Small Strong Epsilon Nets

Pradeesha Ashok* Sathish Govindarajan† Janardhan Kulkarni‡

Abstract

In this paper, we initiate the study of small strong ϵ-nets and prove bounds for axis-parallel rectangles, half spaces, strips and wedges. We also give some improved bounds for small weak ϵ-nets.

1 Introduction

Let P be a set of n points in the plane. $N \subset \mathbb{R}^2$ is a *weak* ϵ*-net* for a family of geometric objects $\mathcal{S}$ if $S \cap N \neq \emptyset$ for any $S \in \mathcal{S}$ such that $|S \cap P| > \epsilon n$. Moreover, N is a *strong* ϵ*-net* if $N \subset P$. The concept of ϵ-nets was introduced by Haussler and Welzl [4] and has found many applications in computational geometry, approximation algorithms, learning theory etc.

It has been proved that for a range space $(P, \mathcal{S})$ with finite VC Dimension d, there exist ϵ-nets of size $O(\frac{d}{\epsilon}\log\frac{1}{\epsilon})$ [4]. Also, ϵ-nets of size $O(\frac{1}{\epsilon})$ exist for half spaces in $\mathbb{R}^2$, $\mathbb{R}^3$ and pseudodisks in the plane [5, 6, 7]. Recently, it is shown that ϵ-nets of size $O(\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$ exist for axis-parallel rectangles in the plane [2].

Small weak ϵ-nets have been studied for convex objects, axis-parallel rectangles, disks and half spaces in [1, 3, 8]. In this problem, the size of a weak ϵ-net is fixed as i and the value of ϵ_i is bounded. Small nets are especially interesting for the range spaces where tight bounds for epsilon nets are not known. Also, small epsilon nets can be seen as a generalization of center points for different range spaces.

In this paper, we initiate the study of small strong ϵ-nets. Let $\epsilon_i^{\mathcal{S}} \in [0, 1]$ represents the smallest real number such that, for any set of points P in the plane, there exists a set $Q \subset P$ of size i which is an $\epsilon_i^{\mathcal{S}}$-net for P with respect to $\mathcal{S}$. We obtain bounds on $\epsilon_i^{\mathcal{S}}$ where $\mathcal{S}$ is the family of axis-parallel rectangles, half spaces, strips or wedges. We also improve some of the small weak ϵ-net bounds given in [1].

*Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India, pradeesha@csa.iisc.ernet.in

†Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India, gsat@csa.iisc.ernet.in

‡Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India, janardhan@csa.iisc.ernet.in

	Rectangles		Halfspaces		Strips	Wedges
	LB	UB	LB	UB	LB	LB
ϵ_1	3/4		1		1	1
ϵ_2	1/2	2/3	5/9	2/3	2/3	1
ϵ_3	3/8	9/16	1/2		1/2	2/3
ϵ_4	2/7	1/2	1/3	1/2	2/5	1/2

Table 1: Summary of lower and upper bounds for $\epsilon_i^{\mathcal{S}}$.

2 General lower bounds for Axis-parallel rectangles and Half spaces

Let $\mathcal{S}$ represents the family of axis-parallel rectangles/ half spaces.

Theorem 1 $\epsilon_i^{\mathcal{S}} \geq \frac{1}{i}$, *for* $i \geq 2$

Proof. Let P be a set of n points arranged uniformly along the boundary of a circle with center c. Let $N = \{p_1, ..., p_i\} \subset P$ be an $\epsilon_i^{\mathcal{S}}$-net. Connect c with p_j for $1 \leq j \leq i$. These lines divide the circle into i sectors and at least one of these sectors contains $\frac{n-i}{i}$ points from P. We can have a range $S \in \mathcal{S}$ such that S contains all points from this sector and does not include any point from N. Therefore, for large values of n, $\epsilon_i^{\mathcal{S}} \geq \frac{1}{i}$. □

In Section 3, we give improved (better than $\frac{1}{i}$) lower bounds for axis-parallel rectangles for $i \leq 4$. For $i \geq 5$, the above bounds improve upon the previously known bounds of $\frac{1}{i+1}$. Section 4 gives better bounds for half spaces.

The above general bound also applies to weak ϵ-nets for the family of axis-parallel rectangles, half spaces and convex sets. The proof for this is similar to Theorem 1.

3 Axis-parallel Rectangles

In this section, we show bounds on $\epsilon_i^{\mathcal{R}}$ for axis-parallel rectangles. The summary of the bounds are given in Table 1.

Let P be a set of n points in $\mathbb{R}^2$. Assume that all points in P have distinct x and y co-ordinates.

Lemma 2 $\epsilon_1^{\mathcal{R}} = \frac{3}{4}$

Proof : Let V_1 and V_2 be vertical lines that divide P such that V_1 has $\frac{n}{4} - 1$ points of P to the left of it and V_2 has $\frac{n}{4} - 1$ points of P to the right of it. Similarly, let H_1 and H_2 be horizontal lines that divide P such that

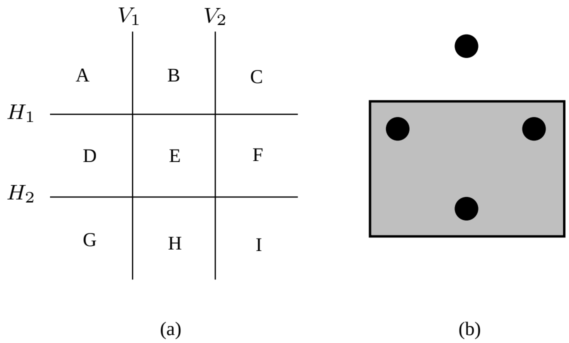

Figure 1: Upper and lower bound for $\epsilon_1^{\mathcal{R}}$

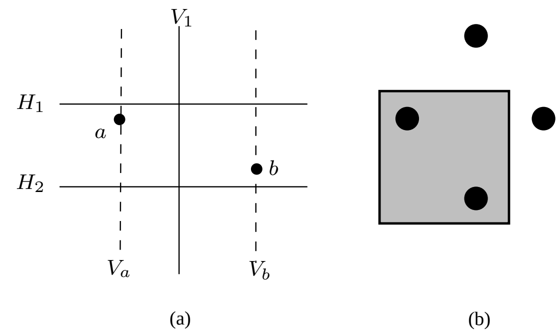

Figure 2: Upper and lower bounds for $\epsilon_2^{\mathcal{R}}$

H_1 has $\frac{n}{4} - 1$ points of P above it and H_2 has $\frac{n}{4} - 1$ points of P below it(See figure 1(a)). Now, the second column contains $\frac{n}{2} + 2$ points. Since regions B and H contain at most $\frac{n}{4} - 1$ points each, the region E is not empty.

Let x be any point in region E. Any axis-parallel rectangle that does not contain x avoids at least one row or column thereby missing out at least $\frac{n}{4}$ points. Thus $\{x\}$ is a $\frac{3}{4}$-net.

To show the lower bound, consider the point set shown in figure 1(b). The n points are arranged as four subsets of equal size. For any point $x \in P$, there exists an axis-parallel rectangle that does not contain x but includes $\frac{3n}{4}$ points from the other three subsets.

Lemma 3 $\frac{1}{2} \leq \epsilon_2^{\mathcal{R}} \leq \frac{2}{3}$

Proof : Let H_1 and H_2 be two horizontal lines that partition P such that each of the three regions contain $\frac{n}{3}$ points. Let V_1 be a vertical line bisecting P. Let a and b be input points in the second row with the least perpendicular distance from V_1 on either side of V_1(See figure 2(a)). We claim that $\{a, b\}$ is a $\frac{2}{3}$-net.

Let R be an axis-parallel rectangle that contains more than $\frac{2n}{3}$ points from P. R should take points from all the three horizontal regions and also should contain points from both sides of V_1. Let V_a and V_b be vertical lines passing through the points a and b respectively. Assume R does not contain any of the points in $\{a, b\}$. Therefore R should be restricted to the region between V_a and V_b and cannot include points from the second row . Hence, a contradiction.

To prove the lower bound, consider the point set shown in figure 2(b). The point set contains four subsets of equal size. For any two points selected from P, there exists an axis-parallel rectangle that contains $\frac{n}{2}$ points and neither of the selected points.

Lemma 4 $\frac{3}{8} \leq \epsilon_3^{\mathcal{R}} \leq \frac{9}{16}$

Proof : Let R be an axis-parallel bounding rectangle containing P. Let $\{p\}$ be $\epsilon_1^{\mathcal{R}}$-net for P, constructed as described in Lemma 2. The vertical line passing through p divides R into two regions of which, at most one, say R_1, can contain more than $\frac{n}{2}$ points from P. Similarly, the horizontal line through p divides R into two regions of which, at most one, say R_2, can have more than $\frac{n}{2}$ points from P. Let $Q = R_1 \cap P$ and $S = R_2 \cap P$. Also, Q and S can have at most $\frac{3n}{4}$ points as $\{p\}$ is a $\frac{3}{4}$-net. Let $\{q\}$ and $\{s\}$ be $\epsilon_1^{\mathcal{R}}$-net for Q and S respectively.

Let $\mathfrak{R}$ be an axis-parallel rectangle that does not contain the points p, q, s. Since it does not contain p, $\mathfrak{R}$ can take points from only one of the regions $R \setminus R_1$, $R \setminus R_2$, R_1 or R_2. In the first two cases, it can contain at most $\frac{n}{2}$ points from P. So assume without loss of generality that it takes points only from the region R_1. Now, since $\mathfrak{R}$ does not contain the point q, it can contain at most $\frac{3}{4}|Q|$ points from P i.e. $\mathfrak{R}$ can contain at most $\frac{3}{4} \cdot \frac{3n}{4} = \frac{9n}{16}$ points from P. Therefore $\{p, q, s\}$ is a $\frac{9}{16}$-net.

To prove the lower bound, consider the point set as shown in figure 3. The point set consists of n points arranged into eight subsets of equal size. We claim that whichever three points we choose from P, there exists an axis-parallel rectangle that avoids these three points and contains three out of the eight subsets. Index these subsets as $1, 2, 3...8$ starting from any subset and moving in the clockwise direction. Consider the axis-parallel rectangles that contain three consecutive subsets. There are eight such rectangles and any point from P can cover only three of them. Hence, three points are needed to cover all the eight rectangles. Figure 3 shows the two ways of picking these three points. In both the cases, there exists an axis-parallel rectangle with at least $\frac{3n}{8}$ points and not containing any of them.

Lemma 5 $\frac{2}{7} \leq \epsilon_4^{\mathcal{R}} \leq \frac{1}{2}$

Proof : Let V_1 and H_1 be lines that bisect P vertically and horizontally respectively and p be the point of intersection. Now bisect the two horizontal slabs so that we get a grid with all the rows containing $\frac{n}{4}$ points each(See figure 4(a)). Let a and b be input points in the second row with the least perpendicular distance from

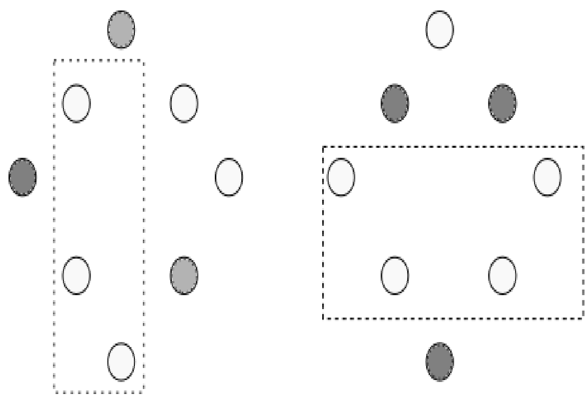

Figure 3: Lower bound for $\epsilon_3^{\mathcal{R}}$

V_1 on either side of V_1. Similarly, let c and d be input points in the third row with the least perpendicular distance from V_1 on either side of V_1. We claim that $\{a, b, c, d\}$ is a $\frac{1}{2}$-net.

Let R be an axis-parallel rectangle which contains more than $\frac{n}{2}$ points from P. Clearly, R contains the point p and includes points from at least three rows. Assume R does not contain any of the points in $\{a, b, c, d\}$. Let V_a, V_b, V_c, V_d be vertical lines passing through the points a, b, c, d respectively. If R includes any point from the first row, then it should be restricted to the region between V_a and V_b and cannot include points from the second row. Similarly if R includes any point from the fourth row, it cannot take points from the third row. Therefore, R can include points from at most two rows, which is a contradiction. Hence, $\epsilon_4^{\mathcal{R}} \leq \frac{1}{2}$.

To prove the lower bound, consider the point set as shown in Figure 4(b), where n points are arranged into seven subsets of equal size inside a grid having three rows and five columns. Let R_{ij} be the subset in the intersection of ith row and jth column where $1 \leq i \leq 3$ and $1 \leq j \leq 5$. A point has to be chosen from R_{23} as part of the $\epsilon_4^{\mathcal{R}}$-net since R_{23} forms an axis-parallel rectangle of size $\frac{2n}{7}$ with all other R_{ij}s. Also, at least two points are needed to cover the four axis-parallel rectangles of size $\frac{2n}{7}$ formed by subsets R_{12}, R_{14}, R_{32} and R_{34}. Assume these two points are chosen from R_{12} and R_{34}. Now there are two disjoint axis-parallel rectangles containing R_{32}, R_{21} and R_{14}, R_{25}. These two rectangles cannot be covered by a single point. Hence, $\epsilon_4^{\mathcal{R}} \geq \frac{2}{7}$.

4 Half spaces

For odd values of i, $\epsilon_i^{\mathcal{H}} = \frac{2}{i+1}$. The lower bound follows from a construction given in [5]. The upper bound follows from a construction given in [6] which reduces the problem of finding an ϵ-net for half spaces in $\mathbb{R}^3$ to the problem of finding such an ϵ-net for points in convex position, by projecting the points on to the convex hull. We can apply the same technique in $\mathbb{R}^2$. For even values

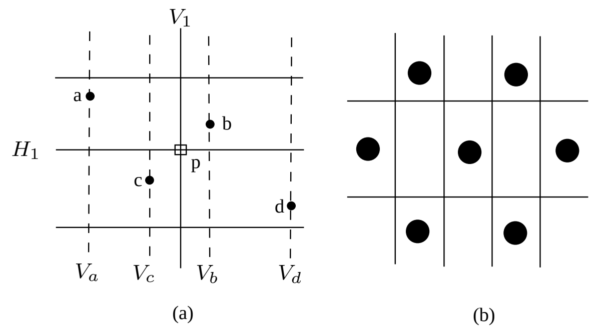

Figure 4: Upper and Lower bounds for $\epsilon_4^{\mathcal{R}}$

of i the best known bounds are $\frac{2}{i+2} \leq \epsilon_i^{\mathcal{H}} \leq \frac{2}{i}$, which follows from the bounds of $\epsilon_{i+1}^{\mathcal{H}}$ and $\epsilon_{i-1}^{\mathcal{H}}$.

Lemma 6 $\frac{5}{9} \leq \epsilon_2^{\mathcal{H}} \leq \frac{2}{3}$

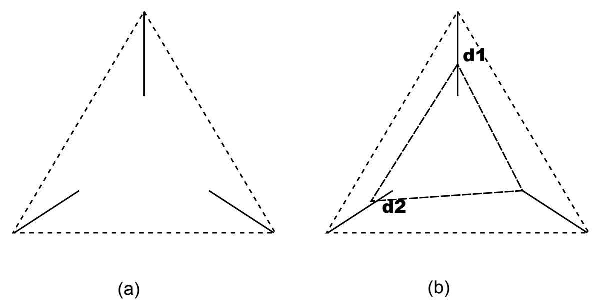

Figure 5: Lower bound for $\epsilon_2^{\mathcal{H}}$

Proof : The upper bound follows from [5]. To show the lower bound, consider the point set as shown in Figure 5(a). The points are arranged as three subsets of equal size near the corners of a triangle along the bold lines. Let a and b be the two points selected. If a and b belong to same subset, then there exists a half space containing all the points from other two subsets i.e, it contains $\frac{2n}{3}$ points. If not, let a and b be the d_1th and d_2th point respectively in their subsets. In this case, there exists a half space that excludes a and b and contains $f(d_1, d_2)$ points, where $f(d_1, d_2) = max(d_1 + d_2, n - (d_1 + d_2), \frac{n}{3} + d_1, \frac{n}{3} + d_2)$. For any d_1, d_2 such that $1 \leq d_1, d_2 \leq \frac{n}{3}$, $f(d_1, d_2)$ is at least $\frac{5n}{9}$.

5 Intersection of two half spaces

In this section we consider small strong nets for ranges defined by intersection of two half spaces. There are two possibilities. First is the family of strips which are formed by two intersecting half spaces with parallel supporting lines. Second is the family of wedges which are formed by two intersecting half spaces with non-parallel supporting lines.

Lemma 7 *For the family of strips,* $\epsilon_i^{\mathcal{T}} \geq \frac{2}{i+1}$

Proof : Let P be a set of n points arranged uniformly on the boundary of a circle and $\{p_1, p_2, ...p_i\}$ be any subset of P, ordered in clockwise direction. Let d_j be the number of points of P between p_j and p_{j+1} and $d_{max} = \max d_j$.

If d_{max} is unique, a strip that does not contain any of the p_is can contain d_j points,$1 \leq j \leq i$ (See strip A in Figure 6) or $2d_j$ points where $d_j < d_{max}$(See strip B in Figure 6). The maximum number of points that can be present in any of these strips are minimized when all d_j except d_{max} are the same and $2d_j = d_{max}$. Therefore, $d_{max} = \frac{2n}{i+1}$. If d_{max} is not unique, then there exists a strip having at least $\frac{2(n-i)}{i}$ points. Therefore, for sufficiently large n, $\epsilon_i^{\mathcal{T}} \geq \frac{2}{i+1}$.

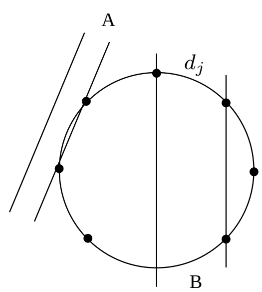

Figure 6: Lower bound for strips

Lemma 8 *For the family of wedges,* $\epsilon_i^{\mathcal{W}} \geq \frac{2}{i}$

Proof : Let P be a set of n points arranged uniformly along the boundary of a circle. Let $N = \{p_1, p_2, ...p_i\} \subset P$ partition P into i intervals. A wedge that does not contain any point from N can still include all the points from any two intervals. Therefore, there exists a wedge that contains $2\frac{n-i}{i}$ points. Therefore, $\epsilon_i^{\mathcal{W}} \geq \frac{2}{i}$.

6 Small Weak Epsilon Nets

In this section, we consider small weak ϵ-nets for disks. We show an improved lower bound of $\epsilon_3^{\mathcal{D}} \geq \frac{2}{7}$. This improves upon the general lower bound of $\frac{1}{4}$.

Lemma 9 $\epsilon_3^{\mathcal{D}} \geq \frac{2}{7}$

Proof : Arrange n points, in seven equal subsets, uniformly along the boundary of a circle. Now we claim that for any three points p, q, r in the plane, we can draw a circle which contains at least two subsets and not containing p, q, r.

Consider the line l passing through p and q. l can intersect at most two subsets. So we assume it intersects exactly two subsets. If there exist two or more subsets on both sides of l, we can draw the needed circle on the side not containing r. So, assume that one side of l contains at most one subset and hence the other side of l has at least four subsets and let r lie in that side. In this case, the line pr contains at least two subsets on one of the sides and we can draw the needed circle.

Conclusions and Future Work

In this paper, we have shown lower and upper bounds for $\epsilon_i^{\mathcal{S}}$ where $\mathcal{S}$ is the family of axis-parallel rectangles, halfspaces, wedges or strips. An interesting open question is to find the exact value of $\epsilon_i^{\mathcal{S}}$ for small values of i. Another interesting question is to obtain non-trivial bounds on ϵ_i for the family of disks.

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