The Stirling triangles

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***Abstract*—The paper is devoted to the elementary discussion on the triangles of Stirling numbers of the first kind and the r-Stirling numbers as well. Aim of our investigations was to extract the numerical sequences connected with these triangles, to verify their presence in OEIS and to try to generate some properties of these numbers. Moreover, we have supplemented with new results the paper written by S. Falcon [4], the research object of which was the triangle of numbers given by the binomial transformations of k-Fibonacci numbers.**

***Index Terms*—Pascal triangle, Stirling numbers of the first kind, r-Stirling numbers of the first kind, Catalan numbers, Fibonacci numbers**

I. Introduction

Creation of this paper was directly influenced by article [17] which intrigued us and, in fact, imposed the subject of our investigations. We became interested in the state of knowledge concerning the Pascal triangles for the Stirling numbers of the first kind and the r-Stirling numbers of the first kind. Thus, as the goal of our research we took the extraction of numerical sequences connected with these triangles (that is mainly the sequences of sums of elements in the rows and along the antidiagonals of the given triangle). Moreover, we verified the presence of such sequences in OEIS and we undertook the attempt to generate some properties of the investigated numbers. It is worth to emphasize that we have also supplemented with completely new results the paper written by Sergio Falcon [4] concerning the triangle of numbers given by the binomial transformations of k-Fibonacci numbers.

II. The Stirling triangle

We begin by presenting the definition of Stirling numbers of the first kind.

Definition 2.1: Stirling numbers of the first kind describe the number of permutations on the $n-$element set possessing k cycles (that is the permutations which may be decomposed into exactly k separated cycles). There is no one standard notation for these numbers and for denoting them one can use one of the following symbols:

- $s(n,k)$,
- $S_n^{(k)}$,
- $S_1(n,k)$,
- $\left[\begin{matrix} n \\ k \end{matrix}\right]$, where $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $k \le n$.

In this elaboration we will use the last one from the above listed notations.

We will use here one more, alternative, definition of the Stirling numbers of the first kind $\left[{n \atop k}\right]$, it means:

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} = {} & \text{sum of all possible products of } (n-k) \\ & \text{different integers taken from among} \\ & \text{the } n \text{ - initial nonnegative integers,} \\ & \text{that is from among numbers } 0, 1, \ldots, n-1. \end{aligned}$$

So we have

$$\begin{bmatrix} n \\ 0 \end{bmatrix} \equiv 0, \quad \begin{bmatrix} n \\ 1 \end{bmatrix} \equiv (n-1)!,$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix} \equiv \sum_{k=1}^{n-1} \prod_{\substack{1 \le i \le n-1 \\ i \ne k}} i = \sum_{k=1}^{n-1} \frac{(n-1)!}{k} = (n-1)! H_{n-1},$$

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \frac{1}{2} n(n-1) = \text{A000217}(n-1), \quad n = 1, 2, \ldots$$

$$\begin{bmatrix} n \\ n-1 \end{bmatrix}^2 = \sum_{k=1}^{n-1} k^3,$$

where H_{n-1} denotes the $(n-1)$th harmonic number, that is $H_n := \sum_{k=1}^{n} \frac{1}{k}$. Moreover we take $\left[{n \atop n}\right] := 1$ for every $n \in \mathbb{N}$.

Let us notice that from this definition, almost immediately after multiplication of the monomials located on the left hand side, the following equality results

$$x\,(1+x)(2+x)\ldots(n-1+x) = x^{\overline{n}} := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad (1)$$

which was taken originally by James Stirling in his monograph *Methodus Differentialis* (1730) as the definition of Stirling numbers of the first kind. Product of the monomials located on the left hand side of equality (1) defines today the so called Pochhammer symbol.

For comparison we have the following widely known Newton binomial formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad (2)$$

describing the generating function for the binomial coefficients.

Let us notice that by using definition 2.1 one can easily derive the formula (recurrence relation for the Stirling numbers of the first kind):

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix}. \quad (3)$$

In other words, the selections of $(n-k)$ different numbers from among n initial nonnegative integers correspond with the sum of selections of $(n-1)-(k-1)=n-k$ different numbers from among $(n-1)$ initial nonnegative integers and selections of $(n-1)-k=n-k-1$ different numbers from among $(n-1)$ initial nonnegative integers with the added number $n-1$.

Let us construct now the Pascal triangle for the Stirling numbers of the first kind which will be called henceforward as the Stirling triangle of the first kind

$$
\begin{array}{ccccccccc}
 & & & & \genfrac{[}{]}{0pt}{}{0}{0} := 1 & & & & \\
 & & & \genfrac{[}{]}{0pt}{}{1}{0} & & \genfrac{[}{]}{0pt}{}{1}{1} & & & \\
 & & \genfrac{[}{]}{0pt}{}{2}{0} & & \genfrac{[}{]}{0pt}{}{2}{1} & & \genfrac{[}{]}{0pt}{}{2}{2} & & \\
 & \genfrac{[}{]}{0pt}{}{3}{0} & & \genfrac{[}{]}{0pt}{}{3}{1} & & \genfrac{[}{]}{0pt}{}{3}{2} & & \genfrac{[}{]}{0pt}{}{3}{3} & \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}
$$

that is the following numerical triangle

$$
\begin{array}{rlllllll}
\text{zero level} \longrightarrow & 1 & & & & & & \\
\text{first level} \longrightarrow & 0 & 0!{=}1 & & & & & \\
\text{second level} \longrightarrow & 0 & 1!{=}1 & 1 & & & & \\
\text{third level} \longrightarrow & 0 & 2!{=}2 & 3 & 1 & & & \\
\text{fourth level} \longrightarrow & 0 & 3!{=}6 & 11 & 6 & 1 & & \\
\text{fifth level} \longrightarrow & 0 & 4!{=}24 & 50 & 35 & 10 & 1 &
\end{array}
$$

Hence, as well as on the basis of formula (3) and in view of the alternative definition 2.1, the following summation formula easily results

$$
\begin{aligned}
\sum_{k=0}^{n} \left[{n \atop k}\right] = n! = (n-1)\sum_{k=1}^{n-1}\left[{n-1 \atop k}\right] + \sum_{k=0}^{n-1}\left[{n-1 \atop k}\right] \\
= n\sum_{k=0}^{n-1}\left[{n-1 \atop k}\right].
\end{aligned} \tag{4}
$$

For contrast, from the Pascal triangle for the binomial coefficients (also called by us the classic Pascal triangle) we have the following summation formula

$$\sum_{k=0}^{n}\binom{n}{k} = 2\sum_{k=0}^{n-1}\binom{n-1}{k}.$$

Obviously, the above formula also arises easily from the classic recurrence relation (equivalent formula for (3) but for the binomial coefficient):

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Executing in the classic Pascal triangle the summation over the antidiagonals we obtain the next unexpected formula (see [12], pages 155-157):

$$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-k}{k},$$

where F_n denotes the nth Fibonacci number.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}
$$

Next, by executing in the Stirling triangle of the first kind the summation of elements along the antidiagonals, as it is presented in the following scheme (the first column containing zeros and one digit one at the zero level is omitted here):

$$
\begin{array}{lllllll}
1 & & & & & & \\
1 & 1 & & & & & \\
2 & 3 & 1 & & & & \\
6 & 11 & 6 & 1 & & & \\
24 & 50 & 35 & 10 & 1 & & \\
120 & 274 & 225 & 85 & 15 & 1 & \\
720 & 1764 & 1624 & 735 & 175 & 21 & 1 \\
5040 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
$$

$1+0=1,$
$2+1=3,$
$6+3=9,$
$24+11+1=36,$
$120+50+6=176,$
$720+274+35+1=1030,$
$5040+1764+225+10=7039,$
etc.

we obtain the sequence of natural numbers labeled by symbol A237653 in the Sloane's OEIS encyclopaedia. In other words we have

$$
\left[{n \atop 0}\right] + \left[{n-1 \atop 1}\right] + \left[{n-2 \atop 2}\right] + \ldots + \begin{cases} 1, \quad \text{when } n = 2r, \\ (r+1)!, \\ \quad \text{when } n = 2r+1, \end{cases}
$$
$$= \text{A237653}(n)$$

for every $n = 1, 2, \ldots$.

Remark 2.1: In the classic Pascal triangle the numbers at the given level n (starting with the zero level) represent the coefficients in the expansion of number $(b+1)^n$ in the given numerical base b (we assume that all the coefficients at level n are $\leq b$) which results from the binomial formula (2).

Whereas in the Stirling triangle of the first kind the numbers at the given level n (starting with the zero level) represent the coefficients in the expansion of number $\prod_{k=0}^{n-1}(bk+1)$ in the given numerical base b (under the assumptions that that all the coefficients at level n are $\leq b$) which results from formula (1) (one should substitute $x = \frac{1}{b}$ and multiply by 10^n on both sides).

What is interesting, Slone's OEIS reports the sequences:

$$A144773(n) = \prod_{k=0}^{n-1}(10k+1), \; A008548(n) = \prod_{k=0}^{n-1}(5k+1),$$
$$A007559(n) = \prod_{k=0}^{n-1}(3k+1).$$

Remark 2.2: Sergio Falcon in paper [4] considers the triangle T of the polynomial coefficients

$$p_n(k) := \sum_{j=0}^{n} \binom{n}{j} F_{k,n-j}, \quad n \in \mathbb{N},$$

which form the binomial transforms of rows of the following triangle of k-Fibonacci numbers where $F_{k,n},\; k \in \mathbb{R},\; k > 0$, and $n \in \mathbb{N}_0$:

$$\begin{array}{ccccccccc}
 & & & & F_{k,0} & & & & \\
 & & & F_{k,1} & & F_{k,0} & & & \\
 & & F_{k,2} & & F_{k,1} & & F_{k,0} & & \\
 & F_{k,3} & & F_{k,2} & & F_{k,1} & & F_{k,0} & \\
\dots & & \dots & & \dots & & \dots & & \dots
\end{array}$$

We have

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \qquad F_{k,0} = 0, \qquad F_{k,1} = 1,$$

for every $n \in \mathbb{N}$. The following results give a supplement for the Falcon's paper [4]. We find that the polynomials $p_n(k)$ satisfy the double recurrence relation

$$\begin{cases} p_{n+1}(k) &= (k+1)p_n(k) + q_n(k), \\ q_{n+1}(k) &= p_n(k) + q_n(k), \end{cases}$$

for every $n \in \mathbb{N}$, where $p_1(k) = q_1(k) \equiv 1,\; p_2(k) = k+2,\; q_2(k) \equiv 2$.

Hence, after simple algebra we get the recursive relation for polynomials $p_n(k)$:

$$p_{n+1}(k) = 1 + kp_n(k) + \sum_{j=1}^{n} p_j(k), \quad n \in \mathbb{N},$$

and the recurence relation for $p_n(k)$

$$p_{n+2}(k) = (k+2)p_{n+1}(k) - kp_n(k), \quad n \in \mathbb{N}.$$

The triangle T has the form

1								
1	2							
1	3	4						
1	4	8	8					
1	5	13	20	16				
1	6	19	38	48	32			
1	7	26	63	104	112	64		
1	8	34	96	192	272	256	128	
1	...	...	...	...	...	...	...	...

We observe also that the sum of elements on antidiagonals of T form three known sequences defined in the OEIS, that is the sequence of all sums

$$\{1, 1, 3, 4, 9, 14, 28, 47, 89, 155, 286, \dots\} = \text{A006053}$$

which satisfies the linear recurrence relation of the third order

$$a_n = a_{n-1} + 2a_{n-2} - a_{n-3},$$

where $\text{A006053}(n) := a_n,\; n = 1, 2, \dots$ And next the following sequences obtained by bisection of $\{a_n\}$:

$$\begin{array}{rcl} \{a_{2n-1}\} &=& \{1, 3, 9, 28, 89, 286, \dots\} = \text{A094790} \\ \{a_{2n}\} &=& \{1, 4, 14, 47, 155, \dots\} = \text{A094789} \end{array}$$

which satisfy both the same recurrence relation of the third order

$$a_n = 5a_{n-1} - 6a_{n-2} + a_{n-3}.$$

Let us set $T^2 = [t_{kn}]_{\mathbb{N}\times\mathbb{N}}$. If $n > k$ then $t_{kn} = 0$. We also verified that

$$\sum_{k=1}^{n} t_{nk} \leq \text{A030240}(n-1),$$

where equality holds only for $n = 1, 2, 3, 4$ and

$$\sum_{k=1}^{5} t_{5k} = \text{A030240}(4) + 1.$$

Remark 2.3: In the context of problems discussed in this section it is also worth to mention the Narayana numbers defined in the way given below (see [6]):

$$N(0,0) = 1, \qquad N(n,0) = 0,$$
$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, \qquad k, n \in \mathbb{N}, \quad k \leq n.$$

Creating the Narayana triangle we find one more beautiful result. The sums of elements in the rows of the Narayana triangle are equal to the Catalan numbers

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

In result of summation over the antidiagonals of Narayana triangle we get the generalized Catalan numbers ($C^*_{n+1} = C^*_n + \sum_{k=1}^{n-1} C^*_k C^*_{n-1-k},\; n \in \mathbb{N}_0$, the sequence $\text{A004148}(n)$ in OEIS), i.e.

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} N(n-k, k+1) = C^*_n.$$

III. R-Stirling triangle of the first kind

By using the analogical properties of r-Stirling numbers we get a little bit more general results. The r-Stirling numbers of the first kind are defined as follows

Definition 3.1: We take

$$\begin{aligned} &\begin{bmatrix} n \\ k \end{bmatrix}_r = 0, \quad n < r, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_r = \delta_{k,r}, \quad n = r, \\ &\begin{bmatrix} n \\ k \end{bmatrix}_r = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}_r + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_r, \quad n > r. \end{aligned} \tag{5}$$

We also take $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_0 = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_1 := \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$.

The combinatoric description of these numbers is presented in paper [1]. So, the number $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_r$ denotes the number of permutations of set $\{1, 2, \ldots, n\}$ possessing exactly k mutually disjoint cycles and such that the numbers $1, 2, \ldots, r$ belong to different cycles.

The above description implies that

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r = \text{number of all permutations of set } \{1, 2, \ldots, n\}$$

so that the numbers $1, \ldots, r$ are in different, mutually disjoint cycles.

Moreover, by using the definition of r-Stirling numbers of the first kind we easily derive the recurrence relation

$$\begin{aligned} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r &= (n-1)\sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_r + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_r \\ &= n \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_r, \end{aligned} \tag{6}$$

for every $r \geq 1$, which is the generalization of identity (4). One can check that we have then

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r = \frac{n!}{r!},$$

for $n \geq r$. Broder in [1] gave the generating function for the r-Stirling numbers of the first kind

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r x^k = \begin{cases} x^r (x+r)^{\overline{n-r}}, & n \geq r \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

where, let us recall, the Pochhammer symbol $x^{\overline{n}}$ is applied. This formula results from the alternative definition of the $r-$ Stirling numbers of the first kind, namely

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = \text{sum of all possible products of exactly}$$

$(n-k)$ different natural numbers from among the numbers $r, r+1, \ldots, n-1$.

Let us notice that

$$\begin{aligned} \begin{bmatrix} n \\ n-1 \end{bmatrix}_2 &= \sum_{k=2}^{n-1} k = \frac{1}{2}(n-2)(n+1), \\ &= \text{A000096}(n-2), \quad n = 3, 4, \ldots \\ \begin{bmatrix} n \\ n-1 \end{bmatrix}_3 &= \sum_{k=3}^{n-1} k = \frac{1}{2}(n-3)(n+2) \\ &= \text{A055998}(n-2), \quad n = 4, 5 \ldots \\ \begin{bmatrix} n \\ n-k-1 \end{bmatrix}_2 &= \text{A00170}k(n-k-1), \end{aligned}$$

for every $k = 1, 2$ and $n = k+3, k+4,$

$$\begin{bmatrix} n \\ n-k+1 \end{bmatrix}_3 = \text{A02418}k(n-k-1),$$

for every $k = 3, 4, 5$ and $n = k+2, k+3, \ldots$. On the basis of formula (5) we can construct the r-Stirling triangle of the first kind

$$\begin{array}{ccccccccc} & & & & \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_r & & & & \\ & & & \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_r & & \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]_r & & & \\ & & \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]_r & & \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]_r & & \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]_r & & \\ & \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]_r & & \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]_r & & \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right]_r & & \left[\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}\right]_r & \\ \ldots & & \ldots & & \ldots & & \ldots & & \ldots \end{array}$$

Let us consider the case for r=2, that is

0								
0	0							
0	0	1						
0	0	2	1					
0	0	6	5	1				
0	0	24	26	9	1			
0	0	120	154	71	14	1		
0	0	720	1044	580	155	20	1	
0	0	...	...	...	...	...	...	...

Now, by summing the elements along the antidiagonals in the above triangle

$0 + 0 = 0,$
$0 + 0 = 0,$
$0 + 0 + 1 = 1,$
$0 + 0 + 2 = 2,$
$6 + 1 = 7,$
$24 + 5 = 29,$
$120 + 26 + 1 = 147,$
$720 + 154 + 9 = 883,$
etc.

we obtain the sequence of natural numbers not existing in OEIS. We have

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_2 + \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_2 + \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_2 + \ldots + \begin{cases} 1, & \text{when } n = 2m, \\ \frac{1}{2}(m-1)(m+2), \\ \text{when } n = 2m+1, \end{cases}$$

$$= \mathrm{A}\emptyset\emptyset\emptyset\emptyset\emptyset 1(n),$$

for every $n = 2, 3 \ldots$. Notation $\mathrm{A}\emptyset\emptyset\emptyset\emptyset\emptyset k,\ k \in \mathbb{N},\ k \leq 9$, with the empty sets, is the notation invented by us to denote the sequences $\{\mathrm{A}\emptyset\emptyset\emptyset\emptyset\emptyset k(n),\ n \in \mathbb{N},\}$, $k \in \mathbb{N}$, $k \leq 9$ not included in OEIS.

We have verified numerically (although basing on premises resulting from the algebraic estimation) that for $5 \leq n \leq 500$ the above sequence satisfies the following inequalities

$$\frac{(n-2)!}{n} < \mathrm{A}\emptyset\emptyset\emptyset\emptyset\emptyset 1(n) < \frac{(n-2)!}{2}.$$

Next we construct the analogical triangle for the case $r = 3$. Thus we have

0									
0	0								
0	0	0							
0	0	0	1						
0	0	0	3	1					
0	0	0	12	7	1				
0	0	0	60	47	12	1			
0	0	0	360	342	119	18	1		
0	0	0	2520	2754	1175	245	25	1	
0	0	0	...	...	...	...	...	...	...

Summing the elements along the antidiagonals in the above triangle we get

0
$0 + 0 = 0,$
$0 + 0 + 0 = 0,$
$0 + 0 + 0 + 1 = 1,$
$0 + 0 + 0 + 3 = 3,$
$12 + 1 = 13,$
$60 + 7 = 67,$
$360 + 47 + 1 = 408,$
and so on.

The obtained sequence of natural numbers is not included either in OEIS. We have

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_3 + \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_3 + \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_3 + \ldots + \begin{cases} 1, & \text{when } n = 2m, \\ \frac{1}{2}(m-2)(m+3), \\ \text{when } n = 2m+1 \end{cases}$$

$$= \mathrm{A}\emptyset\emptyset\emptyset\emptyset\emptyset 2(n),$$

for every $n = 3, 4 \ldots$.

We have verified numerically (but again, basing on premises resulting from the algebraic estimation) that this sequence satisfies for $7 \leq n \leq 500$ the following inequalities

$$(n-5)! < \mathrm{A}\emptyset\emptyset\emptyset\emptyset\emptyset 2(n) < (n-4)!.$$

Remark 3.1: In case of the r-Stirling numbers the numbers at one level n represent the coefficients in the expansion of number $\prod_{k=r}^{n-1}(bk+1)$ in the given numerical base b (under the assumption that all the coefficients at level n are $\leq b$) which results from formula (7) (analogically like in the previous case).

IV. Triangles for the powers of Stirling numbers and the binomial coefficients

Properties of the classic Pascal triangle are studied in [8] together with the presentation of Fibonacci sequence with the aid of binomial coefficients. Let us turn our attention into the Pascal triangle for the powers of binomial coefficients. For example, for the squares of binomial coefficients we receive

$$\begin{array}{ccccccccc} & & & & \binom{0}{0}^2 & & & & \\ & & & \binom{1}{0}^2 & & \binom{1}{1}^2 & & & \\ & & \binom{2}{0}^2 & & \binom{2}{1}^2 & & \binom{2}{2}^2 & & \\ & \binom{3}{0}^2 & & \binom{3}{1}^2 & & \binom{3}{2}^2 & & \binom{3}{3}^2 & \\ \ldots & & \ldots & & \ldots & & \ldots & & \ldots \end{array}$$

or directly

1							
1	1						
1	4	1					
1	9	9	1				
1	16	36	16	1			
1	25	100	100	25	1		
1	36	225	400	225	36	1	
1	...	...	...	...	...	...	...

Summing the elements along the antidiagonals we find

$1,$
$1 + 4 = 5,$
$1 + 9 + 1 = 11,$
$1 + 16 + 9 = 26,$
$1 + 25 + 36 + 1 = 63,$
$1 + 36 + 100 + 16 = 153,$
etc.

This is the sequence labeled by symbol A051286 in the Sloane's OEIS encyclopaedia. In other words we have

$$\binom{n}{0}^2 + \binom{n-1}{1}^2 + \binom{n-2}{2}^2 + \ldots + \begin{cases} 1, \text{ when } & n = 2m, \\ (m+1)^2, & \\ \text{when } n = 2m+1, & \end{cases} = \text{A051286}(n).$$

Summing the elements along the antidiagonals in the analogical triangle for the cubes of binomial coefficients we obtain $1, 2, 9, 29, 92, 343, 1281, 4720, \ldots$, that is the sequence denoted by A181545 in Sloane's OEIS encyclopaedia. In other words

$$\binom{n}{0}^3 + \binom{n-1}{1}^3 + \binom{n-2}{2}^3 + \ldots + \begin{cases} 1, \quad \text{when } n = 2m, \\ (m+1)^3, \\ \text{when } n = 2m+1, \end{cases} = \text{A181545}(n).$$

Let us consider now the analogical triangles for the Stirling numbers of the first kind. For the start we take the squares of these numbers

1							
0	1						
0	1	1					
0	4	**9**	1				
0	36	**121**	36	1			
0	576	**2500**	1225	100	1		
0	14400	**75076**	50625	7225	225	1	
0	...	...	...	...	...	...	...

Executing the summation of elements over the antidiagonals, similarly as in the cases of previous numerical triangles, we get the sequence

$0 + 1 = 1$,
$0 + 1 = 1$,
$4 + 1 = 5$,
$36 + 9 = 45$,
$576 + 121 + 1 = 698$,
$14400 + 2500 + 36 = 16936$,
and so on,

that is the sequence not included in OEIS

$$\begin{bmatrix} n \\ 0 \end{bmatrix}^2 + \begin{bmatrix} n-1 \\ 1 \end{bmatrix}^2 + \begin{bmatrix} n-2 \\ 2 \end{bmatrix}^2 + \ldots + \begin{cases} 1, \quad \text{when } n = 2r, \\ ((r+1)!)^2, \\ \text{when } n = 2r+1, \end{cases} = \text{A}\emptyset\emptyset\emptyset\emptyset\emptyset 3(n),$$

for every $n = 1, 2, \ldots$.

Moreover, we have checked numerically that for $4 \leq n \leq 500$ the above sequence satisfies the inequalities

$$\frac{n!}{5} < \text{A}\emptyset\emptyset\emptyset\emptyset\emptyset 3(n) < \left(\frac{n!}{5}\right)^2.$$

Next, by summing the elements along the antidiagonals in the Pascal triangle for $\begin{bmatrix} n \\ k \end{bmatrix}^3$ we receive the sequence

$0 + 1 = 1$,
$0 + 1 = 1$,
$8 + 1 = 9$,
$216 + 27 = 243$,
$13824 + 1331 + 1 = 15156$,
and so on,

that is

$$\begin{bmatrix} n \\ 0 \end{bmatrix}^3 + \begin{bmatrix} n-1 \\ 1 \end{bmatrix}^3 + \begin{bmatrix} n-2 \\ 2 \end{bmatrix}^3 + \ldots + \begin{cases} 1, \quad \text{when } n = 2r, \\ ((r+1)!)^3, \\ \text{when } n = 2r+1, \end{cases} = \text{A}\emptyset\emptyset\emptyset\emptyset\emptyset 4(n),$$

again not present in OEIS.

As before we have checked numerically that for $4 \leq n \leq 500$ the above sequence satisfies the inequalities

$$\frac{n!}{3} < \text{A}\emptyset\emptyset\emptyset\emptyset\emptyset 4(n) < \left(\frac{n!}{3}\right)^3.$$

V. Conclusion

In the paper the number of elementary properties of the numerical triangles connected with the Stirling numbers and the k-Fibonacci numbers is presented. We linked the discussed sequences with the sequences included in OEIS. The obtained results summarize some state of research. In the future we intend to refer to these numerical triangles from the algebraic side, similarly like, among others, the authors of papers [2], [3], [9], [10] and [14] did it in case of the classic Pascal triangle. In the future we also plan to analyze paper [13] and the drawn conclusions will be included in our next publication.

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