbb{P}, \preceq, \veebar, \barwedge, \mathrm{D_{ob}})$.
A graded fuzzy property-oriented concept lattice is the lattice $\mathbb{P}$ *endowed with operator* $\mathrm{D_{pr}} : L^X \times L^Y \to L$*, that is the tuple* $(\mathbb{P}, \preceq, \veebar, \barwedge, \mathrm{D_{pr}})$.

Going on with the study of graded fuzzy concept lattices in rough set theory, we shall restrict to the case of an object-oriented lattice $(\mathbb{P}, \preceq, \veebar, \barwedge, \mathrm{D_{ob}})$, that is when $\mathrm{D_{ob}}(A, B) = (A^{\Box} \cong B) \wedge (A \cong B^{\blacklozenge})$. The corresponding results for the property-oriented lattice $\mathrm{D_{pr}}(A, B) = (A^{\Box} \cong B) \wedge (A \cong B^{\blacklozenge})$ can be obtained in a similar way just by replacing fuzzy relation $R : X \times Y \to L$ by the fuzzy relation $R^{-1} : Y \times X \to L$ defined by $R^{-1}(x, y) := R(y, x)$ for all $x \in X$, for all $y \in Y$.

The computational complexity of the operator $\mathrm{D_{ob}} : L^X \times L^Y \to L$ laid in the definition of the graded fuzzy object-oriented concept lattice $(\mathbb{P}, \preceq, \veebar, \barwedge, \mathrm{D_{ob}})$ and multidirectional components involved in this definition are the reason for further study of these operators by dividing them into several components.

4.5. Operators $\mathrm{D^1_{ob}}, \mathrm{D^2_{ob}}, \mathrm{D^3_{ob}}, \mathrm{D^4_{ob}} : L^X \times L^Y \to L$

Given a pair $(A, B) \in \mathbb{P}$ let $\mathrm{D^1_{ob}}(A, B) = A^{\Box} \hookrightarrow B$, $\mathrm{D^2_{ob}}(A, B) = A^{\Box} \hookleftarrow B$, $\mathrm{D^3_{ob}}(A, B) = A \hookrightarrow B^{\blacklozenge}$, $\mathrm{D^4_{ob}}(A, B) = A \hookleftarrow B^{\blacklozenge}$. Obviously,

$$\mathrm{D_{ob}}(A, B) = \mathrm{D^1_{ob}}(A, B) \wedge \mathrm{D^2_{ob}}(A, B) \wedge \mathrm{D^3_{ob}}(A, B) \wedge \mathrm{D^4_{ob}}(A, B).$$

Next, we study the properties of the operators $\mathrm{D^1_{ob}}, \mathrm{D^2_{ob}}, \mathrm{D^3_{ob}}, \mathrm{D^4_{ob}}$ separately.

Theorem 6. *Properties of operator* $\mathrm{D^1_{ob}} : \mathbb{P}(X, Y, R, L) \to L$.

1. *Operator* $\mathrm{D}^1_{\mathrm{ob}} : \mathbb{P}(X,Y,R,L) \to L$ *is upper semicontinuous, that is* $\mathrm{D}^1_{\mathrm{ob}}\left(\bigwedge_{i\in I}(A_i,B_i)\right) \geq \bigwedge_{i\in I}\mathrm{D}^1_{\mathrm{ob}}(A_i,B_i)\ \forall\{(A_i,B_i) : i \in I\} \subseteq \mathbb{P}(X,Y,R,L)$.
2. $\mathrm{D}^1_{\mathrm{ob}}(A,1_Y) = 1$ *for every* $A \in L^X$. *In particular* $\mathrm{D}^1_{\mathrm{ob}}(0_X,1_Y) = 1$. *If* R *is left connected, then* $\mathrm{D}^1_{\mathrm{ob}}(0_X,B) = 1$ *for every* $B \in L^Y$.

Proof. Referring to Proposition 6 and Proposition 2 we have
$\mathrm{D}^1_{\mathrm{ob}}\left(\bigwedge_{i\in I}(A_i,B_i)\right) = \mathrm{D}^1_{\mathrm{ob}}\left(\bigwedge_{i\in I}A_i, \bigvee_{i\in I}B_i\right) = (\bigwedge_{i\in I}A_i)^{\square} \hookrightarrow \bigvee_{i\in I}B_i =$
$\bigwedge_{i\in I}A_i^{\square} \hookrightarrow \bigvee_{i\in I}B_i \geq \bigwedge_{i\in I}(A^{\square} \hookrightarrow B_i) \geq \bigwedge_{i\in I}\mathrm{D}^1_{\mathrm{ob}}(A_i,B_i)$.
From the definition it is clear that $\mathrm{D}^1_{\mathrm{ob}}(A,1_Y) = A^{\square} \hookrightarrow 1 \geq 1 \mapsto 1 = 1$. If R is leftconnected, then applying Proposition 5 we have $\mathrm{D}^1_{\mathrm{ob}}(0_X,B) = 0_X^{\square} \hookrightarrow B = 0_Y \hookrightarrow B = \bigwedge_{y\in Y}(0 \mapsto B(y)) = 1$. □

Theorem 7. *Properties of operator* $\mathrm{D}^3_{\mathrm{ob}} : \mathbb{P}(X,Y,R,L) \to L$.

1. *Operator* $\mathrm{D}^3_{\mathrm{ob}} : \mathbb{P}(X,Y,R,L) \to L$ *is upper semicontinuous, that is* $\mathrm{D}^3_{\mathrm{ob}}\left(\bigwedge_{i\in I}(A_i,B_i)\right) \geq \bigwedge_{i\in I}\mathrm{D}^3_{\mathrm{ob}}(A_i,B_i)\ \forall\{(A_i,B_i) : \in I\} \subseteq \mathbb{P}(X,Y,R,L)$.
2. $\mathrm{D}^3_{\mathrm{ob}}(0_X,B) = 1$ *for every* $B \in L$ *in particular,* $\mathrm{D}^3_{\mathrm{ob}}(0_X,1_Y) = 1$. *If* R *is right connected, then* $\mathrm{D}^3_{\mathrm{ob}}(A,1_Y) = 1$ *for every* $A \in L^X$.

Proof. Referring to Proposition 6, Proposition 2 we have
$\mathrm{D}^3_{\mathrm{ob}}\left(\bigwedge_{i\in I}(A_i,B_i)\right) = \mathrm{D}^3_{\mathrm{ob}}\left(\bigwedge_{i\in I}A_i, \bigvee_{i\in I}B_i\right) = \bigwedge_{i\in I}A_i \hookrightarrow \left(\bigvee_{i\in I}B_i\right)^{\blacklozenge} =$
$\bigwedge_{i\in I}A_i \hookrightarrow \bigvee_{i\in I}B_i^{\blacklozenge} \geq \bigwedge_{i\in I}(A_i \hookrightarrow B_i^{\blacklozenge}) \geq \bigwedge_{i\in I}\mathrm{D}^3_{\mathrm{ob}}(A_i,B_i)$.
Just from definitions we have $\mathrm{D}^3_{\mathrm{ob}}(0_X,B) = 0_X \hookrightarrow B^{\blacklozenge} = \bigwedge_{x\in X}(1 \mapsto B^{\blacklozenge}(x)) \geq 1 \mapsto 1 = 1$. In case R is right connected we apply Proposition 5 and have $\mathrm{D}^3_{\mathrm{ob}}(A,1_Y) = A \hookrightarrow 1_Y^{\blacklozenge} = A \hookrightarrow 1_X =$
$\bigwedge_{x\in X}(A(x) \to 1) = 1$ □

Theorem 8. *Properties of operator* $\mathrm{D}^2_{\mathrm{ob}} : \mathbb{P}(X,Y,R,L) \to L$.

1. *Operator* $\mathrm{D}^2_{\mathrm{ob}} : \mathbb{P}(X,Y,R,L) \to L$ *is lower semicontinuous, that is* $\mathrm{D}^2_{\mathrm{ob}}\left(\bigvee_{i\in I}(A_i,B_i)\right) \geq \bigwedge_{i\in I}\mathrm{D}^2_{\mathrm{ob}}(A_i,B_i)\ \forall\{(A_i,B_i) : i \in I\} \subseteq \mathbb{P}(X,Y,R,L)$.
2. $\mathrm{D}^2_{\mathrm{ob}}(A,0_Y) = 1$ *for every* $A \in L^X$ *and* $\mathrm{D}^2_{\mathrm{ob}}(1_X,B) = 1$ *for every* $B \in L^Y$, *in particular* $\mathrm{D}^2_{\mathrm{ob}}(1_X,0_Y) = 1$.

Proof. Referring to Proposition 6, Proposition 2 we have
$\mathrm{D}^2_{\mathrm{ob}}\left(\bigvee_{i\in I}A_i, \bigwedge_{i\in I}\right) = \left(\bigvee_{i\in I}A_i\right)^{\square} \hookleftarrow \bigwedge_{i\in I}B_i \geq \bigvee_{i\in I}A_i^{\square} \hookleftarrow \bigwedge_{i\in I}B_i \geq \bigwedge_{i\in I}(A_i^{\square} \hookleftarrow B_i) =$
$\bigwedge_{i\in I}\mathrm{D}^2_{\mathrm{ob}}(A_i,B_i)$.
$\mathrm{D}^2_{\mathrm{ob}}(A,0_Y) = A^{\square} \hookleftarrow 0_Y = \bigwedge_{y\in Y}(0 \mapsto A^{\square}(y)) \geq 0 \mapsto 1 = 1$. On the other hand $\mathrm{D}^2_{\mathrm{ob}}(1_X,B) =$
$1^{\square} \hookleftarrow B = \bigwedge_{y\in Y}(B(y) \mapsto 1) = 1$. □

Theorem 9. *Properties of operator* $\mathrm{D}^4_{\mathrm{ob}} : \mathbb{P}(X,Y,R,L) \to L$.

1. *Operator* $\mathrm{D}^4_{\mathrm{ob}} : \mathbb{P}(X,Y,R,L) \to L$ *is lower semicontinuous, that is* $\mathrm{D}^4_{\mathrm{ob}}\left(\bigvee_{i\in I}(A_i,B_i)\right) \geq \bigwedge_{i\in I}\mathrm{D}^4_{\mathrm{ob}}(A_i,B_i)\ \forall\{(A_i,B_i) : i \in I\} \subseteq \mathbb{P}(X,Y,R,L)$.
2. $\mathrm{D}^4_{\mathrm{ob}}(A,0_Y) = 1$ *for every* $A \in L^X$ *and* $\mathrm{D}^4_{\mathrm{ob}}(1_X,B) = 1$ *for every* $B \in L^Y$. *In particular* $\mathrm{D}^4_{\mathrm{ob}}(1_X,0_Y) = 1$;

Proof. Referring to Proposition 6, Proposition 2 we have
$D^4_{ob}(\bigvee_{i\in I} A_i, \bigwedge_{i\in I} B_i) = \bigvee_{i\in I} A_i \hookleftarrow (\bigwedge_{i\in I} B_i)^{\blacklozenge} \geq \bigvee_{i\in I} A_i \hookleftarrow \bigwedge_{i\in I} B_i^{\blacklozenge} \geq \bigwedge_{i\in I}(A_i \hookleftarrow B_i^{\blacklozenge}) = \bigwedge_{i\in I} D^4_{ob}(A_i, B_i)$.
Applying Proposition 5 $D^4_{ob}(A, 0_Y) = A \hookleftarrow 0_Y^{\blacklozenge} = 0_Y^{\blacklozenge} \hookrightarrow A = 0_X \hookrightarrow A = \bigwedge_{x\in X}(0 \mapsto A(x)) = 1$.
In turn $D^4_{ob}(1_X, B) = 1_X \hookleftarrow B^{\blacklozenge} = B^{\blacklozenge} \hookrightarrow 1_X = \bigwedge_{x\in X}(B^{\blacklozenge}(x) \mapsto 1) = 1$.

□

Corollary 2. *If fuzzy relation R is connected, then:*

- *Operators* $D^1_{ob}, D^3_{ob} : L^X \times L^Y \to L$ *are upper semicontinuous and* $D^1_{ob}(0_X, B) = D^3_{ob}(0_X, B) = 1$ *for every* $B \in L^Y$ *and* $D^1_{ob}(A, 1_Y) = D^3_{ob}(A, 1_Y) = 1$ *for every* $A \in L^X$.
- *Operators* $D^2_{ob}, D^4_{ob} : L^X \times L^Y \to L$ *are lower semicontinuous and* $D^2_{ob}(A, 0_Y) = D^4_{ob}(A, 0_Y) = 1$ *for any* $A \in L^X$ *and* $D^2_{ob}(1_X, B) = D^4_{ob}(1_X, B) = 1$ *for any* $B \in L^Y$

5. Examples of graded fuzzy concept lattices

In this section, we present two theoretical examples of graded fuzzy concept lattices. The first of these examples as well as its generalizations (obtained by employing fuzzy sets A and B with different number of values) of this example will be used for illustration of possible applications outside "pure" mathematics. One of such applications is presented in the next subsection. The second is a quite "artificial", example. It is added in order to illustrate the fundamental role of the fuzzy relation involved in the definition of graded fuzzy concepts and to stress the importance of the reasonable choice of fuzzy relation when considering special situations.

Example 1. Let X and Y be sets partitioned into three disjoint subsets $X = \bigcup_{i=0}^{2} X_i$ and $Y = \bigcup_{i=0}^{2} Y_i$. Further, let relation $R : X \times Y \to \{0, 1\}$ be defined by

$$R(x, y) = \begin{cases} 1 & \text{if } (x, y) \in (X_0 \times Y_0) \cup (X_1 \times Y_1) \cup (X_2 \times Y_2) \\ 0 & \text{otherwise} \end{cases}$$

We consider fuzzy sets $A : X \to [0, 1]$, $B : Y \to [0, 1]$ and calculate $A^{\square}$ and $B^{\blacklozenge}$:

$$A(x) = \begin{cases} 0 & \text{if } x \in X_0 \\ a & \text{if } x \in X_1 \\ 1 & \text{if } x \in X_2 \end{cases} \qquad A^{\square}(y) = \begin{cases} 0 & \text{if } y \in Y_0 \\ a & \text{if } y \in Y_1 \\ 1 & \text{if } y \in Y_2 \end{cases}$$

$$B(y) = \begin{cases} 0 & \text{if } y \in Y_0 \\ b & \text{if } y \in Y_1 \\ 1 & \text{if } y \in Y_2 \end{cases} \qquad B^{\blacklozenge}(x) = \begin{cases} 0 & \text{if } x \in X_0 \\ b & \text{if } x \in X_1 \\ 1 & \text{if } x \in X_2 \end{cases}$$

Thus we have:
$A^{\square} \hookrightarrow B = a \mapsto b$, $A^{\square} \hookleftarrow B = b \mapsto a$ and $A^{\square} \cong B = a \mapsto b \wedge b \mapsto a$
$A \hookrightarrow B^{\blacklozenge} = a \mapsto b$, $A \hookleftarrow B^{\blacklozenge} = b \mapsto a$ and $A \cong B^{\blacklozenge} = a \mapsto b \wedge b \mapsto a$.
Hence, in particular, for the most important t-norms:

- for the Łukasiewicz t-norm $D_{ob}(A, B) = 1- \mid a - b \mid$;

- in case of the product t-norm $\mathrm{D_{ob}}(A, B) = \min\{\frac{a}{b}, \frac{b}{a}\}$;

- in case of the minimum t-norm $\mathrm{D_{ob}}(A, B) = \min\{a, b\}$.

Example 2. Let X and Y be sets partitioned into three disjoint subsets $X = \bigcup_{i=0}^{2} X_i$ and $Y = \bigcup_{i=0}^{2} Y_i$ and let $A : X \to [0, 1]$, $B : Y \to [0, 1]$ be defined as in the previous example. Further, let relation $R : X \times Y \to \{0, 1\}$ be defined by

$$R(x, y) := \begin{cases} c & \text{if } (x, y) \in X_0 \times Y_1 \\ 1 & \text{if } (x, y) \in X_1 \times Y_2 \cup X_2 \times Y_0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$A^{\square}(y) = \begin{cases} 1 & \text{if } y \in Y_0 \\ c \mapsto 0 & \text{if } y \in Y_1 \\ a & \text{if } y \in Y_2 \end{cases} \qquad B^{\blacklozenge}(x) = \begin{cases} c * b & \text{if } x \in X_0 \\ 1 & \text{if } x \in X_1 \\ 0 & \text{if } x \in X_2 \end{cases}$$

By an easy calculation we get

- $\mathrm{D}^1_{\mathrm{ob}}(A, B) = A^{\square} \hookrightarrow B = 0$, $\mathrm{D}^2_{\mathrm{ob}}(A, B) = A^{\square} \hookleftarrow B = b \mapsto (c \mapsto 0) \wedge a$

- $\mathrm{D}^3_{\mathrm{ob}}(A, B) = B^{\blacklozenge} \hookrightarrow A = (c * b \mapsto 0) \wedge a$, $\mathrm{D}^1_{\mathrm{ob}}(A, B) = B^{\blacklozenge} \hookleftarrow A = 0$,

and hence $\mathrm{D_{ob}}(A, B) = 0$.

6. Application of graded fuzzy concept lattices in assessment of country risks

We consider the imposition of international sanctions (embargoes) to different countries for illustration of possible application of graded fuzzy concept lattices. These sanctions are introduced either by international organizations (like the United Nations) or different countries (including the European Union) to restrict economic activities of particular countries posing different global risks via prohibiting the trading of certain goods or provision of particular services, limiting the access to financial resources, etc. Sanctions are also applied to different private individuals and legal entities. Consequently, several countries and their residents are subject to restrictive measures of lower or higher degree which can be mathematically illustrated using fuzzy sets. Meanwhile only North Korea is considered as fully sanctioned country, but the recent emergence of the war in Ukraine has induced a global wave of new sanctions against Russia making it as almost fully sanctioned country with only few leftovers for very limited non-restricted trades with commodities, e.g. natural gas.
In order to set up a conceptual lattice reflecting this matter we consider the countries as objects and the sanctions as properties. The contents of this lattice can be visualized, e.g. using data from the site www.sanctionsmap.eu comprising information on sanctions imposed by the European Union and the United Nations, and these data can be further enriched by different other sanctions imposed by countries with major global influence, like the United States of America.

Naturally, based on our *Example 1* provided in the previous section all non-sanctioned countries are included in the subset X_0, all fully sanctioned countries are included in the subset X_2, and all partly sanctioned countries are included in the subset X_1 thus creating the set of objects. For the sake of clarity we should admit that terms "partly" and "fully" are conditional and depend on the total number of particular sanctions applied to corresponding countries. Furthermore we introduce the set of properties by applying different importance to particular sanctions. In such case the subset Y_0 is introduced as subset not containing any sanctions, subset Y_2 contains the sanctions imposing the strongest possible restrictions, e.g. prohibition of any financial transactions, and subset Y_1 contains all other types of sanctions targeting particular sectors of economic activities. For the purposes of our example we have chosen the value of a equal to 0.4 and the value of b equal to 0.6. We obtain the following fuzzy sets A and B and corresponding values for $A^{\square}$ and $B^{\blacklozenge}$:

$$A(x) = \begin{cases} 0 & \text{if } x \in X_0 \\ 0.4 & \text{if } x \in X_1 \\ 1 & \text{if } x \in X_2 \end{cases} \qquad A^{\square}(y) = \begin{cases} 0 & \text{if } y \in Y_0 \\ 0.4 & \text{if } y \in Y_1 \\ 1 & \text{if } y \in Y_2 \end{cases}$$

$$B(y) = \begin{cases} 0 & \text{if } y \in Y_0 \\ 0.6 & \text{if } y \in Y_1 \\ 1 & \text{if } y \in Y_2 \end{cases} \qquad B^{\blacklozenge}(x) = \begin{cases} 0 & \text{if } x \in X_0 \\ 0.6 & \text{if } x \in X_1 \\ 1 & \text{if } x \in X_2 \end{cases}$$

The meaning of $A^{\square}$ is explained as relation of particular subsets of sanctions to corresponding countries while the meaning of $B^{\blacklozenge}$ is explained as relation of particular subsets of countries to corresponding sanctions. Furthermore we obtain the following values of $\mathrm{D}_{\mathrm{ob}}(A, B)$: 0.8 in case of Łukasiewicz t-norm, 0.67 in case of product t-norm and 0.4 in case of minimum t-norm. These results basically demonstrate that the level of country sanctioning, i.e. "partial" or "full" cannot exceed the importance of sanctions applied to respective countries. This is also the mathematical reasoning for, e.g. difficulties in taking decision on disconnection of all Russian financial institutions from the global network of the Society for Worldwide Interbank Financial Telecommunication (SWIFT) while still enabling of at least limited trades with commodities (e.g. natural gas). This particular example was used to demonstrate the applicability of concept lattice for analysis of international sanction in a very simplified way. It is obvious that in a more realistic scenario we must refine both the partitioning of the set of objects (countries) and the set of properties (sanctions). As a result we would be dealing with more granulated setup, disjoint subsets $X = \bigcup_{i=0}^{m} X_i$ and $Y = \bigcup_{j=0}^{n} Y_j$ and corresponding values of $a_1, ..., a_m$ and $b_1, ..., b_n$. These values of a_i and b_j are still finite and in practice do not exceed the number of officially recognized countries (currently there are 197 countries) and the number of meaningful sanctions. The analysis of such more granulated landscape is subject to our further research.

7. Conclusion

Operators $\mathrm{D}^1_{\mathrm{ob}}, \mathrm{D}^2_{\mathrm{ob}}, \mathrm{D}^3_{\mathrm{ob}}$ and $\mathrm{D}^4_{\mathrm{ob}} : L^X \times L^Y \to L$ were introduced in order to evaluate various aspects of the correspondence between a fuzzy set of (potential) objects and a fuzzy set of (potential) properties. Based on these operators, we defined graded fuzzy concept lattices and studied their main properties. Our next research will focus on using these operators for the construction of so called, *conceptional hulls of fuzzy preconcepts.* Given a fuzzy preconcept $(A, B) \in L^X \times L^Y$ and $\alpha \in L$, the triple $(A, B^\uparrow, B^\downarrow)$, where $B^\downarrow \leq B \leq B^\uparrow$, is called the concept α-hull of a fuzzy preconcept (A, B), if

$$D^1_{ob}(A, B^\uparrow) \wedge D^3_{ob}(A, B^\uparrow) \geq \alpha \text{ and } D^2_{ob}(A, B^\downarrow) \wedge D^4_{ob}(A, B^\downarrow) \geq \alpha$$

and in addition $B^\uparrow, B^\downarrow$ are the "optimal" fuzzy sets satisfying these conditions. We shall carry out the study of the construction $L^X \times L^Y \Rightarrow L^X \times L^Y \times L^Y$ assigning to each $(A, B) \in L^X \times L^Y$ its conceptional α-hull $(A, B^\uparrow, B^\downarrow)$. Specifically we plan to investigate algebraic (in particular categorical) and topological properties of this construction. In addition, we intend to apply this construction for practical purposes, in particular, for the choice of strategy developed in response to the risks related to global political and economic developments.

The second interesting direction, kindly indicated us by a referee, is the study of the relations between graded fuzzy concept lattices and pattern structures described by certain semi-lattices, see, e.g. [9], [18]. In such structures patterns are used instead of attributes (properties), allowing thus a certain gradation of concepts.

Acknowledgments

The first and the third named authors are grateful for the partial financial support from the project No. Lzp-2020/2-0311 by the Latvian Council of Science. All authors thank the anonymous referees for reading the paper carefully and making useful comments.

References

[1] Bělohlávek, R.: Concept lattices and order in fuzzy logic. Annals of Pure and Applied Logic 128 (2004), 277–298.

[2] Bělohlávek, R., Vychodil, V.: What is a fuzzy concept lattice? Proc. CLA 2005. CEUR WS 162 (2005), 34–45.

[3] Birkhoff, G.: Lattice Theory, AMS Providence, RI, 1995.

[4] Bustinice, H.: Indicator of inclusion grade for interval-valued fuzzy sets, Application for approximate reasoning based on interval-valued fuzzy sets, Int. J. Approx. Reason. 23 (2000), 137–209.

[5] Cornelius, C., Van der Donck, C., Kerre, E.E.: Sinha-Dougherty approach to the fuzzification of set inclusion revisited, Fuzzy Sets Syst. 134 (2003), 283–295.

[6] Lai, H-L., Zhang, D.: Concept lattices of fuzzy context: Formal concept analysis vs. rough set theory. Intern. J. of Approximate Reasoning 50 (2009), 695–707.

[7] Düntch, I., Gediga, G.: Approximation operators in qualitative data analysis, in: Theory and Application of Relational Structures as Knowledge Instruments, de Swart, H., Orlowska, E., Schmidt, G. and Roubens, M. (eds.), Springer, Heidelberg, 216–233, 2003.

[8] Ganter B., Wille R.: Formal Concept Analysis: Mathematical Foundations. Springer Verlag, Berlin, 1999.

[9] Ganter B., Kuznetsov S.: Pattern Structures and their properties. ICCS 2002, 129–142.

[10] Han, S.-E., Šostak, A.: On the measure of *M*-rough approximation of *L*-fuzzy sets, Soft Comput. 22, (2018), 3843–3853.

[11] Goguen, J.A.: *L*-fuzzy sets. J. Math. Anal. Appl. 18 (1967), 145–174.

[12] Höhle, U. *M*-valued sets and sheaves over integral commutative CL-monoids, In: Applications of Category Theory to Fuzzy Subsets, Rodabaugh, S.E., Höhle, U. Klement E.P. (eds.); Kluwer Acad. Publ., Docrecht, Boston, 1992 pp. 33–72.

[13] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M.W., Scott, D.S.: Continuous Lattices and Domains. Cambridge University Press, Cambridge, 2003.

[14] Kitainik, L.M.: Fuzzy inclusions and fuzzy dichotomous decision procedure. Theory and Decision Library. 4 (1987), Springer, Dordrecht, 154–170.

[15] Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms. Kluwer Publ. 2000.

[16] Medina, J.: Relating attribute reduction in formal, object-oriented and property-oriented concept lattices. Computers and Mathematics with Applications 64 (2012), 1992–2002.

[17] Morgan, W., Dilworth, R.P.: Residuated lattices, Trans. Amer. Math. Soc. 45 (1939), 335-–354.

[18] Pankratieva, V.V., Kuznetsov, S.O.: Relations between proto-fuzzy concepts, crisply generalized fuzzy concepts, and interval pattern structures. Fundam. Informaticae 115(4) (2012), 265–277.

[19] Pawlak, Z.: Rough sets, Intern. J. Comp. Inform. Sciences 11 (1982), 341–356.

[20] Rodabaugh, S.E.: Powerset operator foundations for theories and topologies, In: U. Höhle, S.E. Rodabaugh (eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory - Handbook Series, 3, Kluwer Acad. Publ., (1999), 91–117.

[21] Rosenthal, K.I.: Quantales and Their Applications. Pitman Research Notes in Mathematics, vol. 234. Longman Scientific & Technical, 1990.

[22] Sinha, D., Dougherty, E.R.: Fuzzification of set inclusion: Theory and applications. Fuzzy Sets and Systems. 55(1) (1993), 15–42.

[23] Šostak, A.: On a fuzzy mathematical structure, Suppl. Rend. Circ. Matem., Palermo Ser II, 11, (1985), 89–103.

[24] Šostak, A.: Two decades of fuzzy topology: Basic ideas, notions and results, Russian Math. Surveys, 4(4), (1989), 125–186.

[25] Šostak, A., Uļjane, I.: Fuzzy relations: the fundament for fuzzy rough approximation, fuzzy concept analysis and fuzzy mathematical morphology. In. Computational Intelligence and Mathematics for Tackling Complex Problems, 4. M.E. Cornejo, I.A. Harmati, L.T. Koczy, J. Medina (eds.) 2022, Springer.

[26] Šostak, A., Uļjane, I., Krastiņš, M.: Gradations of fuzzy preconcept lattices, (2021) 10(1), 41.

[27] Valverde, L.: On the structure of *F*-indistinguishability operators. Fuzzy Sets and Syst. (1985), 17, 313–328.

[28] Wille R.: Concept lattices and conceptual knowledge systems. Computers and Math. with Applications (1992), 23, 493–515.

[29] Yao, Y.Y.: Concept lattices in rough set theory. In. Proc. 23rd International Meeting of the North American Fuzzy Information Processing Society Annual Meeting. NAFIPS ’04, (2004) pp. 796–801.

[30] Yao, Y.Y.: Rough set approximation in formal concept analysis. In: Transactions on Rough Sets V. J.F. Peters and A. Skowron (eds.) LNCS 4100 (2006), 285–305.

[31] Zadeh, L.A.: Fuzzy Sets. Inf. Control. 8, (1965) 338–353.
[32] Zadeh, L.A.: Similarity relations and fuzzy orderings. Inf. Sci. 3 (1971), 177–200.