On an Improvement of the Numerical Application for Cardano's Formula in Mathematica Software

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***Abstract*—The aim of this paper is to develop a program for determining, by symbolic description, the roots of each real cubic polynomial on the basis of the well known Mathematica software. We have obtained a program completely satisfying our expectations and even more. For example, for many tested cases of the cubic polynomials, on the way of comparing the description of the roots of these polynomials received by using our program with their trigonometric form obtained as a result of geometric discussion or respective trigonometric transformations, we have got some new attractive relations of algebraic and trigonometric nature.**

By applying our elaborated program we can also decide, symbolically, whether the given cubic polynomial is a Ramanujan cubic polynomial (one of two kinds). Moreover, in case of these polynomials we have got many new additional pieces of information essentially completing the facts discovered by now. Additionally, we have solved some, posed ad hoc, theoretical problem.

***Index Terms*—cubic polynomials, Ramanujan cubic polynomials, Cardano's formula.**

I. Introduction

While preparing papers [12], [14] a part of computations has been done with the aid of Mathematica software. To our surprise, the program did not manage to derive, in the way as we expected, the symbolic transformations needed for determining the zeros of polynomials, especially the real cubic and quartic polynomials. We had to make by hand a part of the final transformations. For example, for the equation

$$x^3 - 12x + 13 = 0 \tag{1}$$

Mathematica program gave us the following set of solutions

$$\frac{4}{\sqrt[3]{\frac{1}{2}\left(-13+i\sqrt{87}\right)}} + \sqrt[3]{\frac{1}{2}\left(-13+i\sqrt{87}\right)},$$
$$-\frac{2\left(1+i\sqrt{3}\right)}{\sqrt[3]{\frac{1}{2}\left(-13+i\sqrt{87}\right)}} - \frac{1}{2}\left(1-i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(-13+i\sqrt{87}\right)},$$
$$-\frac{2\left(1-i\sqrt{3}\right)}{\sqrt[3]{\frac{1}{2}\left(-13+i\sqrt{87}\right)}} - \frac{1}{2}\left(1+i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(-13+i\sqrt{87}\right)}.$$

We decided to change this situation which resulted in elaborating the appropriate computer procedure calculating the roots of all real cubic polynomials starting from the Cardano's formula.

The program, developed by us, gave the following form of the solutions of equation (1):

$$-4\cos\left(\frac{1}{3}\arctan\frac{\sqrt{87}}{13}\right), \quad 4\sin\left(\frac{\pi}{6} \pm \frac{1}{3}\arctan\frac{\sqrt{87}}{13}\right).$$

The possibly excessive optimism about the effectiveness of symbolic computations realized by our program has been suppressed by other examples. For instance, for the equation

$$x^3 - 12x + 11 = 0,$$

Mathematica answer are the following solutions

$$1, \quad \frac{1}{2}\left(-1 \pm 3\sqrt{5}\right).$$

To the contrast our program produces the following trigonometric form of the same solutions

$$-4\cos\left(\frac{1}{3}\arctan\frac{3\sqrt{15}}{11}\right), \quad 4\sin\left(\frac{\pi}{6} \pm \frac{1}{3}\arctan\frac{3\sqrt{15}}{11}\right),$$

but only numerically Mathematica verified the equality

$$1 = 4\sin\left(\frac{\pi}{6} - \frac{1}{3}\arctan\frac{3\sqrt{15}}{11}\right) = 4\cos\left(\frac{\pi}{3} + \frac{1}{3}\arctan\frac{3\sqrt{15}}{11}\right).$$

In the third section of this paper we present a number of such completely unexpected trigonometric relations obtained by comparing the "model" decompositions of some cubic polynomials with the decompositions obtained by us by applying the introduced Cardano's type formulae (implemented for our computer procedure).

It should be also emphasized that the mentioned model polynomials and their decompositions were usually obtained in result of some trigonometric transformations of the known trigonometric relations (see [5], [17], [18], [19], [28], [33], [34]). In consequence, these two different ways of decomposing the cubic polynomials resulted in many interesting equalities of trigonometric nature shedding a new light on many quantities, mysterious till now. For example, we discovered that the values of $\cos\frac{2^k\pi}{7}$, $k \in \mathbb{N}$, are strictly connected with the values of $\arctan(3\sqrt{3})$, whereas values of $\left(\cos\frac{2^k\pi}{7}\right) / \left(\cos\frac{2^{k+2}\pi}{7}\right)$, $k \in \mathbb{N}$, are strictly connected with the values of $\arctan\frac{1}{\sqrt[3]{3}}$ – see Section 3.

Finally, in the fourth section we describe the recently popular subject matter concerning the Ramanujan cubic polynomials. Discussion executed for the goal of preparing this paper resulted in creating a new paper devoted to these polynomials [2] – the obtained there original result is announced at the end of Section 4.

As the Authors, we want to emphasize that our aim now is to initiate the investigations on the continuation of this work where we intend to concentrate on developing the algorithm for determining the roots of polynomials belonging to the selected families of polynomials of higher orders, for which such description of the roots is known (the examples are quartic polynomials, the modified Chebyshev polynomials [35] and the selected quintics [8], [26]).

II. Zeros of cubic polynomials – final formulae

From the Cardano procedure for finding the (complex) zeros of real cubic polynomial (see for example [11], [16], [20], [28], [35] – two last papers consider also because of their historical connotations):

$$p(z) := z^3 + az + b$$

it follows that $p(z) = 0$ if and only if $z = u + v$ where $u \in \sqrt[3]{-\frac{b}{2} + \sqrt{\Delta}}$, $v \in \sqrt[3]{-\frac{b}{2} - \sqrt{\Delta}}$ and $\Delta = \left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3$, and where the respective complex roots (of the second and third order) are kept in mind. Let us discuss three cases with respect to the sign of discriminant Δ.

First we assume that $\Delta < 0$ (which implies $a < 0$). Then

$$u \in \sqrt[3]{-\frac{b}{2} + i\sqrt{-\Delta}}, \quad v \in \sqrt[3]{-\frac{b}{2} - i\sqrt{-\Delta}}$$

and $p(z)$ possesses three real roots.

Now let us discuss two cases with respect to the sign of coefficient b.

First case, when $b > 0$: Additionally let us set $\Delta^* := \sqrt{-\left(\frac{2}{b}\right)^2 \Delta}$. We suppose that $\Delta^* \in [0, \infty)$, since $-\Delta \geq 0 \Leftrightarrow (\Delta^*)^2 = -1 - \frac{a}{3}\left(\frac{2a}{3b}\right)^2 \geq 0$.

We have

$$-\frac{b}{2} \textcircled{\pm} i\sqrt{-\Delta} = \sqrt{\left(-\frac{a}{3}\right)^3} \exp\left(\textcircled{\pm} i\left(\pi - \arctan\Delta^*\right)\right)$$

and (each sign from the circle corresponds with two signs $\pm$ not included in the circle):

$$\sqrt[3]{-\frac{b}{2} \textcircled{\pm} i\sqrt{-\Delta}} = \\ = \sqrt{-\frac{a}{3}} \exp\left(i\left(\begin{Bmatrix} 0 \\ \pm\frac{2}{3}\pi \end{Bmatrix} \textcircled{\pm} \frac{1}{3}\left(\pi - \arctan\Delta^*\right)\right)\right),$$

which implies (the formulae below are satisfied by all three real zeros of $p(z)$):

$$\begin{aligned} z = u + v &= 2\sqrt{-\frac{a}{3}} \cos\left(\begin{Bmatrix} 0 \\ \pm\frac{2}{3}\pi \end{Bmatrix} + \frac{1}{3}\left(\pi - \arctan\Delta^*\right)\right) \\ &= 2\sqrt{-\frac{a}{3}} \cos\left(\frac{1}{3}\left(\begin{Bmatrix} \pm\pi \\ 3\pi \end{Bmatrix} - \arctan\Delta^*\right)\right) \qquad (2) \\ &= \begin{cases} -2\sqrt{-\frac{a}{3}} \cos\left(\frac{1}{3}\arctan\Delta^*\right), \\ 2\sqrt{-\frac{a}{3}} \cos\left(\frac{1}{3}\left(\pm\pi + \arctan\Delta^*\right)\right), \end{cases} \qquad (3) \\ &= \begin{cases} -2\sqrt{-\frac{a}{3}} \cos\left(\frac{1}{3}\arctan\Delta^*\right), \\ \sqrt{-\frac{a}{3}}\left(\cos\left(\frac{1}{3}\arctan\Delta^*\right) \pm \sqrt{3}\sin\left(\frac{1}{3}\arctan\Delta^*\right)\right), \end{cases} \qquad (4) \end{aligned}$$

or in the equivalent form

$$\begin{aligned} z = u + v &= \begin{cases} -2\sqrt{-\frac{a}{3}} \cos\left(\frac{1}{3}\arctan\Delta^*\right), \\ 2\sqrt{-\frac{a}{3}} \cos\left(\frac{1}{3}\left(\pi + \arctan\Delta^*\right)\right), \\ 2\sqrt{-\frac{a}{3}} \sin\left(\frac{1}{6}\left(\pi + 2\arctan\Delta^*\right)\right), \end{cases} \qquad (5) \\ &= \begin{cases} -2\sqrt{-\frac{a}{3}} \cos\left(\frac{1}{3}\arctan\Delta^*\right), \\ 2\sqrt{-\frac{a}{3}} \sin\left(\frac{\pi}{6} \pm \frac{1}{3}\arctan\Delta^*\right). \end{cases} \end{aligned}$$

The other form of zeros of $p(z)$ can be deduced if we introduce a new parameter φ setting

$$\Delta^* = \tan\varphi \geq 0.$$

Then $\left(-\frac{a}{3}\right)^3 = \left(\frac{b}{2\cos\varphi}\right)^2$. Hence, if we set $a = -3\alpha^2$, $b = 2\beta^3\cos\varphi$, then $\alpha = \pm\beta$. Finally we obtain the following implication: if $\tan\varphi \geq 0$ and $2\alpha\cos\varphi > 0$, then

$$\begin{aligned} &z^3 - 3\alpha^2 z + (2\cos\varphi)\alpha^3 = 0 \Leftrightarrow \\ &\Leftrightarrow z = \begin{cases} -2\alpha\cos\frac{\varphi}{3}, \\ \alpha\left(\cos\frac{\varphi}{3} \pm \sqrt{3}\sin\frac{\varphi}{3}\right). \end{cases} \qquad (6) \end{aligned}$$

For example, we get

$$\begin{aligned} z^3 - 3\alpha^2 z + \alpha^3 = 0 \Leftrightarrow z &= \begin{cases} -2\alpha\cos\frac{\pi}{9}, \\ \alpha\left(\cos\frac{\pi}{9} \pm \sqrt{3}\sin\frac{\pi}{9}\right), \end{cases} \\ &= 2\alpha\cos\left(\frac{2^k\pi}{9}\right), \quad k = 1, 2, 3, \end{aligned}$$

$$\begin{aligned} &z^3 - 3\alpha^2 z + \sqrt{2}\alpha^3 = 0 \\ &\Leftrightarrow z = \begin{cases} -2\alpha\cos\frac{\pi}{12} = -\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\alpha, \\ \alpha\left(\cos\frac{\pi}{12} \pm \sqrt{3}\sin\frac{\pi}{12}\right) = \begin{cases} \sqrt{2}\alpha, \\ \frac{\sqrt{3}-1}{\sqrt{2}}\alpha, \end{cases} \end{cases} \\ &\quad = \begin{cases} \sqrt{2}\alpha \\ \frac{\pm\sqrt{3}-1}{\sqrt{2}}\alpha, \end{cases} \end{aligned}$$

$$\begin{aligned} z^3 - 3\alpha^2 z + \sqrt{3}\alpha^3 = 0 \Leftrightarrow z &= \begin{cases} -2\alpha\cos\frac{\pi}{18}, \\ \alpha\left(\cos\frac{\pi}{18} \pm \sqrt{3}\sin\frac{\pi}{18}\right), \end{cases} \\ &= 2\alpha(-1)^{k-1}\sin\left(\frac{2^k\pi}{9}\right), \quad k = 1, 2, 3. \end{aligned}$$

Second case, when $b < 0$: We get

$$-\frac{b}{2} \textcircled{\pm} i\sqrt{-\Delta} = \sqrt{\left(-\frac{a}{3}\right)^3} \exp\left(\textcircled{\pm} i\arctan\Delta^*\right)$$

and next

$$\sqrt[3]{-\frac{b}{2}\textcircled{\pm}i\sqrt{-\Delta}} =$$
$$= \sqrt{-\frac{a}{3}}\exp\left(i\left(\left\{\begin{array}{c}0\\ \pm\frac{2}{3}\pi\end{array}\right\}\textcircled{\pm}\frac{1}{3}\arctan\Delta^*\right)\right),$$

which implies

$$z = u+v = 2\sqrt{-\frac{a}{3}}\cos\left(\left\{\begin{array}{c}0\\ \pm\frac{2}{3}\pi\end{array}\right\}+\frac{1}{3}\arctan\Delta^*\right) \tag{7}$$
$$= \begin{cases}2\sqrt{-\frac{a}{3}}\cos\left(\frac{1}{3}\arctan\Delta^*\right),\\ -\sqrt{-\frac{a}{3}}\left(\cos\left(\frac{1}{3}\arctan\Delta^*\right)\mp\sqrt{3}\sin\left(\frac{1}{3}\arctan\Delta^*\right)\right),\end{cases} \tag{8}$$
$$= \begin{cases}2\sqrt{-\frac{a}{3}}\cos\left(\frac{1}{3}\arctan\Delta^*\right),\\ -2\sqrt{-\frac{a}{3}}\cos\left(\frac{1}{3}\left(\pi+\arctan\Delta^*\right)\right),\\ -2\sqrt{-\frac{a}{3}}\sin\left(\frac{1}{6}\left(\pi+2\arctan\Delta^*\right)\right),\end{cases} \tag{9}$$
$$= \begin{cases}2\sqrt{-\frac{a}{3}}\cos\left(\frac{1}{3}\arctan\Delta^*\right),\\ -2\sqrt{-\frac{a}{3}}\sin\left(\frac{\pi}{6}\pm\frac{1}{3}\arctan\Delta^*\right).\end{cases}$$

Similarly as in the previous case, if we set $\Delta^* = \tan\varphi \geq 0$, $a = -3\alpha^2$, $b = 2\alpha^3\cos\varphi < 0$, then

$$z^3-3\alpha^2 z+(2\cos\varphi)\alpha^3 = 0 \Leftrightarrow z = \begin{cases}-2\alpha\cos\frac{\varphi}{3},\\ \alpha\left(\cos\frac{\varphi}{3}\pm\sqrt{3}\sin\frac{\varphi}{3}\right).\end{cases}$$

For example, we obtain (below we assume that $\alpha > 0$, moreover, the respective values of sine and cosine functions can be found in [20], [21], [28]):

$$z^3-3\alpha^2 z-\sqrt{2}\alpha^3 = 0$$
$$\Leftrightarrow z = \begin{cases}2\alpha\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{2}}\alpha,\\ -\alpha\left(\cos\frac{\pi}{12}\pm\sqrt{3}\sin\frac{\pi}{12}\right) = \begin{cases}-\sqrt{2}\alpha,\\ \frac{-\sqrt{3}+1}{\sqrt{2}}\alpha,\end{cases}\end{cases}$$
$$= \begin{cases}-\sqrt{2}\alpha,\\ \frac{1\pm\sqrt{3}}{\sqrt{2}}\alpha,\end{cases}$$

$$z^3-3\alpha^2 z-\frac{\sqrt{5}+1}{2}\alpha^3 = 0$$
$$\Leftrightarrow z = \begin{cases}2\alpha\cos\frac{\pi}{15} = \frac{\alpha}{4}\left(\sqrt{5}-1+\sqrt{3}\sqrt{10+2\sqrt{5}}\right),\\ -\alpha\left(\cos\frac{\pi}{15}\pm\sqrt{3}\sin\frac{\pi}{15}\right) =\\ = \begin{cases}\frac{\alpha}{4}\left(\sqrt{5}-1-\sqrt{3}\sqrt{10+2\sqrt{5}}\right),\\ \frac{\alpha}{2}(1-\sqrt{5}),\end{cases}\end{cases},$$
$$= \begin{cases}\frac{\alpha}{2}(1-\sqrt{5}),\\ \frac{\alpha}{4}\left(\sqrt{5}-1\pm\sqrt{3}\sqrt{10+2\sqrt{5}}\right).\end{cases}$$

We only need to discuss two more (rather trouble-free) cases.

If $\Delta > 0$, then we have precisely one real zero and two conjugate complex roots

$$z_1 = u-v,$$
$$z_2 = e^{i\frac{2}{3}\pi}u - e^{-i\frac{2}{3}\pi}v = -\frac{1}{2}(u-v)+i\frac{\sqrt{3}}{2}(u+v),$$
$$z_3 = \overline{z_2},$$

where $u := \sqrt[3]{-\frac{b}{2}+\sqrt{\Delta}}$, $v := \sqrt[3]{\frac{b}{2}+\sqrt{\Delta}}$.

Hence, on the basis of the same formulae, if $\Delta = 0$, then we get

$$z_1 = -2\sqrt[3]{\frac{b}{2}}, \quad z_2 = z_3 = \sqrt[3]{\frac{b}{2}}.$$

III. Unexpected identities

By using the cubic polynomials (and their decompositions) from papers [17], [18], [28], [30], [33], [34], [35] and by applying the discussed here Cardano's formulae for the roots of cubic polynomials, we obtained the completely unexpected trigonometric identities (of course we present here just few selected examples denoted by A – H). Let us explain only that all the presented equalities are the consequence of the fact that the decompositions of cubic polynomials, given in the cited above papers, result, in the first place, from the appropriate trigonometric transformations (also by geometric discussion) and not from the application of the Cardano's formulae.

We obtain

A)
$$z^3+z^2-2z-1 = \prod_{k=1}^{3}\left(z-2\cos\frac{2^k\pi}{7}\right),$$

which implies

$$6\cos\frac{2\pi}{7} = -1+2\sqrt{7}\cos\left(\frac{1}{3}\arctan(3\sqrt{3})\right),$$
$$6\cos\frac{4\pi}{7} = -1-2\sqrt{7}\cos\left(\frac{1}{3}\left(\pi+\arctan(3\sqrt{3})\right)\right),$$
$$6\cos\frac{8\pi}{7} = -1-2\sqrt{7}\sin\left(\frac{1}{6}\left(\pi+2\arctan(3\sqrt{3})\right)\right),$$

B)
$$z^3-3z^2-4z-1 = \prod_{k=1}^{3}\left(z-\frac{\cos\frac{2^k\pi}{7}}{\cos\frac{2^{k+2}\pi}{7}}\right),$$

which follows

$$\frac{\cos\frac{8\pi}{7}}{\cos\frac{4\pi}{7}} = 1+2\sqrt{\frac{7}{3}}\cos\left(\frac{1}{3}\arctan\frac{1}{3\sqrt{3}}\right),$$
$$\frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}} = 1-2\sqrt{\frac{7}{3}}\cos\left(\frac{1}{3}\left(\pi+\arctan\frac{1}{3\sqrt{3}}\right)\right),$$

As an example we propose to examine the following polynomial

$$\prod_{k=0}^{2}\left(x-2\cos\left(2^k\alpha\right)\right) = x^3+\left(1+2\cos 3\alpha-\frac{\sin(9\alpha/2)}{\sin(\alpha/2)}\right)x^2$$
$$+\left(2\cos 3\alpha-2\cos 4\alpha-1+\frac{\sin(13\alpha/2)}{\sin(\alpha/2)}\right)x-\frac{\sin 8\alpha}{\sin\alpha},$$

which gives for $\alpha = 2\pi/7, 2\pi/9$ the polynomial of type A and polynomial

$$x^3-3x+1 = \prod_{k=1}^{3}\left(x-2\cos\left(2^k\pi/9\right)\right)$$

which can be simply generated on the basis of discussion presented item E, respectively.

$$\frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}} = 1 - 2\sqrt{\frac{7}{3}}\sin\left(\frac{1}{6}\left(\pi + 2\arctan\frac{1}{3\sqrt{3}}\right)\right).$$

We note that (see [34]):

$$\sum_{k=1}^{3}\sqrt[3]{\frac{\cos\frac{2^k\pi}{7}}{\cos\frac{2^{k+2}\pi}{7}}} = 0.$$

C)

$$z^3 - 7z^2 + 7z + 7 = \prod_{k=1}^{3}\left(z - \sqrt{7}\cot\frac{2^k\pi}{7}\right),$$

which implies

$$3\sqrt{7}\cot\frac{8\pi}{7} = 7 + 4\sqrt{7}\cos\left(\frac{1}{3}\arctan 3\sqrt{3}\right),$$

$$3\sqrt{7}\cot\frac{2\pi}{7} = 7 - 4\sqrt{7}\cos\left(\frac{1}{3}\left(\pi + \arctan 3\sqrt{3}\right)\right),$$

$$3\sqrt{7}\cot\frac{4\pi}{7} = 7 - 4\sqrt{7}\sin\left(\frac{1}{6}\left(\pi + 2\arctan 3\sqrt{3}\right)\right);$$

D)

$$z^3 - z^2 - z - \frac{1}{7} = \prod_{k=1}^{3}\left(z - 1 + \frac{4}{\sqrt{7}}\sin\frac{2^k\pi}{7}\right),$$

which follows

$$\frac{6}{\sqrt{7}}\sin\frac{8\pi}{7} = 1 - 2\cos\left(\frac{1}{3}\arctan\frac{3\sqrt{3}}{13}\right),$$

$$\frac{6}{\sqrt{7}}\sin\frac{2\pi}{7} = 1 + 2\cos\left(\frac{1}{3}\left(\pi + \arctan\frac{3\sqrt{3}}{13}\right)\right),$$

$$\frac{6}{\sqrt{7}}\sin\frac{4\pi}{7} = 1 + 2\sin\left(\frac{1}{6}\left(\pi + 2\arctan\frac{3\sqrt{3}}{13}\right)\right).$$

To the contrast, we note that

$$z^3 + z^2 - z + \frac{1}{7} = \prod_{k=1}^{3}\left(z - \frac{\sqrt{7}}{7}\tan\frac{2^k\pi}{7}\right).$$

E)

$$z^3 + 3\sqrt{3}z^2 - 3z - \sqrt{3} = \left(z - \tan\frac{2\pi}{9}\right)\left(z + \tan\frac{4\pi}{9}\right)\left(z - \tan\frac{8\pi}{9}\right),$$

which implies the relation

$$\tan\frac{2^k\pi}{9} = 4\sin\frac{2^k\pi}{9} + (-1)^k\sqrt{3},$$

for $k = 1, 2, 3$, or the equivalent equalities

$$\tan\frac{2\pi}{9} = 4\sin\frac{2\pi}{9} - \sqrt{3}, \quad \tan\frac{4\pi}{9} = 4\cos\frac{\pi}{18} + \sqrt{3},$$
$$\tan\frac{8\pi}{9} = 4\sin\frac{\pi}{9} - \sqrt{3}.$$

To the contrast, we note that

$$z^3 + \sqrt{3}z^2 - 3z - \frac{\sqrt{3}}{3} = \left(z - \cot\frac{2\pi}{9}\right)\left(z + \cot\frac{4\pi}{9}\right)\left(z - \cot\frac{8\pi}{9}\right),$$

which implies the relation

$$(-1)^{k-1}\sqrt{3}\cot\frac{2^k\pi}{9} = 4\cos\frac{2^k\pi}{9} - 1,$$

for $k = 1, 2, 3$, or the equivalent equalities

$$\cot\frac{2\pi}{9} = \frac{4\cos\frac{2\pi}{9} - 1}{\sqrt{3}}, \quad \cot\frac{4\pi}{9} = \frac{1 - 4\sin\frac{\pi}{18}}{\sqrt{3}},$$
$$-\cot\frac{8\pi}{9} = \frac{4\cos\frac{\pi}{9} + 1}{\sqrt{3}}.$$

F)

$$z^3 + \frac{1-\sqrt{13}}{2}z^2 - z + \frac{\sqrt{13}+3}{2} = \left(z - 2\cos\frac{2\pi}{13}\right)\left(z - 2\cos\frac{6\pi}{13}\right)\left(z - 2\cos\frac{8\pi}{13}\right), \tag{10}$$

which implies

$$12\cos\frac{2\pi}{13} = -1 + \sqrt{13} + 2\sqrt{26 - 2\sqrt{13}}\times \times\cos\left(\frac{1}{3}\arctan\frac{5 + 2\sqrt{13}}{3\sqrt{3}}\right),$$

$$12\cos\frac{6\pi}{13} = -1 + \sqrt{13} - 2\sqrt{26 - 2\sqrt{13}}\times \times\cos\left(\frac{1}{3}\left(\pi + \arctan\frac{5 + 2\sqrt{13}}{3\sqrt{3}}\right)\right),$$

$$12\cos\frac{8\pi}{13} = -1 + \sqrt{13} - 2\sqrt{26 - 2\sqrt{13}}\times \times\sin\left(\frac{1}{6}\left(\pi + 2\arctan\frac{5 + 2\sqrt{13}}{3\sqrt{3}}\right)\right).$$

G)

$$z^3 + \frac{1+\sqrt{13}}{2}z^2 - z - \frac{\sqrt{13}+3}{2} = \left(z - 2\cos\frac{4\pi}{13}\right)\left(z - 2\cos\frac{10\pi}{13}\right)\left(z - 2\cos\frac{12\pi}{13}\right), \tag{11}$$

from which we get

$$12\cos\frac{2\pi}{13} = -1 - \sqrt{13} + 2\sqrt{26 + 2\sqrt{13}}\times \times\cos\left(\frac{1}{3}\arctan\frac{5 - 2\sqrt{13}}{3\sqrt{3}}\right),$$

$$12\cos\frac{10\pi}{13} = -1-\sqrt{13}-2\sqrt{26+2\sqrt{13}}\times \times\cos\left(\frac{1}{3}\left(\pi+\arctan\frac{5-2\sqrt{13}}{3\sqrt{3}}\right)\right),$$

$$12\cos\frac{12\pi}{13} = -1-\sqrt{13}-2\sqrt{26+2\sqrt{13}}\times \times\sin\left(\frac{1}{6}\left(\pi+2\arctan\frac{5-2\sqrt{13}}{3\sqrt{3}}\right)\right).$$

Moreover, surprisingly the following relation holds

$$\frac{\sqrt{13}-4}{3}-2\cos\frac{8\pi}{13} \approx \mathbf{0.577}727 \approx C = \mathbf{0.577}21,$$

where C is the Eulerian constant called also as the Euler-Mascheroni constant (see [13]).

H)

$$z^3-z-1 = (z-\tau_0^{-1})(z-i\sqrt{\tau_0}e^{i\Psi})(z+i\sqrt{\tau_0}e^{-i\Psi}).$$

The above polynomial is called the Perrin polynomial, also called as the Siegel's polynomial (see [7], [34]). Constant τ_0^{-1} plays an important role in estimating the Mahler measure $M(f)$ of polynomials f over $\mathbb{C}$, which are not reciprocal. This estimation is of the form $M(f) \geq \tau_0^{-1}$ and is optimal (see [23], [25]). To the contrast let us note that τ_0 is the only positive root of polynomial $\tau^3+\tau^2-1$ (the same fact holds for the polynomial $\tau^5+\tau-1 = (\tau^3+\tau^2-1)(\tau^2-\tau+1)$) and

$$\Psi := \arcsin\frac{1}{2\sqrt{\tau_0^3}}.$$

The number $-\tau_0$ is the only real root of polynomial $\tau^3-\tau^2+1$. Furthermore, we get

$$6\tau_0 = -2+\sqrt[3]{4}\left(\sqrt[3]{25-3\sqrt{69}}+\sqrt[3]{25+3\sqrt{69}}\right),$$

$$\sqrt[3]{18}\tau_0^{-1} = \sqrt[3]{9+\sqrt{69}}+\sqrt[3]{9-\sqrt{69}},$$

$$\sqrt{\tau_0}\sin\Psi = \frac{1}{2}\tau_0^{-1}, \quad \sqrt{\tau_0}\cos\Psi = \frac{1}{2}\tau_0^{-1}\sqrt{4\tau_0^3-1}$$

and

$$i\sqrt{\tau_0}e^{i\Psi} = \frac{1}{2}\tau_0^{-1}\left(-1+i\sqrt{4\tau_0^3-1}\right),$$

$$i2\sqrt[3]{18}\sqrt{\tau_0}e^{i\Psi} = -\left(\sqrt[3]{9+\sqrt{69}}+\sqrt[3]{9-\sqrt{69}}\right) + i\sqrt{3}\left(\sqrt[3]{9+\sqrt{69}}-\sqrt[3]{9-\sqrt{69}}\right).$$

Hence we deduce the equality

$$\sqrt{4\tau_0^3-1} = \sqrt{3}\frac{\sqrt[3]{9+\sqrt{69}}-\sqrt[3]{9-\sqrt{69}}}{\sqrt[3]{9+\sqrt{69}}+\sqrt[3]{9-\sqrt{69}}}.$$

Furthermore, from the equality $z^3 = z+1$ for $z=\tau_0^{-1}$ we obtain the following decomposition of τ_0^{-1} in the nested third roots

$$\sqrt[3]{\frac{1}{2}-\frac{1}{18}\sqrt{69}}+\sqrt[3]{\frac{1}{2}+\frac{1}{18}\sqrt{69}} = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\ldots}}}$$

We note that the convergence of the above nested radical from Theorem 3.2 in [15] holds. Moreover, we observe that the similar equality is fulfilled also for the only positive root z_k of the polynomial z^k-z-1, $k\geq 3$, having the form $z_k = \sqrt[k]{1+\sqrt[k]{1+\sqrt[k]{1+\ldots}}}$.

In order to emphasize the importance of this section let us introduce two more decompositions of the cubic polynomials, which we have not found in literature till now and which essentially complete the decompositions presented in items F) and G).

So from (10) and (11) we can obtain the decompositions

$$\left(z-2\sin\frac{4\pi}{13}\right)\left(z-2\sin\frac{10\pi}{13}\right)\left(z-2\sin\frac{12\pi}{13}\right) = z^3-\sqrt{\frac{13+3\sqrt{13}}{2}}z^2+\sqrt{13}z-\sqrt{\frac{13-3\sqrt{13}}{2}}$$

and

$$\left(z-2\sin\frac{2\pi}{13}\right)\left(z-2\sin\frac{6\pi}{13}\right)\left(z+2\sin\frac{8\pi}{13}\right) = z^3-\sqrt{\frac{13-3\sqrt{13}}{2}}z^2-\sqrt{13}z+\sqrt{\frac{13+3\sqrt{13}}{2}}.$$

Remark III.1. *In this section we have dealt with expressions of the form* $\cos\left(\frac{1}{3}\arctan\Delta^*\right)$ *and* $\sin\left(\frac{1}{3}\arctan\Delta^*\right)$. *Let us notice that one can find in literature some very interesting decompositions of such expressions on the nested square roots. For example, in [3], [4] we can find the following Ramanujan formula*

$$\lim_{n\to\infty} a_n = \frac{A-1}{6}+\frac{2}{3}\sqrt{4a+A}\sin\left(\frac{1}{3}\arctan\frac{2A+1}{3\sqrt{3}}\right),$$

where $A := \sqrt{4a-7}$, $a\geq 2$, *and*

$$a_1 = \sqrt{a}, \quad a_2 = \sqrt{a-\sqrt{a}}, \quad a_3 = \sqrt{a-\sqrt{a+\sqrt{a}}}, \quad a_4 = \sqrt{a-\sqrt{a+\sqrt{a+\sqrt{a}}}}, \quad \ldots$$

and where the sequence of signs $-,+,+,\ldots$, *appearing in this nested radicals, has period 3. In case* $a=44$ *we obtain (see equalities in examples A, C, F for possible connections):*

$$\lim_{n\to\infty} a_n = 2+2\sqrt{21}\sin\left(\frac{1}{3}\arctan 3\sqrt{3}\right).$$

IV. Ramanujan's cubic polynomials

V. Shevelev and R. Wituła in papers [1], [24], [29], [31] have distinguished and discussed the so called Ramanujan's cubic polynomials and Ramanujan's cubic polynomials of the second kind, denoted for shortness by RCP and RCP2, respectively. And all this could happen thanks to the great Indian mathematician Srinivasa Ramanujan who proposed the proof of the following equalities (see [22]):

$$\left(\frac{1}{9}\right)^{1/3}-\left(\frac{2}{9}\right)^{1/3}+\left(\frac{4}{9}\right)^{1/3}=(\sqrt[3]{2}-1)^{1/3},$$

$$\left(\cos\frac{2\pi}{7}\right)^{1/3}+\left(\cos\frac{4\pi}{7}\right)^{1/3}+\left(\cos\frac{8\pi}{7}\right)^{1/3} =\left(\frac{5-3\sqrt[3]{7}}{2}\right)^{1/3},$$

$$\left(\cos\frac{2\pi}{9}\right)^{1/3}+\left(\cos\frac{4\pi}{9}\right)^{1/3}+\left(\cos\frac{8\pi}{9}\right)^{1/3} =\left(\frac{3\sqrt[3]{9}-6}{2}\right)^{1/3}.$$

It is easy to connect the above equations with the following problem: for which cubic polynomials $Q(x)$ (with all real roots) of the form

$$Q(x)=(x-\xi_1)(x-\xi_2)(x-\xi_3)=x^3+px^2+qx+r$$

there exists a function $f(p,q,r)$ that possesses possibly "simple" algebraic form and for which the following equality holds

$$\sqrt[3]{\xi_1}+\sqrt[3]{\xi_2}+\sqrt[3]{\xi_3}=\sqrt[3]{f(p,q,r)}.$$

Although some attempts to solve this interesting problem have been undertaken (see for example the respective Ramanujan Theorem in [4] or in the second Notebook of Ramanujan [22]), only the mentioned above V. Shevelev and R. Wituła succeeded in distinguishing the appropriate families of cubic polynomials and in describing properties of these polynomials (see also the paper [1]). Let us only recall that if the following conditions are satisifed

$$\begin{cases} r\neq 0,\\ p\sqrt[3]{r}+3\sqrt[3]{r^2}+q=0,\\ \left(\frac{b}{2}\right)^2+\left(\frac{a}{3}\right)^3<0, \end{cases} \tag{12}$$

where

$$a:=q-\frac{p^2}{3},\quad b:=\frac{2}{27}p^3-\frac{1}{3}pq+r,$$

then $Q(x)$ is a RCP and the following identities hold (the last expressions in all three formulae below are prepared for the algorithmic applications):

$$\begin{aligned}\sqrt[3]{x_1}+\sqrt[3]{x_2}+\sqrt[3]{x_3}&=\left(-p-6r^{1/3}+3(9r-pq)^{1/3}\right)^{1/3}\\ &=\operatorname{sgn}\left(-p-6r^{1/3}+3\operatorname{sgn}(9r-pq)|9r-pq|^{1/3}\right)\\ &\times\left|-p-6r^{1/3}+3\operatorname{sgn}(9r-pq)|9r-pq|^{1/3}\right|^{1/3},\end{aligned}$$

$$\begin{aligned}\sqrt[3]{x_1x_2}+\sqrt[3]{x_1x_3}+\sqrt[3]{x_2x_3}&=\sqrt[3]{q+6r^{2/3}-3\sqrt[3]{9r^2-pqr}}\\ &=\operatorname{sgn}\left(q+6r^{2/3}-3\operatorname{sgn}(9r^2-pqr)|9r^2-pqr|^{1/3}\right)\\ &\times\left|q+6r^{2/3}-3\operatorname{sgn}(9r^2-pqr)|9r^2-pqr|^{1/3}\right|^{1/3},\end{aligned}$$

as well as the so called Shevelev's formula [24], [31]:

$$\begin{aligned}&\sqrt[3]{\frac{x_1}{x_2}}+\sqrt[3]{\frac{x_2}{x_1}}+\sqrt[3]{\frac{x_1}{x_3}}+\sqrt[3]{\frac{x_3}{x_1}}+\sqrt[3]{\frac{x_2}{x_3}}+\sqrt[3]{\frac{x_3}{x_2}}\\ &=\frac{1}{\sqrt[3]{x_1x_2x_3}}\left(\sqrt[3]{x_1^2x_3}+\sqrt[3]{x_1x_3^2}+\sqrt[3]{x_1^2x_2}+\sqrt[3]{x_1x_2^2}\right.\\ &\left.+\sqrt[3]{x_2^2x_3}+\sqrt[3]{x_2x_3^2}\right)=\operatorname{sgn}\left(\frac{pq}{r}-9\right)\sqrt[3]{\left|\frac{pq}{r}-9\right|}.\end{aligned}$$

To the contrast, if instead of conditions (12) the following conditions are fulfilled

$$\begin{cases} r\neq 0,\\ p^3r+27r^2+q^3=0,\\ \left(\frac{b}{a}\right)^2+\left(\frac{a}{3}\right)^3<0, \end{cases} \tag{13}$$

then $Q(x)$ is a RCP2 and the following relations hold

$$\begin{aligned}\sqrt[3]{\xi_1}+\sqrt[3]{\xi_2}+\sqrt[3]{\xi_3}&=\Big(-p-6\sqrt[3]{r}\\ &-3\sqrt[3]{3\sqrt[3]{r}(q+p\sqrt[3]{r})}-3\sqrt[3]{(p+3\sqrt[3]{r})(q+3\sqrt[3]{r^2})}\Big)^{1/3},\end{aligned}$$

as well as

$$\begin{aligned}&\sqrt[3]{\frac{\xi_1}{\xi_2}}+\sqrt[3]{\frac{\xi_2}{\xi_1}}+\sqrt[3]{\frac{\xi_1}{\xi_3}}+\sqrt[3]{\frac{\xi_3}{\xi_1}}+\sqrt[3]{\frac{\xi_2}{\xi_3}}+\sqrt[3]{\frac{\xi_3}{\xi_2}}\\ &=\sqrt[3]{\frac{3}{r^{2/3}}(q+p\sqrt[3]{r})}+\sqrt[3]{\frac{pq}{r}+\frac{3}{r^{2/3}}(q+p\sqrt[3]{r}+3\sqrt[3]{r^2})}.\end{aligned}$$

Let us notice that a cubic polynomial $p(x)=x^3+px^2+qx+r$, which is either RCP or RCP2, is simultaneously RCP and RCP2 if and only if $pq=0$ which implies that either

$$\begin{aligned}p(x)=x^3-3\sqrt[3]{r^2}x+r&=\left(x-2\sqrt[3]{r}\cos\frac{2\pi}{9}\right)\times\\ &\times\left(x-2\sqrt[3]{r}\cos\frac{4\pi}{9}\right)\left(x-2\sqrt[3]{r}\cos\frac{8\pi}{9}\right)\end{aligned}$$

or

$$\begin{aligned}p(x)=x^3-3\sqrt[3]{r}x^2+r&=\left(x-\frac{1}{2}\sqrt[3]{r}\sec\frac{2\pi}{9}\right)\times\\ &\times\left(x-\frac{1}{2}\sqrt[3]{r}\sec\frac{4\pi}{9}\right)\left(x-\frac{1}{2}\sqrt[3]{r}\sec\frac{8\pi}{9}\right),\end{aligned}$$

where $r\in\mathbb{R}\setminus\{0\}$. On the other hand we know that the polynomials belonging to families RCP and RCP2 share many common analytical-algebraic properties (see [14], [24]).

Let us present now the examples of cubic polynomials with indicating the Ramanujan classes of polynomials they belong:
1° $p(x)=x^3+3x^2-3\sqrt[3]{2}x+1$ is the RCP2 polynomial which is not the RCP one. It has the following zeros (we apply here our formulae (5)):

$$x_1=-1-2\sqrt{1+\sqrt[3]{2}}\cos\left(\frac{1}{3}\operatorname{arccot}\frac{3}{\sqrt{4\sqrt[3]{2}-5}}\right),$$

$$x_2 = -1 + 2\sqrt{1+\sqrt[3]{2}}\cos\left(\frac{1}{3}\left(\pi + \operatorname{arccot}\frac{3}{\sqrt{4\sqrt[3]{2}-5}}\right)\right),$$

$$x_3 = -1+2\sqrt{1+\sqrt[3]{2}}\sin\left(\frac{1}{6}\left(\pi + 2\operatorname{arccot}\frac{3}{\sqrt{4\sqrt[3]{2}-5}}\right)\right).$$

Additionally we observe that

$$\frac{3}{\sqrt{4\sqrt[3]{2}-5}} = \sqrt{3(25+20\sqrt[3]{2}+16\sqrt[3]{4})} = \frac{\sqrt{3}(\sqrt[3]{2}+1)}{\sqrt[3]{2}-1},$$

so by identity $\arctan x = \operatorname{arccot}\frac{1}{x}$ for $x > 0$ we get

$$\operatorname{arccot}\frac{3}{\sqrt{4\sqrt[3]{2}-5}} = \arctan\frac{\sqrt{3}(\sqrt[3]{2}+1)}{\sqrt[3]{2}-1}.$$

Using formulae (3)–(5) from [29] (or the respective ones from [31]) we obtain the following Ramanujan type equalities (for real roots of third order):

$$\sqrt[3]{x_1} + \sqrt[3]{x_2} + \sqrt[3]{x_3} = 0,$$

$$\sqrt[3]{\frac{x_1}{x_2}} + \sqrt[3]{\frac{x_2}{x_1}} + \sqrt[3]{\frac{x_1}{x_3}} + \sqrt[3]{\frac{x_3}{x_1}} + \sqrt[3]{\frac{x_2}{x_3}} + \sqrt[3]{\frac{x_3}{x_2}} = -3. \quad (14)$$

We also have $x_1 + x_2 + x_3 = -3$ and

$$\sqrt[3]{x_1x_2} + \sqrt[3]{x_1x_3} + \sqrt[3]{x_2x_3} = -\sqrt[3]{3(\sqrt[3]{2}+1)}.$$

Hence we deduce the relation

$$\begin{aligned} P(x) &= (x-\sqrt[3]{x_1})(x-\sqrt[3]{x_2})(x-\sqrt[3]{x_3}) \\ &= x^3 - \sqrt[3]{3(\sqrt[3]{2}+1)}x + 1 \end{aligned} \quad (15)$$

and next, by applying the Cardano's formulae (5) we get

$$\sqrt[3]{x_1} = -\frac{2}{\sqrt[3]{3}}\sqrt[6]{1+\sqrt[3]{2}}\cos\left(\frac{1}{3}\arctan\frac{1}{3}\sqrt{4\sqrt[3]{2}-5}\right),$$

$$\sqrt[3]{x_2} = \frac{2}{\sqrt[3]{3}}\sqrt[6]{1+\sqrt[3]{2}}\cos\left(\frac{1}{3}\left(\pi+\arctan\frac{1}{3}\sqrt{4\sqrt[3]{2}-5}\right)\right),$$

$$\sqrt[3]{x_3} = \frac{2}{\sqrt[3]{3}}\sqrt[6]{1+\sqrt[3]{2}}\sin\left(\frac{1}{6}\left(\pi+2\arctan\frac{1}{3}\sqrt{4\sqrt[3]{2}-5}\right)\right).$$

By comparing these relations with the formulae for values of x_1, x_2, x_3 we obtain the identity

$$\frac{x_k+1}{\sqrt[3]{x_k}} = \sqrt[3]{3(\sqrt[3]{2}+1)}, \quad k = 1,2,3,$$

It is well known that each polynomial $q(x)$ from RCP2 class has the from (see [31]):

$$q(x) = x^3 + 3\sqrt[3]{kr}x^2 - 3\sqrt[3]{(k+1)r^2}x + r,$$

where $k, r \in \mathbb{R}$, $r \neq 0$. Then for the roots x_1, x_2, x_3 of $q(x)$ the following Shevelev's identity holds

$$\begin{aligned} &\sqrt[3]{\frac{x_1}{x_2}} + \sqrt[3]{\frac{x_2}{x_1}} + \sqrt[3]{\frac{x_1}{x_3}} + \sqrt[3]{\frac{x_3}{x_1}} + \sqrt[3]{\frac{x_2}{x_3}} + \sqrt[3]{\frac{x_3}{x_2}} \\ &= \sqrt[3]{9}\left(\sqrt[3]{\sqrt[3]{k}-\sqrt[3]{k+1}} + \sqrt[3]{(\sqrt[3]{k}+1)(1-\sqrt[3]{k+1})}\right). \end{aligned}$$

Hence and from (14) we deduce the equality

$$\sqrt[3]{3} = \left(\sqrt[3]{2}+1\right)\sqrt[3]{\sqrt[3]{2}-1}.$$

(this identity results directly from formula (15)). It means that the following "unexpected" polynomial identity holds

$$p\left(\sqrt[3]{3(\sqrt[3]{2}+1)}x - 1\right) = 3(\sqrt[3]{2}+1)P(x). \quad (16)$$

Moreover, by "numerical experiment" we find that

$$\begin{aligned} &\sqrt[3]{x_1x_2} + \sqrt[3]{x_1x_3} + \sqrt[3]{x_2x_3} + \sqrt[3]{2} + \sqrt[3]{\sqrt[3]{2}-1} \\ &= \sqrt[3]{2} + \sqrt[3]{\sqrt[3]{2}-1} - \sqrt[3]{3(\sqrt[3]{2}+1)} \approx 0.00545. \end{aligned}$$

2° $q(x) = x^3 + x^2 - 2x - 1 = (x - 2\cos\frac{2\pi}{7})(x - 2\cos\frac{4\pi}{7})(x - 2\cos\frac{8\pi}{7})$ – an example of the RCP polynomial which is not the RCP2 one.

3° The polynomials from examples B) – H) presented in Section 3 are neither RCP polynomials nor RCP2 ones.

Announcement

While preparing this section we have solved unexpectedly one more problem. We have proven that the polynomials

$$\begin{aligned} R(x;p) &:= x^3 + 9px^2 + 23\left(\frac{6}{1\pm\sqrt{93}}\right)^3 p^2x \\ &- \left(\frac{6}{1\pm\sqrt{93}}\right)^3 p^3 = (x-x_1)(x-x_2)(x-x_3), \end{aligned} \quad (17)$$

where $p \in \mathbb{R}\setminus\{0\}$ and (for the upper signs):

$$\begin{aligned} \frac{x_1}{p} &= -3 - \frac{6\sqrt{6(\sqrt{93}-1)}}{23}\cos\frac{\operatorname{arccot}\sqrt{31}}{3}, \\ \frac{x_2}{p} &= -3 + \frac{6\sqrt{6}(\sqrt{93}-1)}{23}\cos\frac{\pi+\operatorname{arccot}\sqrt{31}}{3}, \\ \frac{x_3}{p} &= -3 + \frac{6\sqrt{6}(\sqrt{93}-1)}{23}\sin\frac{\pi+2\operatorname{arccot}\sqrt{31}}{6} \end{aligned} \quad (18)$$

are the only RCP such that polynomial

$$(x-\sqrt[3]{x_1})(x-\sqrt[3]{x_2})(x-\sqrt[3]{x_3})$$

belongs to the RCP2 family (see [2]). By applying "our" Cardano's formulae and a bit of Mathematica "magic simplifications" we obtain the following relations

$$\sqrt[3]{\frac{x_1}{p}} = -\frac{1+\sqrt{93}+4\sqrt{106-9\sqrt{93}}\cos\left(\frac{1}{3}\varphi\right)}{2\sqrt[3]{182\sqrt{93}-1497}},$$

$$\sqrt[3]{\frac{x_2}{p}} = \frac{-1-\sqrt{93}+4\sqrt{106-9\sqrt{93}}\cos\left(\frac{1}{3}(\pi+\varphi)\right)}{2\sqrt[3]{182\sqrt{93}-1497}},$$

$$\sqrt[3]{\frac{x_3}{p}} = \frac{-1-\sqrt{93}+4\sqrt{106-9\sqrt{93}}\sin\left(\frac{1}{6}(\pi+2\varphi)\right)}{2\sqrt[3]{182\sqrt{93}-1497}},$$

where $\varphi = \arctan\left(\frac{1}{2}\sqrt{3}\left(9+\sqrt{93}\right)\right)$ x_1, x_2, x_3 are defined by formulae (18).

V. Conclusion

In this paper we have presented the algorithms for determining the roots of cubic complex polynomials. The proposed algorithms generate the descriptions of these roots with the aid of radicals of trigonometric functions. The examples of testing polynomials, presented in this paper, have been originally generated by using the direct methods different than the given here algorithms. In consequence, by applying the algorithms presented here we received, the most often, the new and different symbolic descriptions of the sought roots of the given cubic polynomials. It gave us the possibility, by comparing the obtained and the testing descriptions, to reveal many new identities and relations. Additionally some conjecture arose about the possible existence of description of the values of functions $\cos\left(\frac{1}{3}\arctan\alpha\right)$, $\sin\left(\frac{1}{3}\arctan\alpha\right)$ in the form of radicals of variable α, where α is also a radical defined on the set of rational numbers. We intend to discuss this problem in a separate paper. Moreover, let us notice that our algorithm verifies whether the given cubic polynomial belongs to the RCP or RCP2 class.

Also in a separate paper (see [36]) we intend, as we declared at the end of Introduction, to extend the discussion of the symbolic description of the roots of cubic polynomials, undertaken in this paper, for the roots of quartic polynomials and the selected polynomials of higher order. Next, we plan to adapt them numerically since we count on some better results then the ones obtained with the aid of Mathematica software.

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